Thm 1. Let $T \in GL(n, \mathbb{R})$.

(i) $E \in L^n \Rightarrow T(E) \in L^n$ and 
$m(T(E)) = |\det T| m(E)$.

(ii) $f \in L^n$ meas. $\Rightarrow f \circ T \in L^n$ meas.
If $f \in L^1$ or $L^\infty$, then

$\int f \circ T \, dm = |\det T| \int f \, dm$

Pf. We first recall that any $T \in GL(n, \mathbb{R})$
can be written as $T = S_1 \circ S_2 \circ \ldots \circ S_m$, where
the $S_j$ are elementary matrices, i.e. of the form:

(I) $S = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} = 2 \ \text{def} \ S = c$

(II) $S = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 \end{pmatrix} \Rightarrow \text{def} \ S = -1$

(III) $S = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} \Rightarrow \text{def} \ S = 1
For proof, first establish \((\mathcal{C})\) for \(f\) \(B\)-meas., and elementary matrices \(T\). Since \(T\) is cont., \(f \circ T\) is \(B\)-meas.

We check integral when \(f \in L^1\) or \(L^1\) for each elementary type of \(T\):

(I) Using FT we get

\[
\int (f \circ T) \, dm = \int \ldots \int (S f (c x_j)) \, dx_j \wedge dx_j \ldots dx_j \ldots dx_n
\]

\[
= \left| c_1 \right|^2 \int f \, dm = |\text{det } T|^{-1} \int f \, dm
\]

Corollary of Thm 1.21: \(\{m'(t E) = |t| m'(C E)\}

in \(1-D\) as in \(m'(a + E) = m'(CE)\) .

Proof of Thm 1.

(II) \(\int (f \circ T) \, dm = \int f \, dm\) by FT

swap order \(i \leftrightarrow j\).

and \(|\text{det } T| = |-1| = 1\)
\[ (iii) \quad S(f \circ T) \, dm = \underbrace{S \left( \int x_j \, dx_j \right)}_{= S f \, dx_j \text{ by Thm 1.21}} \, \, dx_j \\
= S f \, dm \, , \text{ and } |\det T| = 1. \]

Now, for any \( T = S_1 \circ \ldots \circ S_m \, , \, S_j \text{ elem.} \)

\[ S f \, dm = |\det S_1| \, S f \circ S_1 \, dm = \ldots \]

\[ \underbrace{|\det S_1 \ldots \det S_m|}_1 \, S f \circ S_1 \circ \ldots \circ S_m \, dm \]

\[ = |\det T| \, S f \circ T \, dm \, . \]

Thus, we have established (ii) for \( f \in B_{\mathbb{R}^n} \)-meas. \( \Rightarrow \) (i) for \( E \subset B_{\mathbb{R}^n} \)

\( \chi_E(Tx) = \chi_{T^{-1}E}(x) \)
In particular, \( m(N) = 0 \iff m(IN) = 0 \) for all \( N \subseteq B_R \). The extension of (i) to all \( E \subseteq L^1 \) follows as in pf of Thm 1. \( \Rightarrow \) (i) holds for \( f = \chi_E \) all \( E \subseteq L^1 \) \( \Rightarrow \) for all \( f = \phi \) simple \( L^1 \)-meas. MCT + DCT \( \Rightarrow \forall f \) as in Thm 1.
Change of variables.

Let \( \Omega \subseteq \mathbb{R}^n \) open, \( F: \Omega \to \mathbb{R}^n \) a \( C^1 \) map ( \( \frac{\partial F}{\partial x_j} = F_{x_j} \) exist + cont.). Let \( DF_x \) denote linear map \( \mathbb{R}^n \to \mathbb{R}^n \) given by matrix of partial derivatives at \( x \). If \( \det DF_x \neq 0, \forall x \in \Omega \), then by Inverse Function Theorem, \( F(\Omega) \subseteq \mathbb{R}^n \) is open.

Def! \( F \) is a diffeomorphism \( \Omega \to \Omega' \) if \( \Omega' = F(\Omega) \) and \( F \) is invertible with \( F^{-1} \) being \( C^1 \).

Rev. By IFT, suffices that \( \det DF_x \neq 0 \) and \( F \) injective.
Thm 2. Let \( \Omega \subseteq \mathbb{R}^n \) be open, \( F: \Omega \to \mathbb{R}^n \) a diffeomorphism.

(ii) \( E \subseteq \mathbb{R}^n \Rightarrow F(E) \subseteq \mathbb{R}^n \) and
\[
\mu(F(E)) = \frac{1}{|\det Df|} \int_E \mu(E)
\]

(iii) \( f \in L^1(\mathbb{R}^n) \Rightarrow f \circ F \) is \( L^1 \)-meas. on \( \Omega \). If \( f \in L^1 \) or \( L^\infty \) then
\[
\int_{\Omega'} f \, dm = \int_{\Omega} (f \circ F) |\det Df| \, dm
\]

PF. By similar arguments to the pf of Thm 1, suffices to prove the result for \( B_{\mathbb{R}^n} \) and \( B_{\mathbb{R}^n} \)-meas. \( f \). Also, (i) \( \Rightarrow \) (iii) so suffices to prove (iii).

We shall need the following:

Lemma 1. Let \( \Omega \subseteq \mathbb{R}^n \) open. Then \( \Omega \)

\( \{ B_{\mathbb{R}^n} \} \), closed equilateral cubes \( R_k \) w/ disjoint interiors, s.t. \( \Omega = \bigcup_{k=1}^{\infty} R_k \).

PF. Construction using \( A(\Omega) \) from Lecture 23