For the theory of integration (of measurable functions) we shall need the concept of simple functions to approximate general measurable functions.

Special constructions.

1. If \( f : X \to \mathbb{R} \), then
   \[ f^+ = \max(f, 0), \quad f^- = \max(-f, 0), \]
   the positive and negative parts of \( f \).
   Note \( f^+, f^- \geq 0 \) and \( f = f^+ - f^- \).

2. If \( E \subseteq X \), the indicator function \( f : X \to \mathbb{R} \) is given by
   \[ f(x) = \begin{cases} 0, & x \notin E \\ 1, & x \in E \end{cases} \]
Clearly, for any $a \in \mathbb{R}$, $f^{-1}(a,\infty)$ is either $X$ ($a < 0$), $E$ ($0 \leq a < 1$), or $\emptyset$ ($a \geq 1$) $\Rightarrow$ $f$ $M$-meas. $\Leftrightarrow$ $E \in M$.

3. A simple function $f$ is a linear combination of indicator functions:

$$f = \sum_{k=1}^{n} c_k \mathbf{1}_{E_k}, \quad c_k \in \mathbb{C}$$(or $\mathbb{R}$)

Its standard representation is given as follows, let $G = f(X) \subseteq \mathbb{C}$.

$f$ is simple $\Leftrightarrow$ $G$ is finite $\{z_1, \ldots, z_n\}$.

The SR is then

$$f = \sum_{k=1}^{n} z_k \mathbf{1}_{F^{-1}(z_k)}$$

Ex. If $E = X$, then the SR of $f = X$

$$f = 0 \cdot \mathbf{1}_{E^c} + 1 \cdot \mathbf{1}_{E}$$
Then let \((X, \mathcal{M})\) be a measurable space. If \(f : X \to \mathbb{C}\) is measurable, then \(\exists\) seq. of simple functions \(\varphi_n\) s.t. 
\[0 \leq \varphi_1 \leq \varphi_2 \leq \ldots, \quad \varphi_n(x) \to f(x), \quad \forall x \in X,\]
and \(\varphi_n \to f\) uniformly on any set where \(f\) is bounded.

Proof: It suffices to prove for \(f : X \to \mathbb{R}\) with \(f \geq 0\), by decomposing \(f= h+ig\) and \(h= h^+-h^-; \quad g= g^+-g^-\). Thus, WLOG assume \(f \geq 0\). Idea is simple: decompose the range \([0, \infty]\) into dyadic intervals (as opposed to the domain in the Riemann integral case):

\[
\begin{align*}
3 \cdot 2^{-k} & \\
2 \cdot 2^{-k} & \\
2^{-k} & \\
\end{align*}
\]

The graph illustrates the decomposition of the range into dyadic intervals.
Thus, for \( n \geq 1 \), consider the dyadic decomposition
\[
[0, 2^n] = \bigcup_{k=1}^{2^n} [k-1, 2^{-n}, k2^{-n}].
\]
and let
\[
\psi_n(x) = \sum_{k=1}^{2^n} \frac{1}{n} \chi_{f^{-1}(I_{k,n})} + 2^{-n} \chi_{f^{-1}([2^n, x])}
\]

- If \( E = \{ f(x) \leq M \} \) and \( M < 2^N \), then
  \( E = \bigcup_{k=1}^{2^{2n}} f^{-1}(I_{k,n}) \) and \( \forall x \in E \)
  \[
  |f(x) - \psi_n(x)| < 2^{-n}, \quad n \geq N.
  \]
- \( 0 \leq \psi_1 \leq \psi_2 \leq \ldots \) "Proof by Pic:

Clearly, \( \psi_n(x) \to f(x) \), \( \forall x \).