MATH 240A
Final Exam, December 7, 2022

Instructions: 3 h. You may use without proof results proved in Folland up to and including Chapter 3.2 (unless explicitly stated otherwise). When using a result from the text, be sure to explicitly verify all hypotheses in it. Present your solutions clearly, with appropriate detail.

Notation and terminology: m (or dx) denotes the Lebesgue measure.

1. (10p) Let $X$ be a set, $\mathcal{A}$ an algebra of subsets of $X$, and $\mu_0: \mathcal{A} \to [0, \infty]$ a $\sigma$-finite premeasure. Let $\mu$ be the unique measure induced by $\mu_0$ on $M = M(\mathcal{A})$ (the $\sigma$-algebra generated by $\mathcal{A}$). Show that for each $E \in M$ with $\mu(E) > 0$ and $\alpha < 1$, there exists $A \in \mathcal{A}$ such that $\mu(E \cap A) > \alpha \mu(A)$.

2. (10p) Show that
$$\lim_{n \to \infty} \int_0^\infty \frac{\sin(nx)}{nx(x+1)} \, dx = 0.$$ Justify carefully all steps.

3. (10p) Let $\mathbb{R}_+ = [0, \infty)$, $f, g \in L^1(\mathbb{R}_+, m)$, and consider
$$h(x) = \int_0^\infty f(y)g \left( \frac{x}{y} \right) \frac{dy}{y}.$$ Show that $h$ is well-defined (i.e., $y \to f(y)g(x/y)/y$ is in $L^1(\mathbb{R}_+, m)$) for a.e. $x \in \mathbb{R}_+$, $h \in L^1(\mathbb{R}_+)$, and
$$\|h\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}.$$ Comment: You may use without proof that $g(x/y)$ is Lebesgue measurable on $\mathbb{R}_+^2$.

4. (10p) Let $\mu$ be a finite measure on $(X, \mathcal{M})$. Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions such that $f_n \to f$ a.e.. Show that $f_n \to f$ in measure.

5. (10p) Let $\mu, \nu$ be $\sigma$-finite measures on $(X, \mathcal{M})$ and let $\lambda = \mu + \nu$. Suppose that $\nu << \mu$.

(a) Show that $\mu, \nu << \lambda$ but (e.g., by giving an example) $\lambda$ need not be $<< \nu$.

(b) Let $f = d\nu/d\lambda$. Show that $0 \leq f < 1$ $\mu$-a.e. and
$$\frac{d\nu}{d\mu} = \frac{f}{1 - f}.$$

6. (10p) For $a \in \mathbb{R}$, let $\tau_a: L^1(\mathbb{R}, m) \to L^1(\mathbb{R}, m)$ be given by $(\tau_a f)(x) = f(x+a)$. Show that for any $f \in L^1(\mathbb{R}, m)$, $\tau_a f \to f$ in $L^1$ as $a \to 0$. 

1