Lecture 14

We may now establish additivity of \( \int_X f \, d\mu \) on \( L^+ \). In fact:

**Thm 2.** If \( \{ \phi_n \}_{n=1}^{\infty} \) is a seq. in \( L^+ \) and \( f = \sum_{n=1}^{\infty} \phi_n \), then

\[
\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X \phi_n \, d\mu.
\]

**Pf.** First, let's establish finite additivity. Take \( f_1, f_2 \in L^+ \) and recall \( f \) increasing sequences of simple functions \( \phi_n, \psi_n \) in \( L^+ \) s.t. \( \phi_n \uparrow f_1 \), \( \psi_n \uparrow f_2 \) for all \( x \in X \), and hence \( \phi_n - \psi_n = \phi_n + \psi_n \) is simple for s.t. \( \phi_n \uparrow f_1 + f_2 \). By Monotone Conv. Thm,

\[
\int_X f \, d\mu = \lim_{n \to \infty} \int_X \phi_n \, d\mu = \lim_{n \to \infty} \int_X \phi_n + \lim_{n \to \infty} \int_X \phi_n = \text{add for simple funs}
\]

Thus,

\[
\int_X f \, d\mu = \int_X \phi_n \, d\mu + \int_X \psi_n \, d\mu = \int_X \phi_n \, d\mu + \int_X \psi_n \, d\mu = \sum_{n=1}^{N} \int_X \phi_n \, d\mu + \sum_{n=1}^{\infty} \int_X \psi_n \, d\mu = \sum_{n=1}^{\infty} \int_X \phi_n \, d\mu + \sum_{n=1}^{\infty} \int_X \psi_n \, d\mu = \sum_{n=1}^{\infty} \int_X \phi_n \, d\mu + \sum_{n=1}^{\infty} \int_X \psi_n \, d\mu.
\]

By induction, we have

\[
\sum_{n=1}^{N} \phi_n = \sum_{n=1}^{\infty} \int_X \phi_n.
\]
Since \( \sum_{n=1}^{\infty} p_n \rightarrow f \) as \( N \rightarrow \infty \), MCT \( \Rightarrow \) 
\[ \tilde{S} f = \sum_{n=1}^{\infty} \tilde{S} f_n \text{ as claimed.} \]

The following seems trivial but is important.

Thus, if \( f \in L^1 \), then

\[ \text{if } p \mu = 0 \Leftrightarrow f = 0 \mu \text{-a.e.} \]

except on a set \( N \in \mathcal{N} \) such that \( \mu(N) = 0 \).

**Proof.** \( f \leq \tilde{f} \) \( \Rightarrow \)

\( 0 \leq \tilde{S} f \leq \tilde{S} \tilde{f} \). Also, \( \phi_n = \chi_{-2N} \rightarrow \tilde{f} \) and \( \tilde{S} \phi_n = 0 \Rightarrow \tilde{S} \tilde{f} = 0 \).

\( \Rightarrow \) Consider \( E_n = \{ x : f(x) \geq \frac{1}{n} \} \). Then \( E_1 \subset E_2 \subset \ldots \) and \( E = \bigcap_{n=1}^{\infty} E_n = \{ x : f(x) > 0 \} \).
By def. \( q_n = \frac{1}{n} \sum E_n \leq f \Rightarrow \frac{1}{n} \mu(E_n) \leq Sf = 0. \)

Since \( \mu(E) = \lim_{n \to \infty} \mu(E_n) \) by cont from below, we conclude \( \mu(E) = 0 \), i.e.
\( f = 0 \) \( \mu \)-a.e.

\[ \square \]

Cor.  Let \( \{ f_n \}_{n=1}^\infty \) be seq. in \( L^+ \) s.t.
\( f_1 \leq f_2 \leq \ldots \) \( \mu \)-a.e. and \( f_n \to f \) \( \mu \)-a.e.
Then
\[ Sf = \lim_{n \to \infty} Sf_n. \]

**Pf.** By assumption, \( \exists N \in \mathbb{N} \) s.t. \( \mu(N) = 0 \) and \( f_1 \leq f_2 \leq \ldots \) and \( f_n \to f \) for all \( x \in X \setminus N \). Then \( f_n = f_n \cdot 1_{X \setminus N} \) is increasing and converges to \( f = f \cdot 1_{X \setminus N} \).

Thus, by HCT
\[ Sf_n \to Sf. \]

Thus, \( Sf_n = Sf_n \) and \( Sf = Sf. \) \[ \square \]
If monotonicity in MCT is dropped, the conclusion $Sf_n \to Sf$ may fail.

Ex. Consider $(X=(0,1), \mu)$, $f_n = n \cdot 1_{(0,1/n)}$ and $f = 0$. Then, $f_n(x) \to f(x)$ for all $x \in (0,1)$ but $Sf_n = n \cdot \frac{1}{n} = 1$ and $Sf = 0$.

**Fatou's Lemma**

Assume $\{f_n\}_{n=1}^\infty \leq L^+$.

Then

$$\liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} Sf_n$$

**Proof.** Let $f_N = \inf_{n \geq N} \{ f_n \}$. Then $f_N \leq L^+$,

$$\tilde{f}_1 \leq \tilde{f}_2 \leq \ldots$$

and $f = \liminf_{n \to \infty} f_n$ = $\lim_{N \to \infty} f_N$. Thus, by MCT

$$Sf = \lim_{N \to \infty} Sf_{\tilde{f}_N}.$$
On the other hand, \( \tilde{f}_n \leq f_n \Rightarrow \tilde{s}_{f_n} \leq S_{f_n} \). Taking \( \lim \inf \) we conclude

\[
\tilde{s} \leq \liminf_{n \to \infty} s_{f_n}
\]

as claimed. \( \Box \)