Extreme Points in the Balayage Order

August 12, 1997

Fix μ with σ-finite potential and let $\mathcal{A}$ denote the convex cone

$$\{\nu : \nu U \leq \mu U\}.$$

Using the integral representation theorem of [1], we give a short proof of the following result of Mokobodzki. Let $\mathcal{A}_{\rm ex}$ denote the set of extreme points of $\mathcal{A}$.

Theorem. $\mathcal{A}_{\rm ex} = \{\mu H_B : B \in \mathcal{E}^e, B \textit{ finely closed}\}$.

Proof. (a) Let ν be an extreme point of $\mathcal{A}$. By the main result of [1] there is a decreasing family $\{B(r) : 0 \leq r \leq 1\}$ of finely closed $\mathcal{E}^e$-measurable sets such that

$$\nu = \int_0^1 \mu H_{B(r)}\, dr.$$

The extremality of ν and the monotonicity of $r \mapsto B(r)$ imply that $r \mapsto H_{B(r)}$ is constant on $[0,1]$. Thus $\nu = \mu H_{B(1/2)}$.

(b) Fix ν of the form μH_B, where B is $\mathcal{E}^e$-measurable and finely closed. Suppose there are elements ν_1 and ν_2 of $\mathcal{A}$ such that $\nu = \alpha\nu_1 + (1-\alpha)\nu_2$, where $0 < \alpha < 1$. By Rost 's theorem there are randomized stopping times T_1 and T_2 such that

$$\nu_i = \mu P_{T_i} \qquad i = 1, 2.$$

Each ν_i is carried by B, so

$$\nu_i = \mu P_{T_i} = \mu P_{T_i} H_B = \mu P_{D_i},$$

where $D_i := T_i + D_B \circ \theta_{T_i}$. Thus

$$\mu H_B = \nu = \alpha\nu_1 + (1-\alpha)\nu_2 = \alpha\mu P_{D_1} + (1-\alpha)\mu P_{D_2},$$

which forces $\mu H_B = \mu P_{D_i}$ $(i = 1,2)$ since we clearly have μP_{D_i} downstream from μH_B. Therefore $D_i = D_B$, P^μ-a.s. for $i = 1,2$. Consequently, $\nu = \mu H_B = \mu P_{D_i} = \nu_i$ $(i = 1,2)$, so ν is an extreme point of $\mathcal{A}$. □

Reference

[1] P.J. Fitzsimmons (1991): Skorokhod embedding by randomized hitting times," In: *Seminar on Stochastic Processes 1990*, Birkhauser, Boston, pp. 183-191.