Some Practice Problems involving Green's, Stokes', Gauss' theorems.

1. Let $\mathbf{x}(t) = (a\cos t^2, b\sin t^2)$ with $a, b > 0$ for $0 \le t \le \sqrt{2\pi}$ Calculate $\int_{\mathbf{x}} xdy$. Hint: $\cos^2 t = \frac{1+\cos 2t}{2}$.

Solution1. We can reparametrize without changing the integral using $u = t^2$. Thus we can replace the parametrized curve with $\mathbf{y}(t) = (a\cos u, b\sin u)$, $0 \le u \le 2\pi$. Thus we are being asked to calculate

$$\int_0^{2\pi} a\cos(u)b\cos u du = ab\int_0^{2\pi}\cos^2(u)du = \frac{ab}{2}\int_0^{2\pi}(1+\cos 2u)du = ab\pi.$$

Solution2. The the curve is the boundary of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ oriented counter clockwise. So since $xdy = Mdx + Ndy$ with $M = 0$ and $N = x$ and so $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$ Green's theorem implies that the integral is the area of the inside of the ellipse which is $ab\pi$.

2. Let $\mathbf{F} = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$

a) Use Green's theorem to explain why

$$\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = 0$$

if $\mathbf{x}$ is the boundary of a domain that doesn't contain 0.

In this case we have $M = \frac{-y}{x^2+y^2}, N = \frac{x}{x^2+y^2}$ so $\frac{\partial N}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$, $\frac{\partial M}{\partial y} = \frac{-1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}$ so

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{2}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} - \frac{2y^2}{(x^2+y^2)^2} = 0.$$

We can thus apply Green's theorem and find that the corresponding double integral is 0.

b) Let $\mathbf{x}(t) = (\cos t, 3\sin t)$, $0 \le t \le 2\pi$. and $\mathbf{F} = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$. Calculate $\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s}$. Hint: Consider the domain between x and the circle $\mathbf{y}(t) = (\cos t, \sin t)$. Use part a) to see that $\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{y}} \mathbf{F}\cdot d\mathbf{s}$.

We consider the path gotten by following $\mathbf{x}(t)$ from 0 to π and then $\mathbf{y}(t)$ from π back to 0. This bounds a crescent shaped region that doesn't contain $(0,0)$. Thus the integral around the boundary is 0 by part a). We next follow $\mathbf{y}(t)$ from 2π to π and then $\mathbf{x}(t)$ from π to 2π this path encloses a region that doesn't contain 0. So this path integral is also equal. But the sum of this two path integrals is $\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} - \int_{\mathbf{y}} \mathbf{F}\cdot d\mathbf{s}$. So $\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{y}} \mathbf{F}\cdot d\mathbf{s}$. On the path $\mathbf{y}(t)$, $x^2 + y^2 = 1$ and $-ydx + xdy = \cos^2 tdt + \sin^2 tdt = dt$. Thus the integral is

$$\int_0^{2\pi} dt = 2\pi.$$

3. Which if the following vector fields is of the form ∇f? If it is compute an f.

a) $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$.

If $\mathbf{F} = \nabla f$ then $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ with $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$. In this case $M = x^2$ and $N = xy$ so $\frac{\partial N}{\partial x} - y$ and $\frac{\partial M}{\partial y} = 0$ so $\mathbf{F}$ is not of the form ∇f.

b) $\mathbf{F} = x^2\mathbf{i} - y^2\mathbf{j}$

$M = x^2$ and $N = -y^2$ so $\frac{\partial N}{\partial x} = 0$ and $\frac{\partial M}{\partial y} = 0$ since $\mathbf{F}$ is everywhere defined we know that such an f exists. We have derived a formula:

$$f(x,y) = \int_0^x M(x,0)dx + \int_0^y N(x,y)dy = \frac{x^3}{3} - \frac{y^3}{3}.$$

One can check directly that $\nabla(\frac{x^3}{3} - \frac{y^3}{3}) = \mathbf{F}$.

c) $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$

$M = y$ and $N = -x$ so $\frac{\partial N}{\partial x} = -1$ and $\frac{\partial M}{\partial y} = 1$ this implies there is no f.

d) $\mathbf{F} = \left(3x^2y + 2xy^2\right)\mathbf{i} + \left(x^3 + 2x^2y + 3y^2\right)\mathbf{j}$.

$M = 3x^2y+2xy^2, N = x^3+2x^2y+3y^2$ so $\frac{\partial N}{\partial x} = 3x^2+4xy$ and $\frac{\partial M}{\partial y} = 3x^2+4xy$ so we can use the method:

$f(x,y) = \int_0^x M(x,0)dx + \int_0^y N(x,y)dy = 0 + x^3y + x^2y^2 + y^3$. A direct check shows that the answer is correct.

You might ask why icheckthe answers if we know that the method works? The reason is that everyone makes mistakes and this check is partial insurance that a computational error hasn't been made or that the formula being used is not quite correct.

4. Let S be the surface $z = 4 - x^2 - y^2, z \geq -3$ and let $\mathbf{F} = (2xyz + 3z)\mathbf{i} + x^2y\mathbf{j} + \cos(xyz)e^x\mathbf{k}$. Calculate

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Hint: Observe that ∂S is the boundary of another surface.

The boundary of S consists of the points on the surface with $z = -3$ and $x^2 + y^2 = 7$. This is also the boundary of the circle of radius $\sqrt{7}$ in the plane $z = -3$. Thus Stokes theorem implies that this integral is $\int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$. We can parametrize the boundary as $(\sqrt{7}\cos t, \sqrt{7}\sin t, -3)$ for $0 \leq t \leq 2\pi$. Thus

$$\begin{aligned} \mathbf{F} \cdot d\mathbf{s} &= ((2xyz + 3z)\mathbf{i} + x^2y\mathbf{j} + \cos(xyz)e^x\mathbf{k}) \cdot (-\sqrt{7}\sin t\mathbf{i} + \sqrt{7}\cos t\mathbf{j}) \\ &= (\sqrt{7}(-6 \cdot 7\cos t\sin t - 9)(-\sin t) + 49\cos^3 t\sin t)dt \\ &= 42\sqrt{7}\sin^2(t)\cos t dt + 9\sqrt{7}\sin t dt + 49\cos^3 \sin t dt. \end{aligned}$$

We must integrate this from 0 to 2π. We use $\int \sin^2(t)\cos t dt = \frac{\sin^3 t}{3}$ and $\int \cos^3 \sin t dt = \frac{\cos^4 t}{4}$ so the integral is

$$14\sqrt{7}\frac{\sin^3 t}{3}|_0^{2\pi} - 9\sqrt{7}\cos t|_0^{2\pi} + 49\frac{\cos^4 t}{4}|_0^{2\pi} = 0.$$

Alternatively we can use the hint and not calculate any integrals by noting that the curve is also the boundary of the disk of radius $\sqrt{7}$ centered at the oigin in the plane $z = -3$. Since the line integral doesn't involve the $\mathbf{k}$ component of the vector field we can plug in $z = -3$ in $\mathbf{F}$ and consider the same problem with vector field $(-6xy - 9)\mathbf{i} + x^2y\mathbf{j}$ on the disk, D,of radius $\sqrt{7}$ in the plane. The integral will theorfor be $\int\int_D (2xy + 6x)dA = 2\int\int_D xydA + 6\int\int_D xdA$. We note that the first integral is zero by observing that xy takes paositive values in the first and third quadrant and that it takes the negatives of the same values in the second and fourth quadrant. As for the second integral we argue the same way with the first and fourth and the second and third. Thus both integrals are 0.

5. Let S be the union of the surfaces $z = x^2 + y^2 - 1$ with $z \leq 0$ and $x^2 + y^2 + z^2 = 1, z \geq 0$. Let $\mathbf{x}(t) = (\cos t, \sin t, 0)$, $0 \leq t \leq 2\pi$. Calculate $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$. for $\mathbf{F}$ an arbitrary C^1 vector field using Stokes' theorem. Do the same using Gauss's theorem (that is the divergence theorem).

We note that this is the sum of the integrals over the two surfaces S_1 given by $z = x^2 + y^2 - 1$ with $z \leq 0$ and S_2 with $x^2 + y^2 + z^2 = 1, z \geq 0$. We also note that the unit circle in the xy plane is the set theoretic boundary of both surfaces. However as the boundary of the first surface it is oriented clockwise and as the boundary of the second it is oriented counterclockwise. We have

$$\begin{aligned}\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int\int_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} + \int\int_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \\ &= \int_{\partial S_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\partial S_2} \mathbf{F} \cdot d\mathbf{s} = \mathbf{0}\end{aligned}$$

since the two path integrals are over the same path with opposite orientations.

To use Gauss's theorem we note that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ for every C^2 vector field. Guass't theorem implies:

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int\int\int_V \nabla \cdot (\nabla \times \mathbf{F}) dV = 0$$

with $V = \{(x, y, z)|x^2 + y^2 \leq 1, x^2 + y^2 - 1 \leq z \leq \sqrt{1 - x^2 - y^2}\}$.

6. Let V be the solid cylinder $x^2 + y^2 \leq 1.|z| \leq 1$. Describe the boundary of V. Orient the boundary using the outward normal and use Gauss's theorem to calculate $\int\int_{\partial V} \mathbf{F} \cdot d\mathbf{S}$ with $\mathbf{F} = x\mathbf{k} + y\mathbf{j} + z\mathbf{i}$.

You should draw the picture which looks like a soup can.

$$\nabla \cdot \mathbf{F} = 1.$$

Thus since Gauss's theorem says $\int\int_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \int\int\int_V dV$. That is the volume of this cylinder which is the height times the area of the base that is $2 \times \pi = 2\pi$.

Suppose you decide not to use Gauss's theorem then you must do this. The boundary consists of three parts the disks, S_1 given by $x^2 + y^2 \leq 1$, $z =$

-1 and S_2 given by $x^2 + y^2 \leq 1$ and $z = 1$ and the cylinder S_3 given by $x^2 + y^2 = 1$ and $-1 \leq z \leq 1$. The unit normals that point outward are $-\mathbf{k}$ for S_1, $\mathbf{k}$ for S_2 and $x\mathbf{i} + y\mathbf{j}$ at the point (x, y, z) on S_3. We parametrize S_1 as $(r\cos t, r\sin t, -1), 0 \leq r \leq 1, 0 \leq t \leq 2\pi$,$S_2$ as $(r\cos t, r\sin t, 1), 0 \leq r \leq 1, 0 \leq t \leq 2\pi$ and S_3 as $(\cos s, \sin s, t), -1 \leq t \leq 1, 0 \leq s \leq 2\pi$. For the parametrization of S_1 we have $T_r = (\cos t, \sin t, 0), T_t = (-r\sin t, r\cos t, 0)$, so $\mathbf{T}_r \times \mathbf{T}_t = r\mathbf{k}$ thus $\|\mathbf{T}_r \times \mathbf{T}_t\| = r$ For S_2 we also have $\|\mathbf{T}_r \times \mathbf{T}_t\| = r$ and for S_3 we have $T_s = (-\sin s, \cos s, 0), T_t = (0, 0, 1)$ so $\mathbf{T}_s \times \mathbf{T}_t = \cos s\mathbf{i} + \sin s\mathbf{j}$ so $\|\mathbf{T}_r \times \mathbf{T}_t\| = 1$. We can now compute the integral as the sum of the integrals over the three pieces

$$\begin{aligned} & \int_0^{2\pi} \int_0^1 r^2 \cos t dr dt - \int_0^{2\pi} \int_0^1 r^2 \cos t dr dt + \int_0^{2\pi} \int_{-1}^1 (t\cos s + \sin^2 s) dt ds \\ = \;& 2\int_0^{2\pi} \sin^2 s ds = \int_0^{2\pi} (1 - \cos(2s)) ds = 2\pi. \end{aligned}$$