Note Title 9 Apr 2012 4/9/2012

Maxwell's eqn says that

$$d\omega=0, \quad d{*}\omega=0$$

where $\omega=h_1 dx_2\wedge dx_3 - h_2 dx_1\wedge dx_3 + h_3 dx_2\wedge dx_3 + \sum e_i dt\wedge dx_i$

$*$ relative to $dt^2-\sum_{i=1}^{3} dx_i^2$

orientation chosen: $dt\wedge dx_1\wedge dx_2\wedge dx_3$

V: n-dim

$\langle , \rangle$: non-degenerate symmetric bilinear form

$\lambda_1,\dots\lambda_n$ orthogonal basis of V^* s.t. $\langle\lambda_i,\lambda_j\rangle=\varepsilon_i\delta_{ij}$

$\omega=\lambda_1\wedge\dots\wedge\lambda_n$

$(,)=r\langle , \rangle \quad r>0$

$v\in V, \quad v^{\#}\in V^*$

$$v^{\#}(w)=\langle v,w\rangle$$

$v^{"\#"}$ sharp for $(,)$

$v^{"\#"}=(v,w)=r\langle v,w\rangle \quad \Rightarrow \quad v^{"\#"}=cv^{\#}$

$\lambda^{"b"}=\frac{1}{r}\lambda^{b}$

$"*"\lambda$: $\sqrt{r}\lambda_1,\dots\sqrt{r}\lambda_n$

$\rightsquigarrow \lambda_1',\dots\lambda_n'$

$\langle\lambda_i',\lambda_j'\rangle=\varepsilon_i(\lambda_i',\lambda_j')$

$"\omega"=r^{\frac{n}{2}}\omega$

$T_{s\omega}(\lambda)(\nu)\,s\omega=\lambda\wedge\nu$ for $s>0$

k-form $n-k$ form

$T_\omega(\lambda)(\nu)\,\omega=\lambda\wedge\nu$

$\Rightarrow \quad T_{s\omega}(\lambda)=s^{-1}T_\omega(\lambda)$

$\therefore \quad T_{"\omega"}(\lambda)=r^{-\frac{n}{2}}T_\omega(\lambda) \quad \Rightarrow \quad "*"\lambda=\left(T_{"\omega"}(\lambda)\right)^{"b"}$

for $s=\frac{n}{2}$ $\qquad \lambda\in\Lambda^p V^*$

$=r^{-\frac{n}{2}+p}\left(T_\omega(\lambda)\right)^{b}$

$=r^{-\frac{n}{2}+p}*\lambda$

if $n=2p$, then "$*$" $= *$

$\therefore$ $*$ on 2-form in $\mathbb{R}^4$ is scalar inv. ($2\cdot 2=4$!)

$\varphi(t,x)^2 dt^2 \wedge \sum dx_i^2$ get the same equation.

$d\omega = d{*\omega} = 0$ says that ω is harmonic.

Suppose domain $\mathcal{U} \subseteq \mathbb{R}^4$, $\mathcal{U}$ open.

think $\mathbb{R}\times V$ V open in $\mathbb{R}^3$

Suppose $H^2(\mathcal{U},\mathbb{R}) = 0$

de Rham says if ω is a 2-form $\mathcal{U}$ s.t. $d\omega=0$

$\exists$ a 1-form μ on $\mathcal{U}$ s.t. $\omega = d\mu$

$$\omega = h_1 dx_2 \wedge dx_1 - h_2 dx_1 \wedge dx_3 + h_3 dx_1 \wedge dx_2 + \sum e_i dt \wedge dx_i$$

Assume $\mathcal{U}$ is a contractible set

Then $\exists \nu = \phi dt + \sum a_i dx_i$ s.t. $d\nu = \omega$

$\vec{H} = \vec{\nabla} \times \vec{A}$ $\quad \vec{A} = (a_1, a_2, a_3)$

$\vec{E} = \frac{\partial \vec{A}}{dt} - \nabla\phi$

$\vec{A}$ (or $-\vec{A}$) is called the vector potential

Explain de Rham question

1st fact: $d^2\lambda = 0$ for a form λ

if λ is 0-form, then $\lambda \in C^\infty(\mathcal{U})$, $\mathcal{U} \subseteq \mathbb{R}^n$ open

$$d\lambda = \sum \frac{\partial\lambda}{\partial x_i} dx_i$$

$$d^2\lambda = \sum_{i<j} \left(\frac{\partial^2\lambda}{\partial x_i \partial x_j} - \frac{\partial^2\lambda}{\partial x_j \partial x_i}\right) dx_i \wedge dx_j = 0$$

if $d^2\lambda = d^2\nu = 0$

$$d(\nu \wedge \lambda) = d\nu \wedge \lambda + (-1)^{\deg \nu} \nu \wedge d\lambda$$

$$d^2(\nu \wedge \lambda) = d^2\nu \wedge \lambda + (-1)^{\deg\nu + 1} d\nu \wedge d\lambda + (-1)^{\deg \nu} d\nu \wedge d\lambda + (-1)^{2\deg\nu} \nu \wedge d^2\lambda$$

$$= 0$$

remind to check $d^2(\text{1-form}) = 0$

2nd fact: $\varphi : \underset{\substack{\text{open}\\ \text{in } \mathbb{R}^n}}{U} \longrightarrow \underset{\substack{\text{open}\\ \text{in } \mathbb{R}^n}}{V}$ is C^∞

$\varphi^* d\lambda = d\varphi^*\lambda$ $\qquad \lambda$: k-form on V

$$\lambda = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k} \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$$\varphi^*\lambda = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k} \, d\varphi_{i_1} \wedge \cdots \wedge d\varphi_{i_k}$$

Lemma: (Poincaré lemma)

Suppose that U is contractible, then if ω is a k-form on U, $d\omega = 0$ then $\exists\, \nu$, k-1 form, s.t. $d\nu = \omega$.

$\varepsilon > 0$

$\exists\, \varphi : (-\varepsilon, 1+\varepsilon) \times U \to U$ s.t. $\varphi(1, x) = x \quad x \in U$

$\varphi(0, x) = p_0 \in U$

$\varphi^*\omega$ $\qquad$ differential form on $(-\varepsilon, 1+\varepsilon) \times U$

$x_0 = t, x_1, \ldots x_n$ coord.

$$\varphi^*\omega = \sum_{0 \le i_1 < \cdots < i_k \le n} b_{i_1 \cdots i_k} \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

μ: differential form of deg k on V

then $$H^k(\mu) = \sum_{1 \le i_1 < \cdots < i_k \le n} \left(\int_0^1 b_{0, i_1 \cdots i_{k-1}}(t, x)\, dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}}$$

this is a k-1 form on U

$\psi_1(x) = x$

$\psi_0(x) \equiv p_0$

$$\psi_1^* w - \underset{\overset{\shortparallel}{0}}{\psi_0^* w} = d\underbrace{\boxed{(H^k(\psi^*(w))}}_{k-1 \text{ form}} + H^k(d\psi^*(w))$$

$d\psi^* w = \psi^* dw = 0$

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U open in $\mathbb{R}^n$

$\Omega^k(U)$ k-forms on U

$d : \Omega^k(U) \to \Omega^{k+1}(U)$, $d^2 = 0$

$\therefore$ $\Omega^k(U)$ is a complex in sense of homological algebra.

$H^k_{dR}(U, \mathbb{R}) = H^k(\Omega^k(U), d)$ $\quad$ dR = de Rham

Poincaré lemma

If U is contractible, then $H^k_{dR}(U) = 0$ for all $k > 0$.

Conclusion: If $\omega \in \Omega^r(U)$, $r \geq 1$, and $d\omega = 0$
then $\omega = d\eta$ for some $\eta \in \Omega^{r-1}(U)$

Maxwell's eqns say that $\underset{\Omega^2(\mathbb{R}\times\mathbb{R}^3)}{\omega \in} \!\!\; \omega = \sum e_i \, dt \wedge dx_i + \sum h_i \, dx_{\{i\}}$. $\quad dx_{\{1\}} = dx_2 \wedge dx_3$
$\quad dx_{\{2\}} = -dx_1 \wedge dx_3$

$d\omega = d{*}\omega = 0$ $\quad$ $*$ is relative to the metric $dt - \sum_{i=1}^{3} dx_i$

$d\omega = 0 \Rightarrow \omega = d(\phi \, dt - \sum a_i \, dx_i)$ $\quad$ $\vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$
vector potential

$\Phi : (-\varepsilon, 1+\varepsilon) \times \underset{\text{open in } \mathbb{R}^n}{U} \longrightarrow \underset{\text{open in } \mathbb{R}^m}{V}$ $\quad$ is C^∞

$\Phi_0(x) := \Phi(0, x)$

$\Phi_1(x) := \Phi(1, x)$

If $\lambda \in \Omega^k(V)$, then $\Phi_1^*(\lambda) - \Phi_0^*(\lambda) = d\eta$ for some $\eta \in \Omega^{k-1}(U)$ $\quad$ potential come from

$\Phi^* d\nu = d\Phi^*\nu$

This implies Poincaré lemma

(Since If $\Phi : (-\varepsilon, 1+\varepsilon) \times U \to U$ C^∞ s.t. $\Phi_1(x) \equiv x$, $\Phi_0(x) \equiv p_0 \in U$
$\Phi_1^* = \mathrm{id}$ on cohomology & $\Phi_0^* = 0$)

Homotopy operator

$$\mu \in \Omega^k((-\varepsilon, 1+\varepsilon) \times U)$$
$$\underbrace{(-\varepsilon, 1+\varepsilon)}_{x_0 = t}$$

$$\mu = \sum_{0 \le i_1 < \cdots i_k \le n} a_{i_1 \cdots i_k}(t,x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$$h^k(\mu) = \sum_{1 \le i_1 < \cdots < i_k \le n} \left(\int_0^1 a_{0,i_1,\ldots,i_{k-1}}(t,x)\,dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$$h^k : \Omega^k(W) \to \Omega^{k-1}(U)$$

Lemma

$$dh^k(\mu) + h^k d(\mu) = \boxed{\sum_{1 \le i_1 < \cdots < i_k \le n} \left(a_{i_1 \cdots i_k}(1,x) - a_{i_1 \cdots i_k}(0,x) \right) dx_{i_1} \wedge \cdots \wedge dx_{i_k}} \quad (*)$$

$$dh^k(\mu) = \boxed{\sum_{1 \le i_1 < \cdots < i_k \le n} \left(\int_0^1 \frac{\partial a_{0,i_1 \cdots i_k}}{\partial t}(t,x)\,dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_k}} \quad (**)$$

$$d\mu = \sum_{1 \le i_1 < \cdots < i_k \le n} \frac{\partial a_{i_1 \cdots i_k}}{\partial t}\, dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$
$$+ \sum_{1 \le i_1 < \cdots j < \cdots < i_{k-1} \le n} \frac{\partial a_{0,i_1 \cdots i_{k-1}}}{\partial x_j}(t,x)\, dx_j \wedge dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$
$$+ \text{terms without } dt$$

$$h^{k+1}(d\mu) = (*) - (**)$$
$$dh^k(\mu) + h^{k+1}(d\mu) = (*)$$

Q.E.D.

Speed of light c (constant!)

Want Maxwell eqn look like

$$\frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{H}$$
$$\frac{1}{c}\frac{\partial \vec{E}}{\partial t} = -\vec{\nabla} \times \vec{H}$$

If $t = cu$

$$\frac{\partial}{\partial u} f(cu) = c \frac{\partial f}{\partial t}(cu)$$
$$\therefore \quad \frac{\partial}{\partial t} f = \frac{1}{c}\frac{\partial}{\partial u} f$$

$t \rightsquigarrow cu$, then renormalized

$dt \rightsquigarrow c\,dt$

$dt^2 - \sum dx_i^2 \rightsquigarrow c^2dt^2 - \sum dx_i^2$

(Ch. 5, 7 Frankel ($\vec{H} = \vec{B}$))

Newton's 2nd Law

$$\frac{d\vec{P}}{dt} = \vec{F}$$

Particle

Electro-Magnetic field

Lorentz Force

e: electric charge

$$\vec{F} = -\frac{e}{c}\frac{\partial \vec{A}}{dt} - e\nabla\phi + e\vec{v}\times(\vec{\nabla}\times\vec{A}) = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{H}$$

(recall $\omega = d(\phi dt - \sum a_i dx_i)$)

If $\|\vec{v}\|$ is small relative to c, can pretend $m\frac{d\vec{v}}{dt} = \vec{F}$

if not, $\frac{m}{\sqrt{1-\frac{\|v\|^2}{c^2}}}\frac{d\vec{v}}{dt}$

relativistic interpretation of mass.

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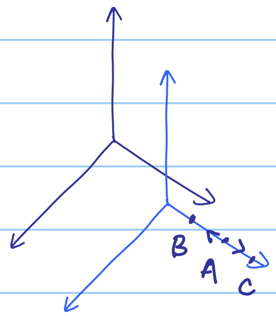
Special Relativity



Only thing can say: Speed of light is constant in all frame

time $-t_2$ $P_2 = (x_2, y_2, z_2)$

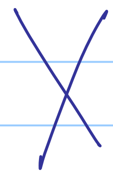
t_1

$P_1 = (x_1, y_1, z_1)$

dist: $c|t_2 - t_1| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

$$c^2 \cdot (\Delta t)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

means - it lies on bdry of a cone.



$c^2t^2 - x^2 - y^2 - z^2 = 0$ call Null cone

$c^2t^2 - x^2$

$(ct - x)(ct + x) = 0$

i.e. looking at $c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 = 0$

Null cone in tangent space

Only transformation allow = those preserve the cone.

$= SO(1,3)$

Tangent space

in $\mathbb{R}^n$, tangent vector at $p\in\mathbb{R}^n$ is (p,v), $v\in\mathbb{R}^n$

this data gives v_p. directional derivative at p.

$$v_p f = \frac{d}{dt} f(p+tv)$$

$$v_p : C^\infty(\mathbb{R}^n) \to \mathbb{R} \text{ is linear.}$$

$$v_p(fg) = f(p)\, v_p g + g(p)\, v_p f \quad \text{Leibniz rule.}$$

<u>Lemma</u> If $U \subseteq \mathbb{R}^n$ is open and contains 0.

Then if $L : C^\infty(U) \to \mathbb{R}$ is linear and satisfies Leibniz rule at 0.

then $\exists\, v \in \mathbb{R}^n$ so that $L = v_0$

Pf:
$$f(x) - f(0) = \int_0^1 \frac{\partial}{\partial t} f(tx)\, dt$$
$$= \int_0^1 \sum x_i \frac{\partial f}{\partial x_i}(tx)\, dt$$
$$f(x) = f(0) + \sum x_i g_i(x), \qquad g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx)\, dx$$
$$g_i(0) = \frac{\partial f}{\partial x_i}(0)$$

if L satisfies Leibniz rule and linear

then $L1 = 0$ $\quad \left(\because \underset{L(1)}{\underset{\parallel}{L(1^2)}} = L(1)1 + 1\cdot L(1) = 2L(1)\right)$

$\Rightarrow Lf(0) = 0 \quad (\because \text{linear})$

$$\Rightarrow Lf = \sum \Big(L(x_i)\, g_i(0) + \underbrace{x_i(0)}_{\parallel \atop 0} \cdot L g_i \Big)$$
$$= \sum L(x_i) \frac{\partial f}{\partial x_i}(0)$$
$$= \big(L(x_1), \dots, L(x_n)\big)_0 f$$

Manifold

X: topological space (Hausdroff)

An n-dim chart for X is a pair $U \subseteq X$, U open &

$\varphi: U \to \mathbb{R}^n$ s.t. $\varphi(U) = V$ is open in $\mathbb{R}^n$

$\varphi: U \to V$ is homeomorphism

An n-atlas for X is a collection $(U_\alpha, \varphi_\alpha)_{\alpha \in I}$ of charts s.t. $\bigcup_{\alpha \in I} U_\alpha = X$

(U, φ), (V, ξ), n-charts, are said to be C^∞-related if

$\xi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \xi(U \cap V)$ are C^∞

$\varphi \circ \xi^{-1} : \xi(U \cap V) \longrightarrow \varphi(U \cap V)$

A n-dim C^∞-atlas is a n-altas s.t. any pair of charts in atlas are C^∞ related

Two atlas are C^∞ equivalent if the union of the two is a C^∞-atlas.

A C^∞ manifold is a topological space (Hausdroff), X, paired with an equivalence class of C^∞ n-atlas on X

M: C^∞ manifold (X, class)

(U, φ) a chart for M is an elt. of one of the class of atlases.

(U, φ). $\varphi(p) = (x_1(p), \ldots, x_n(p))$

$V = \varphi(U)$ is open, $\underbrace{x_1, \ldots, x_n}_{\text{call coordinate system on } U}$ fcn on U

f a fcn on M

$f \circ \varphi^{-1}\underbrace{(x_1, \ldots, x_n)}_{\varphi(p)}$ write as $f(x_1, \ldots, x_n)$ though of function on U

We say f is C^∞ if for all such coord system $x_1, \ldots, x_n$ on M.

$f(x_1, \ldots, x_n)$ is C^∞ on $(x_1, \ldots, x_n)(U)$

Example: $\quad f : \mathbb{R}^n \to \mathbb{R}$ is C^∞

$$X = \{ p \in \mathbb{R}^n \mid f(p) = 0,\ df_p \neq 0 \}$$

$p \in X$ s.t. $f(p) = 0$, $\frac{\partial f}{\partial x_i}(p) \neq 0$ for some i.

$$\psi = (x_1, \dots, x_{i-1}, f, x_{i+1}, \dots, x_n)$$

$$\left(\frac{\partial \psi_i}{\partial x_j} \right) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ * & & \bullet & & * \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \qquad \frac{\partial f}{\partial x_i} \neq 0$$

$$\therefore \det \left(\frac{\partial \psi_i}{\partial x_j} \right) \neq 0$$

By inverse fcn thm

$\Rightarrow (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)\big|_{V \cap X}$

V open in $\mathbb{R}^n$, $p \in V$, V sufficiently small

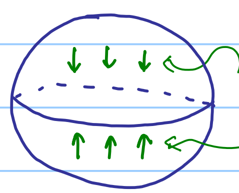
defines an $(n-1)$-chart at an open set containing p.

e.g. $\quad f(x) = \sum x_i^2 - 1 \quad \longleftrightarrow \quad S^{n-1}$

$\frac{\partial f}{\partial x_i} = 2x_i$

Take $p \in S^{n-1} \subseteq \mathbb{R}^n$

when $p_i \neq 0$, take $\left(x_1|_{S^{n-1}}, \dots, x_{i-1}|_{S^{n-1}}, x_{i+1}|_{S^{n-1}}, \dots \right)$



projection