On the Laplace Transform
of a Continuous Additive Functional

by P.J. Fitzsimmons

The theorem below was established in [**1**] in the context of a stable Markov chain on $\{0, 1, 2, \ldots\}$. We present here a simple proof based on the domination principle.

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbf{P}^x)$ be a right Markov process with state space E, finite lifetime ζ, and cemetery state Δ. Let $A = (A_t)_{t\ge 0}$ be a continuous additive functional of X such that $A_\zeta < \infty$ almost surely. Define $\varphi(x) := \mathbf{P}^x[\exp(-A_\zeta)]$, with the understanding that $\varphi(\Delta) = 1$. Observe that

$$1 - \exp(-A_\zeta) = \int_0^\zeta \exp(-A_\zeta \circ \theta_t)\, dA_t, \tag{1}$$

whence the identity

$$1 - \varphi(x) = \mathbf{P}^x \int_0^\zeta \varphi(X_t)\, dA_t = U_A\varphi(x), \qquad x \in E_\Delta, \tag{2}$$

where U_A is the potential operator associated with A: $U_A\varphi(x) := \mathbf{P}^x \int_0^\zeta \varphi(X_t)\, dA_t$. In particular,

$$M_t^\varphi := \mathbf{P}^x \left[\int_0^\zeta \varphi(X_s)\, dA_s \middle| \mathcal{F}_t \right] = \int_0^t \varphi(X_s)\, dA_s + 1 - \varphi(X_t), \qquad t \ge 0,$$

is a uniformly integrable martingale under $\mathbf{P}^x$, for each $x \in E$. The intuitive content of this statement becomes clearer when A has the simple form $A_t = \int_0^t a(X_s)\, ds$ for some bounded positive Borel function a. Writing L for the infinitesimal generator of X, we see that the martingale property of M^φ amounts to saying that $L\varphi = a \cdot \varphi$. The theorem stated below asserts that φ is the maximal solution of this equation with values in $[0, 1]$.

Theorem. *Let $w : E_\Delta \to [0, 1]$ be a finely continuous function, and suppose that*

$$M_t := w(X_t) - \int_0^t w(X_s)\, dA_s$$

is a uniformly integrable martingale under $\mathbf{P}^x$, for each $x \in E$. Then $w \le \varphi$.

Proof. Define $h := w + U_A w$. It is easy to check that h is harmonic, in the sense that $\mathbf{P}^x[h(X_T)] = h(x)$ for each stopping time T. Observe that

$$U_A(\varphi - w) = 1 - h - (\varphi - w) \le 1 - h \quad \text{on } \{\varphi > w\}.$$

The harmonicity of $1 - h$ and the domination principle now imply that

$$U_A(\varphi - w) \le 1 - h \quad \text{on } E.$$

That is,

$$-(\varphi - w) \le 0 \quad \text{on } E,$$

which means that $w \le \varphi$ on all of E_Δ. ∎

Reference

[1] P.K. Pollett and V.T. Stefanov: Path integrals for continuous-time Markov chains. *J. Appl. Prob.* **39** 2002 901-904.