Lecture 8. \( F: \mathbb{R} \to \mathbb{R} \)

Let \( F \) be increasing + right cont. and \( \mathcal{M}_F \) the Lebesgue-Stieltjes measure.

\( \mathcal{M}_F \) denotes the \( \sigma \)-algebra (containing \( \mathcal{B}_\mathbb{R} \)) of \( \mathcal{M}_F \). Recall that, by construction, for \( E \in \mathcal{M}_F \),

\[
\mathcal{M}_F(E) = \inf \left\{ \sum_{k=1}^{\infty} (F(b_k) - F(a_k)) : E \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k] \right\}
\]

Note that using half-open intervals was convenient for the construction of \( \mathcal{M}_F \), but they are special to \( \mathbb{R} \) so would be useful to replace them at this point by open intervals.
In fact, we have

**Theorem.** For all $E \in \mathcal{M}_F$, \[
\mu_F(E) = \inf \left\{ \mu(U) : E \subseteq U, U \text{ open} \right\} \]
\[
\geq \sup \left\{ \mu(K) : K \subseteq E, K \text{ compact} \right\}.
\]

**Proof.** First, note that this is trivial if $\mu_F(E) = \infty$. Pick $\varepsilon > 0$ and cover $E$ by $\bigcup_{n=1}^{\infty} (a_n, b_n)$ s.t. $\mu_F(E) \geq \sum_{n=1}^{\infty} (F(b_n) - F(a_n)) - \varepsilon/2$.

Next, we might want to find $b'_n > b_n$ s.t. $F(b_n) > F(b'_n) - \varepsilon/2^k$. Then $E \subseteq \bigcup_{n=1}^{\infty} (a_n, b'_n) = : U$ open, \[
\mu_F(U) \leq \sum_{n=1}^{\infty} (F(b'_n) - F(a_n)) \leq \sum_{n=1}^{\infty} (F(b_n) - F(a_n)) + \varepsilon/2 \leq \mu_F(E) + \varepsilon/2 + \varepsilon/2 \quad \Rightarrow \quad \mu_F(U) \leq \mu_F(E) + \varepsilon.
\]
Since \( \mu_f(E) \leq \mu_f(U) \) for all \( U \supseteq E \), we conclude
\[
\mu_f(E) = \inf \{ \mu_f(U) : E \subseteq U \text{ open} \}.
\]

(2) Let \( E_n = E \cap [E_n, \infty) \). Then, each \( E_n \) is closed, \( E_1 \subseteq E_2 \subseteq \cdots \), \( E = \bigcup_{n=1}^{\infty} E_n \)
\[
\Rightarrow \quad \mu_f(E) = \lim_{n \to \infty} \mu_f(E_n).
\]

If we can show \( \mu_f(E_n) = \sup \{ \mu_p(K) : E_2 \in \mathcal{K} \text{ compact} \} \)

then \( \mu_f(E) = \sup \{ \mu_f(K) : \cdots \} \) follows easily. Thus, WLOG, assume \( E \subseteq \mathbb{R} \)

is closed, say \( E \subseteq [E, R] \). Let \( A = [E, R] \setminus E \) and, given \( \varepsilon > 0 \), take \( U \)

open s.t. \( A \subseteq U \) and \( \mu_f(A) \geq \mu_f(U) - \varepsilon \).
$U^c$ is closed and $U^c \subseteq A^c = E \cup [r, R]^c$

Since $E \subseteq [r, R]$, $K = U^c \cap [r, R] \subseteq E$

and $K$ is closed + bounded $\Rightarrow$ compact.

$\mu_f([r, R]) \leq \mu_f(U) + \mu_f(K)$

$\mu_f(E) + \mu_f(A) = \mu_f([-r, R]) \leq \mu_f(U) + \mu_f(K)$

$\leq \mu_f(A) + \varepsilon + \mu_f(K) \Rightarrow$

$\mu_f(E) \leq \mu_f(K) + \varepsilon \Rightarrow \mu_f(E) = \sup \{ \mu_f(K) : E \supseteq K \text{ compact} \}$

This "regularity" of the Lebesgue-Stieltjes leads to a description of the sets in $\mu_f$ as follows.