Lecture 21.

Product measures. We shall define the product measure of 2 measure spaces \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\). An inductive procedure works for \(n\) spaces but we shall restrict to \(n=2\).

Recall. On \(X \times Y\), we have introduced the product \(\sigma\)-algebra \(\mathcal{M} \otimes \mathcal{N}\). In the \(\sigma\)-finite case, it is generated by \(A \times Y, A \in \mathcal{M}, B \in \mathcal{N}\). We shall define \(\mu \otimes \nu\) on \(\mathcal{M} \otimes \mathcal{N}\) by first defining a premeasure \(\pi\) on an algebra \(\mathcal{A}\) that generates \(\mathcal{M} \otimes \mathcal{N}\).

1. The collection of rectangles \(A \times B\) as above is an elementary family. 

Prop. 7 \(\Rightarrow\) \(\mathcal{A} = \{\text{finite disjoint unions of such}\}\) is an algebra.
Given disjoint union $\bigcup_{n=1}^{\infty} A_n \times B_n$ in $\mathcal{A}$, let

$$\Pi\left(\bigcup_{n=1}^{\infty} A_n \times B_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \nu(B_n).$$

Then, $\Pi$ is well-defined on $\mathcal{A}$ (check drift representation gives same result; cf. before) and $\Pi$ is a premeasure; Note that if $\bigcup_{n=1}^{\infty} A_n \times B_n$ a disjoint union in $\mathcal{A}$

then $\varphi_n(x,y) = \sum_{n=1}^{\infty} \chi_{A_n}(x) \chi_{B_n}(y)$ is

a simple function separately in $x,y$, in $L^+(X), L^+(Y)$, and $\varphi_n \rightarrow \varphi = \sum_{n=1}^{\infty} \chi_{A_n} \chi_{B_n}.$

Since $\bigcup_{n=1}^{\infty} A_n \times B_n$ in $\mathcal{A}$ it is a finite disjoint union of rectangles. Assume, for simplicity that it is just $A \times B$ ($m=1$; the general case can be reduced to this). Then, $\chi_{A \times B}(x,y) = \sum_{n=1}^{\infty} \chi_{A_n}(x) \chi_{B_n}(y)$.
Also, \( X_{A \times B}(x, y) = X_A(x) X_B(y) \).

Integrating in \( x \), by MCT \( \Rightarrow \)

\[
\mu(A) X_B(y) = \sum_{n=1}^{\infty} \mu(A_n) X_{B_n}(y)
\]

Integrating in \( y \) \( \Rightarrow \)

\[
\mu(A) \mu(B) = \sum_{n=1}^{\infty} \mu(A_n) \mu(B_n)
\]

Thus,

\[
\Pi(A \times B) = \mu(A) \mu(B) = \sum_{n=1}^{\infty} \Pi(A_n \times B_n).
\]

\( \Rightarrow \) \( \Pi \) is premeasure on \( \mathcal{A} \).

By "standard procedure", \( \Pi \rightarrow \) outer measure \( \rightarrow \) measure on \( M \otimes N \) which coincides w/ \( \Pi \) on \( \mathcal{A} \).

Def. This \( \Pi \) is the product measure \( \mu \times \nu \) on \( M \otimes N \).
Rem. When $\mu, \nu$ are $\sigma$-finite, so is $\mu \times \nu$. In this case, $\mu \times \nu$ is the unique measure on $\mathcal{M} \mathcal{N}$ that coincides with $\nu$ on $\mathcal{N}$.

**General set.** Let $E \in \mathcal{M} \mathcal{N}$. Define $x$- and $y$-sections by

- $E_x = \{ y \in Y : (x,y) \in E \}$
- $E_y = \{ x \in X : (x,y) \in E \}$

For $f : X \times Y \to C$ (or $\mathbb{R}$ or ...), let

$$f_{x}(y) = f(x,y) = f_{y}(x).$$

**Prop.** $E \in \mathcal{M} \mathcal{N} \Rightarrow \exists E_x \in \mathcal{N}, E_y \in \mathcal{M} \text{ for all } x, y$.

**Pf.** Let $\mathcal{C} = \{ \text{all } E \in X \times Y \text{ s.t. concl. holds} \}$.

Then, $\mathcal{C}$ is a $\sigma$-algebra and contains $\mu \times \nu$. 

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all rectangles $\Rightarrow E \in M \otimes N$. Details are DIY.

Cor 1. $f \in M \otimes N$-meas. $\Rightarrow f_x, f_y$
are $N$- resp. $M$-meas. for all $x, y$.

Thm. Suppose $(X, M, \mu), (Y, N, \nu)$ are
finite. If $E \in M \otimes N$, then
$x \mapsto \nu(E_x), y \mapsto \mu(E_y)$ are $M$- resp.
$N$-measurable and

$$(\mu \times \nu)(E) = \int \mu(E_y) \, d\nu(y)$$

$$= \int \nu(E_x) \, d\mu(x).$$