Lecture 16

Recall: We are given a measure space $(X, \mathcal{M}, \mu)$. $L^1 = L^1(X, \mu)$ denotes the normed (metric) space of equivalence classes $[f]$ ($\nu$ means equal $\mu$-a.e.) of integrable $f : X \to \mathbb{C}$. We still think of elements of $L^1$ as being functions (representatives), but only determined up to null sets. We can turn any statement "$\mu$-a.e." to everywhere by multiplying those fns by $1_{\mathcal{N}}$, where $\mathcal{N}$ is the null set. A fundamental result:

**Dominated Convergence Theorem**. Let

$\{f_n\}_{n=1}^{\infty} \subseteq L^1(X, \mu)$ and assume

1. $f_n \to f$ $\mu$-a.e. and
2. $\exists g \in L^\infty(X, \mu)$ s.t. $|f_n| \leq g$ $\mu$-a.e.

Then, $f \in L^1$ and $\int f = \lim_{n \to \infty} \int f_n$. 

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\textbf{Pf.} Per above, \( f \) is meas. (after possibly modifying it on a null set), and \( \|f\| \leq g \Rightarrow \|f\| \leq \|g\| \Rightarrow f \in L^1 \). If we write \( f_n = u_n + iv_n \Rightarrow f = u + iv \), then \( u_n \to u, v_n \to v \) a.e. and it suffices to show \( su_n \to su, sv_n \to sv \). We have, by assumption, that \( g = u_n \in L^1 \). By Fatou's,

\[ Sg = Su + Sv \leq \liminf_{n \to \infty} S_{u_n} \]

\[ \Rightarrow Su \leq \liminf_{n \to \infty} S_{u_n} \]

\[ \Rightarrow Su \geq \limsup_{n \to \infty} S_{u_n} \]

\[ \Rightarrow \limsup_{n \to \infty} S_{u_n} \leq Su \leq \liminf_{n \to \infty} S_{u_n} \]

which of course yields \( Su = \lim_{n \to \infty} S_{u_n} \). Similarly for \( v_n, v \).
As a result, we have powerful convergence results not available in Rieman’s theory of integrals.

Thus 1. Let \( f : [a, b] \rightarrow \mathbb{C} \) be s.t.

\( f_t = f(\cdot, t) \) integrable for each \( t \in [a, b] \)

and let \( F : [a, b] \rightarrow \mathbb{C} \) be given by

\[
F(t) = \int_{a}^{b} f_t(x) \, dx \tag{1}
\]

Then:

(i) If \( \lim_{t \to t_0} f_t = f_{t_0} \quad \forall x \)

\( \exists \, g \text{ s.t. } |f_t| \leq g, \quad \forall x, t, \) then

\[
\lim_{t \to t_0} F_t = F_{t_0}
\]

(i.e. \( \lim_{t \to t_0} \int_{a}^{b} f_t(x) \, dx \) = \( \int_{a}^{b} f_{t_0}(x) \, dx \)
(ii) If $t \to f_t(x)$ is diff. $\forall x$ and $\exists g(\text{ s.t. } \left| \frac{\partial f_t}{\partial t} \right| \leq g(\text{ } \forall x, t)$, then $F$ is diff. and

$$\frac{dF}{dt} = \int f_t \frac{df_t}{dt} \, dx$$

(i.e. $\frac{d}{dt} \int f_t(x,t) \, dx(x) = \int \frac{df_t}{dt}(x,t) \, dx(x)$)

**Rem.** In this time, we are assuming that conditions hold $\forall x$ and not a.e. $x$. Why? For seq. $|f_n| \leq g$ a.e. = $\exists \text{ null set } N_n \text{ s.t. } |f_n| \leq g$ on $N_n$. Then $N = \bigcup_{n=1}^{\infty} N_n$ is also null and $|f_n| \leq g$ on $N \setminus N_n \forall n$. But if $|f_n| \leq g$ a.e. for all $t \in [a,b]$, then $\exists N_n \text{ s.t. } |f_n| \leq g$ on $N_n$ but $N = \bigcup_{n=1}^{\infty} N_n$ need not be null. One should formulate a.e. versions but would need to be more careful.
Pf. (i) follows immediately from DCT once you recall that \( \lim_{t \to 0} h(t) = Y \)
\[ \Rightarrow \forall \{t_n\}_{n=1}^{\infty}, \text{ s.t. } t_n \to 0 \text{ we have } \lim_{n \to \infty} h(t_n) = Y. \]

(ii) Need to check, for \( t_n \to 0 \), that
\[ F(t_0 + t_n) - F(t_0) - \int_{t_n}^{t_0} \frac{\partial F}{\partial x}(x, t_0) \, dx (x) \to 0 \]
\[ \left( \int_{t_n}^{t_0} \left( \frac{f(x, t_0 + t_n) - f(x, t_0)}{t_n} - \frac{\partial F}{\partial x}(x, t_0) \right) \, dx (x) \right) \]
\[ h_n(x) \]
Since \( t \to f(x) \) is diff., by Mean Value Theorem
\[ \left| f(x, t_0 + h) - f(x, t_0) \right| \leq \sup_{h \in [t_0, t_0 + h]} \left| \frac{\partial f}{\partial x}(x, t_0) \right| \]
\[ \Rightarrow |h_n(x)| \leq 2g. \text{ Since } h_n \to 0, \forall x, \text{ the conclusion (i) follows}, \text{ again by DCT.} \]