Levels of Entaglement

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Entanglement

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- Thus there are two groups that act naturally on $\mathcal{H}$, $U(\mathcal{H})$ the unitary transfomations and $GL(\mathcal{H})$ the colineations.
- We will call a transformation local if it is of the form $T_1 \otimes \cdots \otimes T_m$.

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- Thus if $m > 1$ and all $d_i > 1$ almost all states are entangled.

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- Is there a natural ordering of entanglement and if so is there a way to place an entangled state in the order?

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- ϕ is a product state if and only if any pair of the ϕ_{ij} with the same first index is linearly independent. This leads to our measure Q.

- $Q(\phi) = \sum_{i=1}^{m} \sum_{k<l} \left\|\phi_{ik} \wedge \phi_{il}\right\|^2$ up to normalization.

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- We define for each i a linear map from $\mathcal{H}_i$ to the tensor product with $\mathcal{H}_i$ deleted. by

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- $Q(\phi) = m \|\phi\|^4 - \sum_i trA_i(\phi)^2$. This expression is usually called the total linear entropy. In the case when all of the d_i have dimension 2 this formula is attributed to Brennan.

Entropy as a measure.

- If $m = 2$ then it is standard to define the Von Neumann entropy of ϕ by

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- These two measures have the same extreme states: if $d_1 \leq d_2$ then the maximal value of these entropies is attained if and only if

$$A_1(\phi) = \frac{1}{d_1} I.$$

- We now return to the case of m factors but assume that all the $d_i = d$. If $J \subset \{1, ..., m\}$ then we can divide $\mathcal{H}$ into a tensor product of the spaces whose index is in J and one with the rest of the indices. We can thus look at ϕ as bipartite in this way. We can thus define $A_J(\phi)$ a semidefinite matrix of size $d^{|J|}$.

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  then we can argue that ϕ is maximally entangled.
- We look at some examples. Here we will only look at qubits ($d = 2$).
- $m = 2$. Then a state, ϕ, satisfies the condition for maximal entanglement if and only if there is a transformation of the form $u = u_1 \otimes u_2$ with u_i unitary such that $u\phi = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. That is, a local unitary transformation transforms it to one state (usually called Bell or GHZ).

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- $m = 5$. Define $\langle i_0 i_1 i_2 i_3 i_4 \rangle = |i_0 i_1 i_2 i_3 i_4\rangle + |i_4 i_0 i_1 i_2 i_3\rangle + ... + |i_1 i_2 i_3 i_4 i_0\rangle$. That is cycle over the tensor factors. Let $\phi_0 = \frac{1}{4}(|00000\rangle + \langle 11000\rangle - \langle 10100\rangle - \langle 11110\rangle)$. Then there exists $u = u_1 \otimes u_2 \otimes u_3 \otimes u_4 \otimes u_5$ such that $u\phi = \phi_0$ (Rains).

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- In part due to this Gilad Gour and I decided to determine the "maximally entangled states for 4 qubits".
- As it turns out there is a vast physics literature on 4 qubits. For example, Verstrade and his coworkers.

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- Thus in four qubits there are several answers to the question of maximal entanglement.

- In 2 qubits Bell introduced the basis $v_0 = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $v_1 = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, $v_2 = \frac{1}{\sqrt{2}}(|01\rangle + |01\rangle)$, $v_3 = \frac{1}{\sqrt{2}}(|01\rangle - |01\rangle)$.

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- If we set $G = SL(2, \mathbb{C})^4$ acting on $\mathcal{H}$ by the tensor product action then the algebra of polynomials on $\mathcal{H}$ invariant under the action of G is a polynomial algbra in 4 homogeneous generators, f_1, f_2, f_3, f_4, of degrees 2.4.4.6.

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- Given by $\sum z_i u_i \to \sum z_i^2, \sum z_i^4, z_0 z_1 z_2 z_3, \sum z_i^6$.
- Furthermore, $G\mathfrak{a}$ is dense in $\mathcal{H}$ and contains interior.
- A specific state that is singled out in our study is one introduced by Love

$$L = \frac{1}{\sqrt{3}}(u_0 + \zeta u_1 + \zeta^2 u_2)$$

with $\zeta = e^{\frac{2\pi i}{3}}$.

- For the simple Lie algebra of type D_4 there is an involution (corresponding to the real form $SO(4,4)$ with the fixed algebra $\mathfrak{k} \cong A_1 \oplus A_1 \oplus A_1 \oplus A_1$ and the -1 eigenspace $\mathfrak{p} \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ as a $\mathfrak{k}$ module.

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- If we add the permutations of the qubits to the action of $K = SL(2,\mathbb{C})^4$ We get a subgroup of F_4. The corresponding invariants are of degrees $2, 6, 8, 12$ and there is a corresponding cyclic element.
- We have $D_4 \subset B_4 \subset F_4$ and for B_4 there is a cyclic element. Except for one orbit the special elements that Gour and I found are cyclic for these groups.

Hyperdeterminants.

- An invariant of degree 24 for the group, G, (the descriminant of D_4) of transformations of 4 qubits of the form $g_1 \otimes g_2 \otimes g_3 \otimes g_4$ with $g_i \in SL(2, \mathbb{C})$ pointed to way of separating out certain states as "most generic".

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- As it turns out we rediscovered Cayley's hyperdeterminant.
- In the mid 19th century Cayley invented a generalization of the determinant to tensors $A_{i_1 i_2 \ldots i_m}$ which in the context of qubits exists for all $m \geq 2$. The degrees are respectively $2, 4, 24, 128, 880$ for $m = 2, 3, 4, 5, 6$. The degrees grow extremely rapidly.

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- In particular, there is now an effective method of seeing if a hyperdetrminant is zero using Groebner Bases.