Integration of $C$-valued fees. $(X, M, \mu)$ given.

First, if $f : X \to \mathbb{R}$ is decomposed $f = f^+ - f^-$, then $f^+, f^- \in L^+$, so $Sf^+, Sf^-$ are defined.

If $Sf^+ < \infty$ or $Sf^- < \infty$, we may define

$$Sf = Sf^+ - Sf^-.$$ 

Def 1. (i) If $Sf^+ < \infty$ and $Sf^- < \infty$ (or equivalently $S|f| < \infty$), then we say $f$ is integrable.

(ii) If $f : X \to \mathbb{C}$, we may decompose $f = u + iv$, where $u, v : X \to \mathbb{R}$. We say $f$ is integrable if $u, v$ are integrable and define

$$Sf = Su + i Sv.$$
We shall use notation \( L^{1}(X,\mu) = L^{1}(\mu) = L^{1} \) for space of \( \mathbb{C} \)-valued integrable fns.

**Obs.** \( L^{1}(X,\mu) \) is a vector space / \( \mathbb{C} \) and \( S: L^{1} \to \mathbb{C} \) is linear.

**Prop.** Let \( f, g \in L^{1}(X,\mu) \).

(i) \( |Sf| \leq |Sg| \)

(ii) \( \{ x : f(x) \neq 0 \} \) is \( \sigma \)-finite.

(iii) \( Sf = Sg, \forall E \in \mathcal{M} \iff f = g \mu \text{-a.e.} \)

**Pr.** (i) \( \text{D.I.Y.} \)

(ii) Part of HW.

(iii) Let \( h = f - g \). Since \( h = 0 \iff |h| = 0 \), we see \( h = 0 \iff S|h| = 0 \) by Folland Prop. 2.16 and hence \( S|h| = 0, \forall E \in \mathcal{M} \) by monotonicity. Since \( S|h| = 0 \), \( \forall E \) then follows by (i). Thus, "\( \iff \)" is proved.
To establish converse, suppose \( h \neq 0 \) on \( E \) w/ \( \mu(E) > 0 \). Writing \( h = u^+i^+v = u^-i^-i(v+v) \) we note at least one of \( u^+, u^-, v^+, v^- \) must be > 0 on \( E \). Suppose \( u^+ > 0 \) on \( E \). Then, \( u^- = 0 \) on \( E \) and \( Su > 0 \) by Prop 2.16
\[ \Rightarrow \text{Re} \left( \frac{\text{Sh}}{E} \right) > 0 \Rightarrow \frac{\text{Sh}}{E} \neq 0 \text{; proving } \Rightarrow \text{.} \]

At this point, let us modify our def. of \( L^1(x, \mu) \). For \( f, g : X \to C \) integrable, we say \( f \sim g \) if \( f = g \; \mu\text{-a.e.} \) and let
\[ L^1(x, \mu) = \left\{ \left[ f \right] : f : X \to C \text{ integrable} \right\}. \]

Then \( L^1 \) is still a vector space / \( C \) and \( S : L^1 \to C \) is a well-defined linear functional. We make \( L^1 \) into a normed (metric) space by.
\[ \|f\|_{L^1} = \sup_{x \in X} |f(x)|. \]

(Prop?) by
Thus, the metric is \( d(f, g) = \| f - g \|_{L^1} \).

Note: Even though elements of \( L^1 \) are eq. classes, we speak of "functions" in \( L^1 \) referring to some repr. of the eq. class.

(2) This avoids the issue that may arise if \( \mu \) is not complete: If \( \{f_n\} \) is a seq. of meas. funct. s.t. \( f_n \rightarrow f \) \( \mu \)-a.e., \( f \) need not be meas. but if \( f_n \rightarrow f \) on \( E \) w/ \( \mu(E^c) = 0 \), then \( \mathcal{X} f_n \rightarrow \mathcal{X} f \) everywhere, so \( \mathcal{X} f \) is meas. and \( \mathcal{X} f = f \) \( \mu \)-a.e.

A fundamental convergence result:

**Dominated Convergence Theorem.** Let \( \{f_n\} \subseteq L^1(\mathcal{X}, \mu) \) s.t. \( f_n \rightarrow f \) \( \mu \)-a.e. and \( \exists g \in L^1 \) s.t. \( |f_n| \leq g \) \( \mu \)-a.e. Then, \( f \in L^1 \) and \( \mathcal{S} f = \lim_{n \rightarrow \infty} \mathcal{S} f_n \).