On a Problem of Kingman

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Let $(E, \mathcal{E})$ be a Lusin metrizable space and let Φ be a random measure on $(E, \mathcal{E})$. More precisely, Φ is a random variable, defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with values in the (positive, σ-additive) measures on $(E, \mathcal{E})$. We assume that Φ is *completely random* [**3**], in the sense that $\{\Phi(A_n) : n \in \mathbf{N}\}$ is an independent sequence of random variables whenever $\{A_n : n \in \mathbf{N}\}$ is a collection of pairwise disjoint $\mathcal{E}$-measurable sets.

It is shown in [**3**], under a mild σ-finiteness condition, that such a random measure Φ can be decomposed as

$$\Phi = \Phi_f + \Phi_d + \Phi_0, \tag{1}$$

where Φ_f is a purely atomic measure with atoms of independent sizes located at the points of a deterministic subset of E, and Φ_d is a non-atomic deterministic measure. The remaining component Φ_0 is the most interesting of the three, and is the subject of most of the discussion in [**3**]. In particular, it is shown there that Φ_0 is equal in distribution to a purely atomic measure Φ_*. In [**4**] an argument due to D. Blackwell [**1**] is adapted to prove that Φ_0 is itself purely atomic (with probability 1), under a broad additional condition. Our aim in this note is to point out that there is a simple direct argument showing that Φ_0 is purely atomic in general.

For simplicity, we assume in what follows that $\mathbf{P}[\Phi(E) < \infty] = 1$. This finiteness condition is stronger than the σ-finiteness condition mentioned earlier; in particular, the decomposition (1) is valid.

Let M denote the class of finite measures on $(E, \mathcal{E})$, and let M_a denote the subclass of purely atomic measures. Let $\mathcal{M}$ be the σ-algebra on M generated by the maps $\mu \mapsto \mu(B)$, $B \in \mathcal{E}$. In saying that Φ_0 and Φ_* have the same distribution, we mean that $(\Phi_0(B_1), \Phi_0(B_2), \ldots, \Phi_0(B_n))$ has the same distribution as $(\Phi_*(B_1), \Phi_*(B_2), \ldots, \Phi_*(B_n))$ for all n-tuples $(B_1, B_2, \ldots, B_n)$ of elements of $\mathcal{E}$ and all $n \in \mathbf{N}$. It then follows from the monotone class theorem that

$$\mathbf{P}[\Phi_0 \in C] = \mathbf{P}[\Phi_* \in C], \qquad \forall C \in \mathcal{M}. \tag{2}$$

In view of (2), we need only verify that M_a is $\mathcal{M}$-measurable. But this is an immediate consequence of Lemma 2.3 on page 20 of [**2**]. Indeed, each $\mu \in M$ admits a unique decomposition $\mu = \mu_d + \mu_a$ into diffuse and purely atomic parts, and the mapping $\mu \mapsto \mu_d$ is $\mathcal{M}$-measurable. Thus $M_a = \{\mu \in M : \mu_d(B) = 0, \forall B \in \mathcal{E}\}$ is $\mathcal{M}$-measurable because $\mathcal{E}$ is countably generated.

References

[1] Blackwell, D.: Discreteness of Ferguson selections, *Ann. Statist.* **1** (1973) 356–358.

[2] Kallenberg, O.: *Random Measures* (Third edition). Akademie-Verlag, Berlin, 1983.

[3] Kingman, J. F. C.: Completely random measures, *Pacific J. Math.,* **21** (1967) 59–78.

[4] Kingman, J. F. C.: *Poisson Processes,* Oxford University Press, New York, 1993.