Introductory lectures on automorphic forms

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1 Orbital integrals and the Harish-Chandra transform.

This section is devoted to a rapid review of some of the basic analysis that is necessary in representation theory and the basic theory of automorphic forms. Even though the material below looks complicated it is just the tip of the iceberg.

1.1 Left invariant measures.

Let X be a locally compact topological space with a countable basis for its topology. Let $C(X)$ denote the space of all continuous complex valued functions on X. If f is a function on X then we denote by $\text{supp}(f)$ the closure of the set

$$\{x \in X | f(x) \neq 0\}.$$

We set

$$C_c(X) = \{f \in C(X) | \text{supp}(f) \text{ is compact}\}.$$

If $K \subset X$ is a compact subset then we set $C_K(X) = \{f \in C(X) | \text{supp}(f) \subset K\}$. Whe endow each space with the norm topology induced by

$$\|f\|_K = \max_{x \in K} |f(x)|.$$

We endow $C_c(X)$ with the union topology. That is, a subbasis of the topology is the set consisting of the sets that are open subsets of some $C_K(X)$. With this notation in place a complex measure on X is a continuous linear map $\mu : C_c(X) \to \mathbb{C}$. A measure is a complex measure μ such that $\mu(f)$ is real if f is real valued and non-negative if f takes on only non-negative values.

Example 1 *Let $x \in X$ and let $\delta_x(f) = f(x)$. This measure is called the Dirac delta function supported at x.*

Example 2 *Let Γ be a closed subset of X that is discrete in the subspace topology. In particular, the intersection of Γ with a compact subset is finite. Then $\mu_\Gamma(f) = \sum_{\gamma \in \Gamma} f(\gamma)$ defines a measure on X.*

Example 3 *Let X and Y be locally compact topological spaces. Let μ and ν be respectively measures on X and Y. We now show how to associate a measure $\mu \times \nu$ on $X \times Y$ If $f \in C_c(X \times Y)$ and if $y \in Y$ is fixed then we write $f_y(x) = f(x, y)$. Set $g(y) = \mu(f_y)$. One can show that $g \in C_c(Y)$. Thus we can define $(\mu \times \nu)(f) = \nu(g)$. One can show that if we first integrate out Y (rather than X) one obtains the same measure. (We have just described Fubini's theorem.)*

We now assume that $X = G$, a locally compact group with a countable basis for its topology. Then a measure, μ , on G is said to be left invariant if

$$\mu(f \circ L_g) = \mu(f).$$

Here $L_g : G \to G$ is given by $L_g x = gx$. Similarly if $R_g x = xg$ then μ would be said to be right invariant if

$$\mu(f \circ R_g) = \mu(f).$$

It is a theorem of Haar that there always exists left (hence right) invariant measure, μ, on G such that if $f \in C_c(X)$ is non-negative and non-zero then $\mu(f) > 0$ and furthermore if ν is any left invariant complex measure then ν is a scalar multiple of μ. We fix a non-zero left invariant measure, μ, on G. We will generally write

$$\mu(f) = \int_G f(g)dg.$$

if μ is a left invariant measure on G.

If $f_1 \in C_C(G)$and $f_2 \in C(G)$ (or vice-versa) then we define

$$f_1 * f_2(g) = \int_G f_1(x) f_2(x^{-1}g)dx.$$

This function is in $C(G)$ and if $f_1, f_2 \in C_c(G)$ then so is $f_1 * f_2$. We call $f_1 * f_2$ the convolution of f_1 and f_2. We note that the obvious calculation implies that if $f_1, f_2 \in C_c(G)$ then $f_1 * (f_2 * f_2) = (f_1 * f_2) * f_3$.

One direct application of this theorem is to locally compact fields. Our reference for this material is to A. Weil, *Basic Number Theory*. Let F be a locally compact non-discrete field. Then as a group under addition it is a locally compact group. We therefore have a unique, up to positive multiple, left (that is translation) invariant measure on F. In other words

$$\int_F f(x+y)dx = \int_F f(x)dx$$

and the integral of a non-zero positive function is strictly positive. If F is either $\mathbb{R}$ or $\mathbb{C}$ then we can take μ to be Lebesgue measure. One knows from the classification of locally compact non-discrete fields that the other alternative is that F be totally disconnected this implies that F has a neighborhood basis of 0 consisting of compact open subsets . Set $M_a x = ax$ for $a \in F^\times = F - \{0\}$. We write $\mu_a(f) = \mu(f \circ M_{a^{-1}})$. Then the distributive rule implies that μ_a is a translation invariant measure thus we have

$$\mu_a = c(a)\mu, a \in F^\times.$$

Furthermore, we have $c(ab) = c(a)c(b)$. If $F = \mathbb{R}$ then $c(a) = |a|$ in the case $F = \mathbb{C}$ then $c(a) = |a|^2$. For a general locally compact, nondiscrete, totally disconnected field one has (here $c(0) = 0$)

$$c(a+b) \leq \max(c(a), c(b)).$$

Thus in all cases but $F = \mathbb{C}$, c satisfies the triangle inequality on F. We will call $c(a)$ the *norm* of a and write $|a| = c(a)$.

Now consider the group $G = GL(n, F)$ the group of invertible $n \times n$ matrices over F with the subspace topology in $M_n(F)$ thought of as the n^2-fold product of F with itself. On F^n we put the n-fold product measure of μ with itself and denote it by dx. We have

Lemma 4 *If F is a local field and $g \in GL(n, F)$ then*

$$\int_{F^n} f(gx)dx = |\det(g)|^{-1} \int_{F^n} f(x)dx.$$

Proof. We observe that $GL(n,F)$ is generated by elementary matrices. That is matrices of the form

$$D_i(a) = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & a & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

(with zeros off of the main diagonal, a nonzero a in the i-th diagonal position, all the rest of the diagonal entries 1) and $T_{ij}(y)$ which is the linear transformation of F^n such that if $T_{ij}(y)x = z$ then $z_k = x_k$ for $k \neq j$ and $z_j = x_j + yx_i$. The translation invariance of the measure and the previous lemma imply

$$\int_{F^n} f(T_{ij}(y)x)dx = \int_{F^n} f(x)dx$$

and

$$\int_{F^n} f(D_i(a)x)dx = |a|^{-1} \int_{F^n} f(x)dx.$$

Since $\det(T_{ij}(y)) = 1$ and $\det(D_i(a)) = a$ the lemma follows. ∎

Corollary 5 *Left invariant measure on $GL(n,F)$ is given by*

$$\int_{GL(n,F)} f(X)\frac{dX}{|\det(X)|^n}.$$

Here $\int_{GL(n,F)} f(X)dX$ means the restriction of the translation invariant measure on $M_n(F)$

In general if G is an n-dimensional Lie group with a finite number of connected components then G is a locally compact, separable topological group. In this case left invariant measures can be described using a bit of differential geometry. Let $Lie(G)$ denote, as usual, the Lie algebra of G. Let ω be a differential form of degree n on G such that $L_g^*\omega = \omega$ for all $g \in G$. Here we recall that ω is an assignment $x \longmapsto \omega_x$ with $\omega_x \in \bigwedge^n T(G)_x^*$ $(T(G)_x$

the tangent space at x) such that if $X_1, ..., X_n$ are vector fields on G then the map

$$x \longmapsto \omega_x((X_1)_x, ..., (X_n)_x)$$

is of class C^∞. The condition $L_g^*\omega = \omega$ means that

$$\omega_{gx}((dL_g)_x(X_1)_x, ..., (dL_g)_x(X_n)_x) = \omega_x((X_1)_x, ..., (X_n)_x)$$

for all $x \in G$ (here d, as usual, stands for differential) The standard way of constructing such an ω is to choose a basis $X_1, ..., X_n$ looked upon as left invariant vector fields ($(dL_g)_x X_x = X_{gx}$) and choosing $\eta \in \bigwedge^n Lie(G)^*$ such that $\eta(X_1, ..., X_n) = 1$ then we identify $Lie(G)$ with $T(G)_1$. We set for $Y_1, ..., Y_n$ vector fields on G

$$\omega_x((Y_1)_x, ..., (Y_n)_x) = \eta((dL_x)^{-1}(Y_1)_x, ..., (dL_x)^{-1}(Y_n)_x).$$

We fix an orientation on G (we can choose the one corresponding to ω as above). Then if $f \in C_c(G)$ we can integrate f with respect to ω defining

$$\mu(f) = \int_G f\omega.$$

This defines a Haar measure on G.

1.2 Some integration formulas

Let G be a locally compact separable topological groups. and let H be a closed subgroup of G with a fixed left invariant measure dh. On $X = G/H$ and let $\pi : G \to X$ be the natural projection given by $\pi(g) = gH$. We write $\tau(g)(xH) = gxH$, this defines the standard action of G on X. We endow G/H with the quotient topology. That is, the open sets are the subsets whose inverse images in G are open. A measure, ν, on X will be said to be G-invariant if $\nu(f \circ \tau(g)) = \nu(f)$ for all $g \in G$. We have

Theorem 6 *Let G be unimodular (that is left invariant measures are right invariant). There exists a unique measure ν on G/H such that*

$$\int_G f(g)dg = \int_{G/H} \int_H f(gh)dhd\nu(gH).$$

Furthermore, ν is a G-invariant measure on G/H.

The displayed formula needs some explanation. The inner integral is

$$\nu((f \circ L_g)_{|H}).$$

Which makes sense since $(f \circ L_g)_{|H} \in C_c(H)$. The main point in the proof is the fact that if $\phi \in C_c(G/H)$ then there exists $f \in C_c(G)$ such that $\phi(gH) = \int_H f(gh)dh$. This is usually proved using a partition of unity argument.

The following lemma is a special case of a much more general result but it will be sufficient for our purposes.

Lemma 7 *Let X be a locally compact, separable topological space and let G be a locally compact, separable group acting on X. We assume that the action is transitive ($Gx_o = X$ for some, hence all, $x_o \in X$). We also assume that if $x_o \in X$ then*

$$G_{x_o} = \{g \in G | gx_o = x_o)$$

is compact. Then up to positive multiple there is at most one G-invariant measure on X.

Proof. For this we need the following fact (which is not completely trivial). Let $x_o \in X$ then X is homeomorphic with G/G_{x_o}under the map $gG_{x_o} \longmapsto gx_o$ (for a proof see Helgason,S., *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, New York, 1978). Set $K = G_{x_o}$. Fix a Haar measure on K which will be denoted with the usual integral notation. We define for $f \in C_c(G;\mathbb{C})$,

$$T(f)(gx_o) = \int_K f(gk)dk.$$

We leave it to the reader to check that $T : C_c(G;\mathbb{C}) \to C_c(X;\mathbb{C})$ is linear and continuous. Let μ be a G-invariant measure on X then we define

$$\nu(f) = \mu(T(f))$$

for $f \in C_c(G;\mathbb{C})$. This is easily checked to be left invariant on G. We leave it to the reader to see that it is a measure on G. The lemma now follows from Haar's theorem. ■

We note that if μ is a nonzero left invariant measure on G then the measure $f \longmapsto \mu(f \circ R_g)$ is also a left invariant measure. Thus Haar's theorem implies that there exists a function δ on G such that

$$\mu(f \circ R_g) = \delta(g)\mu(f)$$

for all $g \in G$ and $f \in C_c(G)$.

Definition 8 *The function δ is called the* modular function *of G. If G is not understood then we will write δ_G for δ. It easily seen to be continuous. If $\delta \equiv 1$ then we say that G is* unimodular.

If G is a Lie group then the left invariant measure can be gotten as in the end of the previous section. Observing that $x \mapsto L_g R_{g^{-1}} x$ is an automorphism of G with differential $Ad(g)$. We conclude

Lemma 9 *If G is a Lie group then $\delta(g) = |\det(Ad(g))|$.*

Exercise 10 *Show that the measure $f \longmapsto \mu(\delta^{-1} f)$ is invariant under the right regular action.*

Proposition 11 *Let G be a locally compact, separable, unimodular topological group and suppose that A and B are two closed subgroups of G such that*

$$AB = \{ab | a \in A, b \in B\} = G$$

and $A \cap B$ is compact. Then if dg denotes invariant measure on G, da denotes left invariant measure on A and db right invariant measure on B then up to constants of normalization

$$\int_G f(g)dg = \int_A \int_B f(ab)dadb.$$

In other words if $T : C_c(G) \to C_c(A \times B)$ is given by $T(f)(a,b) = f(ab)$, if μ is left invariant on A, ν is right invariant on B then $(\mu \times \nu) \circ T$ is invariant on G.

Proof. Let $A \times B$ act on G by $(a,b) \cdot x = axb^{-1}$. Then the stability group of 1 is $\{(k,k)|k \in A \cap B\}$ which is compact. The measure $(\mu \times \nu) \circ T$ and Haar measure are both invariant under this action of $A \times B$. Lemma 7 now implies the result. ■

Our main application of this result involves parabolic subgroups of reductive groups over local fields. However we will take this chance to talk about algebraic and reductive groups. Let F be a field and let $\overline{F}$ denote it's algebraic closure. Then an algebraic group is a subgroup, $\mathbf{G}$, of $GL(n, \overline{F})$ whose elements are the locus of zeros of a finite number of polynomials on $M_n(\overline{F})$ let I be the ideal of polynomials in $\overline{F}[x_{ij}, \det^{-1}]$ that vanish on $\mathbf{G}$. We say that $\mathbf{G}$ is defined over F if the ideal I is generated by elements of $F[x_{ij}, \det^{-1}]$. We denote by $\mathbf{G}_F$ the subgroup $GL(n, F) \cap \mathbf{G}$ and call the subgroup the F-rational points. We say that $\mathbf{G}$ is reductive if the only normal subgroup consisting of unipotent elements ($1 - X$ is nilpotent) is $\{1\}$. We now assume that F is a nondiscrete locally compact field and $\mathbf{G}$ be is a reductive algebraic group defined over F and let G be the F-rational points. We will now give some representative examples. We will think of $\mathbf{G}$ as a group with points in the algebraic closure. We first look at the case of $F = \mathbb{R}$. For a more detailed discussion of this case ones hould also consult, N.Wallach, *Real Ruductive Groups (RRGI)* I, Chapter 2.

Example 12 $F = \mathbb{R}$. $\mathbf{G} = GL(n, \mathbb{C})$ *and* $G = GL(n, \mathbb{R})$.

Example 13 F *as above* $\mathbf{G} = SL(n, \mathbb{C})$ *and* $G = SL(n, \mathbb{R})$.

One can have algebraic groups isomorphic over the algebraic closure but having non-isomorphic F-rational points.

Example 14 $F = \mathbb{R}$. $\mathbf{G} = O(n, \mathbb{C})$ *and* $G = O(p, n-p)$. *Here on* $\mathbb{R}^n$ *we put symmetric form of signature* $(p, n-p)$ *and* G *is the corresponding indefinite orthogonal group. These are isomorphic over* $\mathbb{R}$ *if and only if the signatures are the same or reversed.*

Example 15 *The real forms of the classical groups over* $\mathbb{C}$. *See for example Goodman, Wallach, Representations and invariants of the classical groups..*

We will now look at F a non-archimedian locally compact field with algbraic closure $\bar{F}$. Then the first examples are essentially the same

Example 16 $\mathbf{G} = GL(n, \bar{F})$ *and* $G = GL(n, F)$.

Example 17 $\mathbf{G} = SL(n, \bar{F})$ *and* $G = SL(n, F)$.

More generally we have for the other classical groups over an algebraically closed field O, SO, Sp we note that each is given as the group leaving invariant a bilinear form (and in the case of SO of determinant 1). If the form has coefficients in the field F then one can take the group leaving invariant that form.as G.

The parabolic subgroups are defined to be the F-rational points algebraic subgroups of $\mathbf{G}$ that are defined over F that contain a maximal connected (in the Zariski topology– that is, the closed subsets are the loci of zeros of polynomials) solvable subgroup (Borel subgroup). We will look at some of the examples. For $G = GL(n, F)$ or $SL(n, F)$ then up to conjugacy in G we are talking about sunbroups of the following form

$$\begin{bmatrix} g_1 & * & * & * & * \\ 0 & g_2 & * & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & g_{k-1} & * \\ 0 & 0 & 0 & 0 & g_k \end{bmatrix}$$

with g_i an $n_i \times n_i$ invertible matrix (the product of the determinants must be 1 in the SL case) and $n_1 + ... + n_k = n$. For the orthogonal and symplectic cases if we take the symmetric form to have matrix of the symetric form to be

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ \vdots & \vdots & J & \vdots & \vdots \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with J having ones on the indicated diagonal and zeros elsewhere and skewsymmetric form to be

$$\begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}$$

Then up to conjugacy the parabolic subgroups are the intersections of parabolic subgoups of $GL(n, F)$.

In all cases if we chose a maximal compact subgroup, K, of G and a parabolic subgoup, P, over F then we have

$$G = KP.$$

One can show that reductive groups and compact groups are unimodular. (If $F = \mathbb{R}$ then $\delta(g) = |\det Ad(g)|$ which is 1 for a reductive group.) We can thus apply Proposition 10 and we see that if dk and $d_r p$ are respectively invariant and right invariant measures on K and P then up to constants of normalization we have

$$\int_G f(g)dg = \int_{K\times P} f(kp)dkd_rp.$$

Also of dp is left invariant measure then we have

$$\int_G f(g)dg = \int_{P\times K} f(pk)dpdk.$$

A parabolic group over F can be written in the form MN with M a reductive group over F and N the unipotent radical of P. For our context a group over F is said to be unipotent if there is an ascending series $\{1\} = N_0 \subset N_1 \subset ... \subset N_r = N$ with each N_i closed and normal and N_i/N_{i-1} isomorphic as an algebraic group with the additive group of F. This in particular implies that δ_P is identically equal to 1 on N. One can refine the above integration formulas as follows.

Proposition 18 *Let G be a reductive group over F and let P be a parabolic subgroup with $P = MN$ a Levi decomposition. Then up to constants of normalization we have*

$$\int_G f(g)dg = \int_{N\times M\times K} \delta_P(m)^{-1} f(nmk)dpdk.$$

The following standard material is used in the definition of parabolic induction.

Lemma 19 *If $f \in C(K \cap P\backslash K) = C(P\backslash G)$ then there exists $\varphi \in C_c(G)$ such that*

$$f(k) = \int_P \varphi(pk)dp.$$

Proof. Consider the map $p, k \longmapsto pk$ of $P \times K$ onto G. Then if $pk = p_1k_1$ with $p, p_1 \in P$ and $k, k_1 \in K$ then $(p_1)^{-1}p = k_1k^{-1} = m \in K \cap P$. Hence $p_1 = pm^{-1}$ and $k_1 = mk$. This implies that if we consider the left action $m(p,k) = (pm^{-1}, mk)$, $m \in K \cap P$ then G is homeomorphic with $K \cap P \backslash (P \times K)$. Let $\phi \in C_c(P/K \cap P)$ and define $\varphi(p,k) = \phi(p)f(k)$. Then $\varphi(pm.m^{-1}k) = \varphi(p,k)$ for all $p \in P$, $k \in K$, $m \in P \cap K$. Assume that

$$\int_P \phi(p)dp = 1.$$

Then the formula in the statement is satisfied. ∎

Here is the key integration formula in this context. First we need some notation. If $u \in C(P)$ and $u(pm) = u(p)$ for $p \in P$ and $m \in P \cap K$ then we extend u to G by $u(pk) = u(p)$, for $p \in P$ and $k \in K$. If $h \in C(K)$ and $h(mk) = h(k)$ for $m \in K \cap P$ and $k \in K$ then we extend h to G by setting $h(pk) = h(k)$, $p \in P, k \in K$. Let δ be the modular function of P. Then since $\delta_{|K \cap P} = 1$ we may extend it to G as above.

Lemma 20 *Let $f \in C(K \cap P \backslash K)$ then*

$$\int_K f(k)dk = \int_K f(kg)\delta_P(kg)dk$$

for all $g \in G$.

Proof. Let φ be as in the previous lemma. Then

$$\int_K f(k)dk = \int_G \varphi(x)dx = \int_G \varphi(xg)dg.$$

If $x \in G$ then we write $x = p(x)k(x)$ for some choice of $p(x)$ and $k(x)$ the ambiguity of the choice will be irrelevant in the rest of the argument. We continue

$$\int_G \varphi(xg)dx = \int_{P \times K} \varphi(pkg)dpdk = \int_{P \times K} \varphi(pp(kg)k(kg))dpdk$$

$$= \int_{P \times K} \delta(p(kg))\varphi(pk(kg))dpdk = \int_K \delta(p(kg))f(k(kg))dk.$$

Since $\delta(p(kg)) = \delta(kg)$ and $f(k(kg)) = f(kg)$ the lemma follows. ∎

In the case when F is $\mathbb{R}$ then all of the groups we have been studying are all Lie groups. Let $\mathfrak{p} = Lie(P), \mathfrak{m} = Lie(M)$ and $\mathfrak{n} = Lie(N)$. Then $[p, \mathfrak{n}] \subset \mathfrak{n}$ and if $n \in N$ then $Ad(n)$ on $\mathfrak{p}$ is unipotent (the only eigenvalue over the algebraic closure is 1). This implies that if $n \in N$ and $m \in M$ then $\theta\theta$

$$\delta_P(mn) = |\det(Ad(m)_{|\mathfrak{n}})|.$$

1.3 Orbital Integrals I. SL(2,$\mathbb{R}$)

Let G be a locally compact, seperable, unimodular, topological group with invariant measure dg. Let $\gamma \in G$ and set $G_\gamma = \{g \in G | g\gamma g^{-1} = \gamma\}$. Then G_γ is closed subgroup of G. We will use the notation dxG_γ for G-invariant measure on G/G_γ asserted to exist in Proposition 6. Note that it is uniquely determined by fixed choices of left invariant measures on G and on G_γ.The integrals

$$F(\gamma) = \int_{G/G_\gamma} f(g\gamma g^{-1}) dg G_\gamma$$

will be called orbital integrals. We note that this makes formal sense since $gh\gamma h^{-1}g^{-1} = g\gamma g^{-1}$ for $h \in G_\gamma$. It is, however, not easy to determine when (if ever) this integral converges. In this section we will concentrate on the case when $G = SL(2, \mathbb{R})$ and the next we will concentrate on the case when G is a more general reductive group over $\mathbb{R}$. There are many references for this material but we will use Chapter 7 in Real Reductive Groups I (RRGI). We will not give details in the case of $G = SL(2, \mathbb{R})$ (exept for the socalled nilpotent orbital integrals), rather we will write out formulae details can be found in RRGI section 7.5. The formulae are relatively easy consequences of the more general results later in this section. Let

$$h = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then every element in G is conjugate to one of the form e^{xu} with $x \in \mathbb{R}$ and $u \in \{h, H, X\}$. We chose for a maximal compact subgroup of G the group $K = SO(2)$. That is $K = \{e^{\theta h} | \theta \in \mathbb{R}\}$ since

$$e^{\theta h} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

We also note that since dg is invariant we will get the same orbital integrals from

$$\bar{f}(g) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{\theta h} g e^{-\theta h}) d\theta.$$

We will thus assume that $f = \bar{f}$ until further notice (this is no loss in generality since all orbital integrals of f are exactly the same as those of $\bar{f}$. If $\theta = 0$ or $\theta = \pi$ then $\gamma = \pm I$ and the integral is just $F(\pm I) = f(\pm I)$. We will now concetrate on $\theta \neq 0, \pi$. It is also convenient to normalize the integral and define

$$F_f(\theta) = \sin\theta F(e^{\theta h}).$$

Using standard integration formulas one finds that (there are constants of normalization that we will ignore)

$$F_f(\theta) = \sin\theta \int_0^{\infty} \sin(2t) f\left(\exp\left(\theta \begin{bmatrix} 0 & e^{2t} \\ -e^{-2t} & 0 \end{bmatrix}\right)\right) dt.$$

And

$$F_f(\theta) = \frac{\sin\theta}{|\theta|}\int_{|\theta|}^{\infty} f\left(\exp\left(\operatorname{sgn}\theta \begin{bmatrix} 0 & u + (u^2-\theta^2)^{\frac{1}{2}} \\ -u + (u^2-\theta^2)^{\frac{1}{2}} & 0 \end{bmatrix}\right)\right) du.$$

Notice that for $\theta \neq 0$ the function $F_f(\theta)$ is easily seen to be smooth (continuous derivatives of all orders). Thus there is no problem with the convergence of the corresponding orbital integral in for $\theta \neq 0$ or π (notice that $\sin\pi = 0$ so F_f says nothing about the orbital integral at $-I$). The second formula yields

$$\lim_{\theta \to 0+} F_f(\theta) = \int_0^{\infty} f\begin{pmatrix} 1 & 2u \\ 0 & 1 \end{pmatrix} du$$

and

$$\lim_{\theta \to 0-} F_f(\theta) = -\int_0^{\infty} f\begin{pmatrix} 1 & -2u \\ 0 & 1 \end{pmatrix} du.$$

Thus$\neq$

$$\lim_{\theta \to 0+} F_f(\theta) - \lim_{\theta \to 0-} F_f(\theta) = \frac{1}{2}\int_{-\infty}^{\infty} f\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} du.$$

A direct calculation of derivatives

$$\lim_{\theta\to 0}\frac{d}{d\theta}F_f(\theta)=2f(I).$$

This formula is a critical first step in the proof of the Plancherel fomula for G.

We next look at the integrals corresponding to $\gamma=\exp tH=\begin{bmatrix} e^t & 0\\ 0 & e^{-t}\end{bmatrix}$. Then if $t=0$ then $\gamma=I$ thus we will consider the case when $t\neq 0$. We will do a similar normalization setting for $\gamma=\exp xH$, $x\neq 0$

$$F_f^A(x)=|\sinh x|\int_{G/G_\gamma}f(g\gamma g^{-1})dg.$$

The superscript A corresponds to $A=\{\exp xH|x\in\mathbb{R}\}$ (to be consistant we should set $F_f^T=F_f$ with $T=\{\exp\theta h|\theta\in\mathbb{R}\}$. Then we have

$$F_f^A(x)=\frac{e^x}{2}\int_{-\infty}^{\infty}f\left(\exp xH\exp yX\right)dx.$$

As before this implies that the orbital integral converges for all $x\neq 0$. This implies the following beautiful formula

$$\lim_{\theta\to 0+}F_f^T(\theta)-\lim_{\theta\to 0-}F_f^T(\theta)=\lim_{x\to 0}F_f^A(x).$$

There is one more case to do. We set $N=\{\exp xX|x\in\mathbb{R}\}$. This time we look at $\gamma=\exp(xX)$ with $x\neq 0$. We will give a bit more detail in this case since this case is not covered in RRGI. We first note that if $P=\pm AN$ then P is a parabolic subgroup of G. Since $\pm I\in K$ we see that $G=KAN$. We note that the right invariant measure on AN can be described as follows

$$\int_{AN}f(s)d_rs=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{2t}f\left(\begin{bmatrix} e^t & 0\\ 0 & e^{-t}\end{bmatrix}\begin{bmatrix} 1 & y\\ 0 & 1\end{bmatrix}\right)dtdy.$$

One can check this formula directly observing that it is obviously invariant under right multiplication by $\begin{bmatrix} 1 & u\\ 0 & 1\end{bmatrix}$ for $u\in\mathbb{R}$ and if we multiply on the right by $\begin{bmatrix} e^s & 0\\ 0 & e^{-s}\end{bmatrix}$ then we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{2t}f\left(\begin{bmatrix} e^t & 0\\ 0 & e^{-t}\end{bmatrix}\begin{bmatrix} 1 & y\\ 0 & 1\end{bmatrix}\begin{matrix} e^s & 0\\ 0 & e^{-s}\end{matrix}\right)dtdy=$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2t} f\left(\left[\begin{array}{cc} e^{t+s} & 0 \\ 0 & e^{-t-s}\end{array}\right]\left[\begin{array}{cc} 1 & e^{-2s}y \\ 0 & 1\end{array}\right]\right) dtdy =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2(t+s)} f\left(\left[\begin{array}{cc} e^{t+s} & 0 \\ 0 & e^{-t-s}\end{array}\right]\left[\begin{array}{cc} 1 & y \\ 0 & 1\end{array}\right]\right) dtdy =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2t} f\left(\left[\begin{array}{cc} e^{t} & 0 \\ 0 & e^{-t}\end{array}\right]\left[\begin{array}{cc} 1 & y \\ 0 & 1\end{array}\right]\right) dtdy.$$

We next observe that $G_\gamma = \pm N$. The invariant measure on G/N is defined up to normalization by the condition that

$$\int_{G/N}\int_N f(gn)d(gN) = \int_G f(g)dg.$$

Thus by the above, if $\phi \in C_c(G/N)$ then up to normalization we have

$$\int_{G/N} \phi(gN)d(gN) = \int_K \int_{-\infty}^{\infty} \phi(k\exp(tH)N)e^{2t}dtdk.$$

Thus in this case we have if $x \neq 0$ then (recall that $f = \bar{f}$)

$$F\left(\left[\begin{array}{cc} 1 & x \\ 0 & 1\end{array}\right]\right) = \int_{-\infty}^{\infty} f\left(\left[\begin{array}{cc} 1 & e^{2t}x \\ 0 & 1\end{array}\right]\right) e^{2t}dx =$$

$$\frac{1}{|x|}\int_0^{\infty} f\left(\left[\begin{array}{cc} 1 & \mathrm{sgn}(x)u \\ 0 & 1\end{array}\right]\right) du.$$

The upshot is that every orbital integral for $SL(2,\mathbb{R})$ can be obtained by some sort of limiting procedure from the orbital integrals for the semisimple elements (γ conjugate to an element of K or $\pm A$).

1.4 Orbital integrals II. Elliptic elements in groups over $\mathbb{R}$

In the last section we saw that there was an apparent hierarchy of orbital integrals. The elliptical (i.e. $\gamma \in K$) at the top of the heap. We will begin this section with that class. Let K be a maximal compact subgroup of G.

Let $\mathfrak{g} = Lie(G)$ and $\mathfrak{k} = Lie(K)$ We note that G can be realized as a closed subgroup of $GL(n, \mathbb{R})$ (for some n) that is invariant under transpose ($g \mapsto g^T$) such that $K = \{g \in G | g^T g = I\}$. Indeed, this may be taken to be a definition of a linear reductive group, in any event we will assume this and note that it is true for all of our examples. We therefore assume that $\mathfrak{g}$ is a Lie subalgebra of $M_n(\mathbb{R})$ that is invariant under transpose. We set $B(X, Y) = \mathrm{tr}(XY)$. Then B is a symmetric and G-invariant ($B(Ad(g)X, Ad(g)Y) = B(X, Y)$ for $g \in G$) bilinear form. We set $\mathfrak{p} = \{X \in \mathfrak{g} | B(X, Y) = 0 \text{ for } Y \in \mathfrak{k}\}$. Let $\theta(g) = (g^T)^{-1}$ then θ defines an involutive automorphism of G and K is the fixed point set. We also note that $\theta(X) = -X$ for $X \in \mathfrak{p}$. We set

$$K'' = \{k \in K | \det((I - Ad(k))_{|\mathfrak{p}}) \neq 0\}.$$

This is the set of all $k \in K$ such that $G_k \subset K$. In general this set may be empty even if the center of G is contained in K.

Example 21 *$G = SL(n, \mathbb{R})$ with $n \geq 3$. Then $K = SO(n)$. Every element is conjugate (in K) to one of the form*

$$\begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & \ddots \end{bmatrix}$$

with each k_i in $SO(2)$ if n is even and if n is odd all but the last (which must be 1×1 and equal to 1). The centralizer of any such element contains elements of the form

$$\begin{bmatrix} a_1 I & 0 & 0 \\ 0 & a_2 I & 0 \\ 0 & 0 & \ddots \end{bmatrix}$$

with $a_i > 0$.

Harish-Chandra observed that the situation with $K'' \neq \emptyset$ is the basic case and developed methods to reduce to this case. Before we pursue them we will analyze this case. We note that the set

$$G[K''] = \{gkg^{-1} | g \in G, k \in K''\}$$

is open in G. We will be assuming until further notice that it is non-empty. One has the following integratal formula (which is a consequence of the Weyl integration for G and K).

Lemma 22 *There exists a positive constant such that*

$$\int_{G[K'']} f(g)dg = c\int_K |\det((1-Ad(k))_{|\mathfrak{p}})| \int_G f(gkg^{-1})dgdk.$$

This integration formula is the basis of the estimation that is used in finding general classes of functions for the convergence of orbital integrals. We will jump to the end. If $f \in C_c^\infty(G)$ then we define a function

$$Q_f(k) = \int_G f(gkg^{-1})dg$$

on K with domain those k such that the integral converges absolutly. Then we have

Lemma 23 *If $f \in C_c^\infty(G)$ then K'' is contained in the domain of Q_f and $Q_f \in C^\infty(K'')$.*

As in the case of $SL(2,\mathbb{R})$ it is useful to add a factor to the orbital integral which we will now describe. We will assume that the group $\mathbf{G}$ is of inner type. This means that $Ad(\mathbf{G}) = Ad(\mathbf{G}^o)$ with $\mathbf{G}^o$ the identity component of $\mathbf{G}$. Our assumption that $K'' \neq \emptyset$ implies that a maximal torus (compact, connected abelian subgroup) of K is a Cartan subgroup of G (that is the complexified Lie algebra is maximal abelian and consists of diagonalizable elements). Fix a maximal torus, T in K. Let $\mathfrak{h}$ be the complexification of $Lie(T)$. Then the elements of $ad(\mathfrak{h})$ are simultaneously diagonalizable. Let $\mathfrak{g}_C$ denote the complexified Lie algebra of G and for each $\alpha \in \mathfrak{h}^*$ we set $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | ad(h)X = \alpha(h)X, h \in \mathfrak{h}\}$. We set $\Phi = \{\alpha \in \mathfrak{h}^* - \{0\} | \mathfrak{g}_\alpha \neq \{0\}\}$. Then the standard theory implies that the spaces $\mathfrak{g}_\alpha$ are all one dimensional and

$$\mathfrak{g} = \mathfrak{h} \bigoplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha.$$

We make a choice, Φ^+, of positive roots for Φ (i.e. Φ^+ is closed under as much addition as Φ is and such that if $\alpha \in \Phi$ that exactly one of the elements $\pm\alpha$ is in Φ^+). We note that $Ad(t)_{|\mathfrak{g}_\alpha} = t^\alpha I$ for $t \in T$. We write $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$ and we assume that ρ is the differential of a group homomorphism $(t \longmapsto t^\rho)$ of T into the circle group (we note that this is possible by possibly replacing G by a two-fold covering).

We write

$$\Delta(t) = t^{\rho} \prod_{\alpha\in\Phi^{+}} (1 - t^{-\alpha}) = \prod_{\alpha\in\Phi^{+}} (t^{\frac{\alpha}{2}} - t^{-\frac{\alpha}{2}}).$$

Notice that if $G = SL(2, \mathbb{R})$

$$t = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

then

$$\Delta(t) = 2i \sin\theta.$$

We write

$$F_f(t) = \Delta(t) Q_f(t)$$

and note that this function is defined on $T \cap K''$. Except for a factor of $2i$ this is consistant with the notation of the previous section. We set $T' = \{t \in T | t^{\alpha} \neq 1, \alpha \in \Phi\} = \{t \in T | \Delta(t) \neq 0\} \subset T \cap K''$. We look upon $Lie(T)$ as left invariant differential operators (as usual) and $U(Lie(T))$ as the algebra of differential operators generated by $Lie(T)$. We denote by $B(T')$ the space of all $f \in C^{\infty}(T')$ such that

$$\sigma_D(f) = \sup_{t\in T'} |Df(t)| < \infty$$

for all $D \in U(Lie(T))$. We endow $B(T')$ with the topology induced by the seminorms σ_D for $D \in U(Lie(T))$.

The fundamental theorem of Harish-Chandra in this context is

Theorem 24 *If $f \in C_c^{\infty}(G)$ then $F_f \in B(T')$. Furthermore, the map $f \longmapsto F_f$ is continuous into $B(T')$ (indeed it extends to a continuous mapping of the Harish-Chandra Schwartz space to $B(T')$).*

The hardest part of this result is the parenthetic remark and the topologies in question will be explained in the next section. Harish-Chandra's proof uses the theory of the discrete series for G. The proof in RRGI is straight analysis and similar ideas have been used in the non-Archimedian case by Rader and Silberger. The point here is that this implies that the singularities of F_f thought of as a function on all of T are at worst jump singularities (e.g. they are of the same nature as we encountered for $SL(2, \mathbb{R})$.

The next step is to look at more general diagonalizable elements (as in the case of $SL(2, \mathbb{R})$) For this and for later developments in representation theory we will need the notion of the Harish-Chandra transform.

1.5 The Harish-Chandra transform

For simplicity we will assume that $\mathbf{G}$ is connected. We will drop the assumption that $K'' \neq \emptyset$. The form $B(X,Y)$ =tr(XY) on $\mathfrak{g}_C$ is symmetric, invariant and non-degenerate. Let $n = \dim G$. We will think of $(\mathfrak{g}_C, B)$ as the pair $\mathbb{C}^n$ and the usual dot product. Thus $Ad : \mathbf{G} \to SO(n,\mathbb{C})$ is a group homomorphism. Let $\tilde{\mathbf{G}}$ denote the (algebraically) simply connected covering group of $\mathbf{G}$. Then Ad lifts to a homomorphism, μ, of $\tilde{\mathbf{G}}$ into $Spin(n,\mathbb{C})$. Let (δ, S) denote the spin representation of $Spin(n,\mathbb{C})$. Fix a Hilbert space structure on S that is invariant under the action of a maximal compact subgroup of $\tilde{\mathbf{G}}$ containing the inverse image of K. We denote by $\|...\|$ denote the corresponding Hilbert-Schmidt norm on $End(S)$. We note that $g \longmapsto \|\delta(\mu(g))\|$ is well defined on $\mathbf{G}$. Hence on G. We write $\|g\|$ for $\|\delta(\mu(g))\|$. We note that

$$\|k_1 g k_2\| = \|g\|\,, k_1, k_2 \in K, g \in G.$$

If you are uncomfortable with the spin representation but like the theorem of the highst weight then we have defined a multiple of the irreducible representation with highest weight ρ. For our purposes there are four important spaces of functions on G which we will now describe. We look upon $\mathfrak{g}$ as the left invariant vector fields on G (as is usual). We can also look at the Lie algebra as right invariant vector fields, for the right invariant fields (involving differentiation on the *left*) we will write X_L The algebra of differential operators generated by $\mathfrak{g}$ is denoted $U(\mathfrak{g})$. When we use right invariant differential operators we will write D_L. For the sake of consistancy we will use the notation D_R for the operator D to emphasize the fact that it involves differentiation on the right. Also, we write $G = {}^oGA$ with ${}^oG$ the intersection of the kernels of all continuous homomorphisms of G into $\mathbb{R}_{>0}$ and A is a subgroup of G isomorphic with the additive group of $\mathbb{R}^n$ for some n (but should be thought of as the multiplicative group $(\mathbb{R}_{>0})^n$). The group A is called the split component of G. Then the exponential mapping of $Lie(A)$ to A is an isomorphism. We denote by log the inverse map. Fix a Euclidean norm, $|...|$, on $Lie(A)$. We set $\sigma(g) = |\log(a)|$ if $g = g_1 a$ with $g_1 \in {}^oG$ and $a \in A$.

Example 25 *If $G = GL(n.\mathbb{R})$ or $SL(n.\mathbb{R})$ and H is the subgroup of diagonal elements of G then A_H is the group of diagonal elements with positive diagonal entries and T_H is the subgroup of the diagonal with diagonal entries*

± 1. $n\sigma(g) = |\log(|\det(g)|)|$. Finally, there are constants $C_1, C_2 > 0$ such that $C_1\phi(g) \leq \|g\| \leq C_2\phi(g)$ with

$$\phi(g)^2 = \int_K \frac{(\Delta_1^2\Delta_2^2\cdots\Delta_{n-1}^2)(kgk^{-1})}{|\Delta_n(g)|^{n-1}}dk$$

Here $\Delta_i(g)$ is the determinant of the upper left corner $i \times i$ block in g.

The first space is $C_c^\infty(G)$. Here if Ω is a compact subset of G then we set $C_\Omega^\infty(G) = \{f \in C_c^\infty(G)|\text{supp}(f) \subset \Omega\}$. We endow $C_\Omega^\infty(G)$ with the topology induced by the seminorms

$$p_{\Omega,D}(f) = \sup_{x\in\Omega} |Df(g)|$$

for $D \in U(\mathfrak{g})$. Here we could use either the right invariant or the left invariant version (or both). We put the union topology on $C_c^\infty(G)$. The next is the Harish-Chandra Schwartz space. Let $\mathcal{C}(G)$ denote the space of all smooth functions on G such that

$$q_{k,l,D,D'}(f) = \sup_{g\in G}(1 + \log\|g\|)^k(1 + \sigma(g)^l \|g\| |D_L D'_R f(g)| < \infty$$

for all k, l and all $D, D' \in U(\mathfrak{g})$. We endow $\mathcal{C}(G)$ with the topology induced by these seminorms. The next space is similar, let $\mathcal{C}^1(G)$ denote the space of all smooth functions on G such that

$$q_{1,k,l,D,D'}(f) = \sup_{g\in G}(1 + \log\|g\|)^k(1 + \sigma(g))^l \|g\|^2 |D_L D'_R f(g)| < \infty$$

for all k, l and all $D, D' \in U(\mathfrak{g})$. We endow $\mathcal{C}^1(G)$ with the topology induced by these seminorms. This is the socalled L^1-Schwartz space. The final example was introduced by Casselman and the author. We denote by $\mathcal{S}(G)$ the space of all smooth functions on G such that

$$\nu_{k,l,D}(f) = \sup_{g\in G} \|g\|^k (1 + \sigma(g))^l |Df(g)| < \infty$$

for all k, l and all $D \in U(\mathfrak{g})$. We endow $\mathcal{S}(G)$ with the topology induced by these seminorms.

The hierachy of these spaces is

$$C_c^\infty(G) \subset \mathcal{S}(G) \subset \mathcal{C}^1(G) \subset \mathcal{C}(G)$$

with continuous inclusion. One can show that every one of these spaces is closed under the left and right action of $U(\mathfrak{g})$.

We note that $\|g\| = \|g^{-1}\|$ so $\|g\|^2 = \|g\|\,\|g^{-1}\| \geq \|1\| \geq 1$. Using standard integration formulas for the socalled KAK decomposition one can show

Lemma 26 *If k is sufficiently large then*

$$\int_G (1+\sigma(g))^{-k}(1+\log\|g\|)^{-k}\,\|g\|^{-2}\,dg < \infty.$$

In particular, $\mathcal{C}(G) \subset L^2(G)$ and $\mathcal{C}^1(G) \subset L^1(G)$.

We will use the notation $\|...\|_G$ for the norm defined above for G (note that it is invariant under left and right multiplication by elements of the center of G. Let P be a parabolic subgroup of G with unipotent radical N. We choose a Levi factor, M, of P and we use the decomposition $M = {}^oMA_M$ we did for G with A_M the split component. Then an element of P can be written uniquely in the form $g = n(g)a(g)m(g)$ with $m(g) \in {}^oM$, $a(g) \in A_M$ and $n(g) \in N$. Also, $G = KP$ thus if $g \in G$ then we can write

$$g = n(g)a(g)m(g)k(g)$$

with $k(g) \in K$ and $m(g) \in {}^oM$, $a(g) \in A_M$ and $n(g) \in N$. We note that only the cosets $M \cap Kk(g)$ and $m(g)M \cap K$ are well defined. The following is a variant of a result of Harish-Chandra (for a proof see 4.5.6 in RRGI). We write $\bar{N} = \theta(N)$.

Theorem 27 *If $r > 0$ is sufficiently large then for fixed d we have*

$$\int_{\bar{N}} \delta_P(a(\bar{n}))^{\frac{1}{2}}\,\|m(\bar{n})\|_M^{-1}\,(1+\log\|m(\bar{n})\|)^d(1+|\log(\delta_P(a(\bar{n})))|)^{-k} < \infty.$$

The point of this ugly result is that if $f \in \mathcal{C}(G)$ then the integral

$$f^P(m) = \delta_P(m)^{-\frac{1}{2}}\int_N f(mn)dn$$

converges absolutely for $m \in M$. One has (cf. Theorem 7.2.1 in RRGI).

Theorem 28 *The map $f \longmapsto f^P$ defines a continuous linear map from $\mathcal{C}(G)$ into $\mathcal{C}(M)$.*

We will call f^P the Harish-Chandra transform of f. It is ubiquitous in Harish-Chandra's approach to representation theory and automorphic forms.

1.6 Orbital Integrals III. General semisimple elements.

We retain the assumptions of the previous section. A subgroup H of G is said to be a Cartan subgroup of G if it is the set of real points of a Cartan subgroup of $\mathbf{G}$ defined over $\mathbb{R}$. That is to say, there is a maximal abelian subalgebra of $\mathfrak{g}_C$, $\mathfrak{h}$, consisting of diagonalizable (e.g. semisimple) elements such that $\mathfrak{h} \cap \mathfrak{g}$ is a real form of $\mathfrak{h}$ and finally H is the subgroup $\{g \in G | Ad(g)_{|\mathfrak{h}} = I\}$. Fix such a group. Then H is a reductive group of the type we are studying. Hence $H = T_H A_H$ with T_H a compact with its identity component a torus and A_H isomorphic with $(\mathbb{R}_{>0})^n$.

Example 29 *$G = GL(n, \mathbb{R})$. Let $2k \leq n$. Let H denote the subgroup of G consisting of block diagonal matrices*

$$\begin{bmatrix} a_1 t_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 t_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_k t_k & 0 \\ 0 & 0 & \cdots & 0 & D \end{bmatrix}$$

with t_i an element of $SO(2)$ and $a_i > 0$ for $i = 1, ..., k$ and D a diagonal $n - 2k \times n - 2k$ matrix.

Fix such a Cartan subgroup. Let Φ be the root system of $\mathfrak{g}_C$ with respect to $\mathfrak{h}$. Let $\bar{X}$ denote complex conjugation in $\mathfrak{g}_C$ with respect to $\mathfrak{g}$. If $\alpha \in \Phi$ then set $\bar{\alpha}(h) = \overline{\alpha(\bar{h})}$. Choose a system of positive roots Φ^+ such that if $\alpha \in \Phi^+$ and if $\bar{\alpha} \neq -\alpha$ then $\bar{\alpha} \in \Phi^+$. Let $\Sigma = \{\alpha \in \Phi^+ | \bar{\alpha} \neq -\alpha\}$. We set

$$\Delta_H = h^\rho \prod_{\alpha \in \Phi^+ - \Sigma} (1 - h^{-\alpha}) \prod_{\alpha \in \Sigma} |1 - h^{-\alpha}|.$$

Note that if $H = T$ as in section 1.4 we have the same factor. In the case of $G = SL(2, \mathbb{R})$ and if H is the group of diagonal elements of G then $\Delta_H \left(\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \right) = |\sinh t|$. As before $H' = \{h \in H | \Delta_H(h) \neq 0\}$.We note that if $\gamma \in H'$ then $G_\gamma = H$.

We define for $\gamma \in H'$

$$F_f^H(\gamma) = \Delta_H(\gamma) \int_{G/H} f(g\gamma g^{-1}) dg.$$

Let $\mathfrak{n}_C = \sum_{\alpha\in\Sigma}\mathfrak{g}_\alpha$ and let $P = \{g \in G|Ad(g)\mathfrak{n}_C \subset \mathfrak{n}_C\}$. Then P is a parabolic subgroup of G with unipotent radical N and n_C is the complexification of $Lie(N)$. Furthermore, if $M = \{g \in G|ga = ag, a \in A\}$ then M is a Levi factor of P. Finally, A is a split component of M and T is a Cartan subgroup of ${}^oM$.

With this notation we have (see RRGI 7.4.10).

Proposition 30 *Let $f \in \mathcal{C}(G)$. Define $\bar{f}(g) = \int_K f(kgk^{-1})dk$. Let $u_a(m) = \bar{f}^P(ma)$ for $m \in {}^oM$ and $a \in A$. Then*

$$F_f^H(ta) = F_{u_a}^T(t)$$

for $t \in T'$ and $a \in A$.

Let $H'' = T'A$. Then we see that the integral defining $F_f^H(\gamma)$ converges absolutely for $\gamma \in H''$. Let σ be as above for H. We denote by $\mathcal{C}(H'')$ the space of all smooth f on H'' such that

$$\xi_{k,D}(f) = \sup_{h\in H''}(1+\sigma(h))^k|Df(h)| < \infty$$

for all k and all $D \in U(Lie(H))$ endowed with the topology induced by the seminorms $\xi_{k,D}$. We have (cf. RRGI, Theorem 7.4.10)

Theorem 31 *The map $f \longmapsto F_f^H$ defines a continuous linear map of $\mathcal{C}(G)$ into $\mathcal{C}(H'')$.*

2 Representation Theory

In this section we we develop just enough representation theory to study the terms that occur in the trace formula.

2.1 Definitions and preliminary results.

Let G be a locally compact, separable, topolgical group. We will write the left invariant measure as dg (as usual). If H is a topological vector space (over $\mathbb{C}$) then a representation of G on H is a group homomorphism, π, of G into $GL(H)$ (the continuous, linear bijections with continuous inverses) such that the map of $G \times H$ to H

$$g, h \longmapsto \pi(g)h$$

is continuous. That is, we will, at a minimum, be looking at strongly continuous representations. Generally, H will either be a Hilbert space or a Frèchet space. In either case a continous bijection is automatically an element of $GL(H)$. If H is a Hilbert space then $(\pi, H$ is called a Hilbert representation and if $\pi(g)$ is a unitary operator for each $g \in G$ then (π, H) will is called a unitary representation of G.

We will need a result of Banach (the principal of uniform boundedness).

Theorem 32 *Let V, W be Banach spaces and let $\mathcal{U}$ be a subset of $L(V, W)$. Suppose that for each $v \in V$ there exists $C_v < \infty$ such that $\|Tv\| \leq C_v \|v\|$ for all $T \in \mathcal{U}$. Then there exists $C < \infty$ such that $\|T\| \leq C$ for all $T \in \mathcal{U}$.*

This surprising theorem is a consequence of the Baire category theorem. A proof can be found in K.Yoshida, Functional Analysis. A useful criterion for when a group homomorphism from G to $GL(H)$ is a representation is

Lemma 33 *Let H be a Hilbert space, let G a locally compact, separable topological group and let π be a homomorphism of G into bounded, invertible operators on H. Then π defines a representation of G (that is, it is continuous in the strong topology) if and only if the following two conditions are satisfied*

1. There is a dense subspace, $V \subset H$ such that if $v \in V, w \in H$ the function $c_{v,w}(g) = \langle \pi(g)v, w \rangle$ is continuous.

2. If K is a compact subset of G then there exists a positive constant C_K such that $\|\pi(g)\| \leq C_K$ for all $k \in K$.

Proof. If (π, H) is a representation then condition 1. is clearly satisfied. The strong continuity implies that the functions

$$g \longmapsto \|\pi(g)v\|$$

are continuous on G. Condition 2 now follows from the principle of uniform boundedness. We now consider the converse. So we assume the two conditions. We observe that they imply

$1'$. $c_{v,w}$ is continuous for all $v, w \in H$.

This is proved by a "3ε argument". Let $v \in H$, $g \in G$, $\{v_j\}$ a sequence in V such that $\lim v_j = v$, $\{g_n\}$ a sequence in G such that $\lim g_n = g$. Then there exists a compact subset K of G containing g and each of the g_n. We must show that $\lim_{n\to\infty} c_{v,w}(g_n) = c_{v,w}(g)$. We note that Condition 2 implies that $\|\pi(k)\| \le C_K < \infty$ for all $k \in K$. Set $C = C_K$. We have

$$
\begin{aligned}
&|c_{v,w}(g_n) - c_{v,w}(g)| \\
&= |c_{v,w}(g_n) - c_{v_j,w}(g_n) + c_{v_j,w}(g_n) - c_{v_j,w}(g) + c_{v_j,w}(g) - c_{v,w}(g)| \\
&\le |c_{v,w}(g_n) - c_{v_j,w}(g_n)| + |c_{v_j,w}(g_n) - c_{v_j,w}(g)| + |c_{v_j,w}(g) - c_{v,w}(g)| \\
&= |\langle \pi(g_n)(v - v_j), w\rangle| + |c_{v_j,w}(g_n) - c_{v_j,w}(g)| + |\langle \pi(g)(v_j - v), w\rangle| \\
&\le 2C \|v - v_j\| \|w\| + |c_{v_j,w}(g_n) - c_{v_j,w}(g)|.
\end{aligned}
$$

Now let $\varepsilon > 0$ be given then there exists N such that if $j \ge N$ then $\|v - v_j\| < \varepsilon$. Fix one such j. There exists N_1 such that if $n \ge N_1$ then $|c_{v_j,w}(g_n) - c_{v_j,w}(g)| < \varepsilon$. Putting all of this together we have that if $n \ge N_1$ then $|c_{v,w}(g_n) - c_{v,w}(g)| < (2C + 1)\varepsilon$. This proves $1'$.

We will now begin the proof of the lemma. Ideas in this proof will be used in the next section. Let H_o be the subspace of all $v \in H$ such that the map $g \longmapsto \pi(g)v$ is continuous from G to H. Then using an argument as in the proof of $1'$. one can show that condition 2. implies that H_o is closed. Also, it is not hard to see that if we can show that $H_o = H$ then the result is proved.

If $f \in C_c(G)$ then we set

$$\mu_f(v, w) = \int_G f(g) \langle \pi(g)v, w\rangle \, dg.$$

If the support of f is contained in the compact set K and if $\phi \in C_c(G)$ is such that $\phi(k) = 1$ for all $k \in K$ (such a ϕ exists by Urysohn's theorem) then we have

$$|\mu_f(v, w)| \le C_K \|\phi\|_1 p_K(f) \|v\| \|w\|.$$

Thus the Riesz representation theorem implies that for each $v \in H$ there exists an element $T_f(v) \in H$ such that $\langle T_f(v), w\rangle = \mu_f(v, w)$. It is easy to see that T_f is a linear map of H to H. The estimate above now shows that $\|T_f(v)\| \le C_K \|\phi\|_1 p_K(f) \|v\|$. Thus the map $f \longmapsto T_f$ of the completion of $C_c(G)$ into $L(H, H)$ is strongly continuous. We note that

$$T_{L_g f} = \pi(g) T_f.$$

Hence $T_f(H) \in H_o$ for all $f \in C_c(G)$. Now, since G is separable and locally compact we can find a sequence of open subsets $U_j \subset G$ such that $\overline{U_j}$ is compact, $U_j \supset \overline{U}_{j+1}$ and $\cap_j U_j = \{1\}$. Uryson's lemma implies that there exists $\phi_j \in C_c(G)$ such that the support of ϕ_j is contained in U_j, $\phi_j(x) \geq 0$ for all $x \in G$ and $\phi_j(x) = 1$ for all $x \in \overline{U}_{j+1}$. Set $u_j(x) = \frac{\phi_j(x)}{\|\phi_j\|_1}$. Then if $v, w \in H$

$$\lim_{j \to \infty} \left\langle T_{u_j}(v), w \right\rangle = \langle v, w \rangle .$$

Before we prove this we will show how it completes the proof.

We need to show that $H_o^{\perp} = 0$. But if $w \in H_o^{\perp}$ then $\left\langle T_{u_j}(w), w \right\rangle = 0$ for all j. Hence the limit formula implies that $\langle w, w \rangle = 0$.

To prove the limit formula we note that

$$\left\langle T_{u_j}(v), w \right\rangle - \langle v, w \rangle = \int_G u_j(g)(c_{v,w}(g) - c_{vw}(1))dg.$$

Let $\varepsilon > 0$ be given then there exists N such that if $j \geq N$ then

$$|c_{v,w}(g) - c_{vw}(1)| < \varepsilon \text{ for } g \in U_j.$$

Thus if $j \geq N$ then

$$\int_G u_j(g)(c_{v,w}(g) - c_{vw}(1))dg \leq \varepsilon \int_G u_j(g)dg = \varepsilon.$$

This completes the proof. ■

Definition 34 *If (π, H) is a Hilbert representation of G then the operator T_f as defined in the above proof will be denoted $\pi(f)$.*

We note that if (π, H) is unitary then the operator norm of $\pi(f)$ is less than or equal to the L^1-norm. Thus in this case $\pi(f)$ is meaningful for f in $L^1(G)$.

Definition 35 *If G is a locally compact separable topological group then a sequence $\{u_j\}$ of non-negative functions in $C_c(G)$ such that $\|u_j\|_1 = 1$ for all j and there exist open subsets U_j of G such that $\overline{U_j}$ is compact, $U_j \supset \overline{U}_{j+1}$ and $\cap_j U_j = \{1\}$ and the support of u_j is contained in U_j then $\{u_j\}$ will be called a* delta sequence *or* approximate identity *on G.*

In the course of the proof of Lemma 33 we have also proved

Lemma 36 *Let (π, H) be a Hilbert representation of G. Let $\{u_j\}$ be a delta sequence on G. Then if $u, v \in H$ then*

$$\lim_{i\to\infty} \langle \pi(u_j)u, v\rangle = \langle u, v\rangle .$$

The following is known as the Weyl unitary trick (however, Weyl called it the unitarian trick)..

Lemma 37 *Assume that K is a compact topological group and that (π, H) is a Hilbert representation of K. Let $\langle ..., ...\rangle$ be the Hilbert space structure on H. Then there is an inner product $(..., ...)$ on H such that $(\pi(k)v, \pi(k)w) = (v, w)$ for all $k \in K, v, w \in H$ and such that the topology on H induced by $(..., ...)$ is the same as the original topology. In particular, if (σ, V) is a finite dimensional represntation of K then we may assume that it is unitary.*

Proof. We assume that $\int_K dk = 1$. We define

$$(v, w) = \int_K \langle \pi(k)v, \pi(k)w\rangle \, dk, v, w \in H.$$

Then since K is compact there exists a constant $C > 0$ such that

$$\|\pi(k)\| \le C, k \in K.$$

This implies that if v is a unit vector then

$$\langle \pi(k)v, \pi(k)v\rangle \le C \|\pi(k)v\| , k \in K.$$

Thus

$$\langle \pi(k)v, \pi(k)v\rangle \ge C^{-2} \langle v, v\rangle , k \in K, v \in H.$$

We conclude that

$$C^{-2} \langle v, v\rangle \le (v, v) \le C^2 \langle v, v\rangle , v \in H.$$

The last assertion follows by observing that any finite dimensional vector over $\mathbb{C}$ space is isomorphic with a Hilbert space. ■

We will show how these results allow us to show that certain standard group actions yield representations.

Let X be a locally compact space on which G acts continuously on the right. We also assume that there is a measure dx on X and a continuous function $c : X \times G \to \mathbb{R}_{>0}$ such that

$$\int_X f(xg)c(x,g)dx = \int_X f(x)dx$$

for all $g \in G$. One checks that if dx is a regular measure (if $f(x) \geq 0$ and $f \neq 0$ then $\int_X f(x)dx > 0$) then c must satisfy the cocycle condition

$$c(x, g_1)c(xg_1, g_2) = c(x, g_1g_2).$$

In any event we assume this condition and that the measure is regular. The function c is called the cocycle determined by the action and the measure. If $f \in C_c(X)$ and $g \in G$ then we set

$$\pi(g)f(x) = c(x,g)^{\frac{1}{2}}f(xg).$$

One checks that

$$\pi(g_1g_2)f = \pi(g_1)\pi(g_2)f.$$

As usual, we define

$$\langle f, g\rangle = \int_X f(x)\overline{g(x)}dx$$

for $f, g \in C_c(X)$. We will use the usual notation $L^2(X)$ for the Hilbert space completion of $C_c(X)$ relative to this inner product. The factor in the action is designed so that

$$\langle \pi(g)f_1, \pi(g)f_2\rangle = \langle f_1, f_2\rangle$$

for $f_1, f_2 \in C_c(X)$. Thus the operators $\pi(g)$ extend to unitary operators on $L^2(X)$. Since it is easy to see that the hypothesis of Lemma 33 are satisfied with $V = C_c(X)$ we have defined a unitary representation of G.

Example 38 *$X = G$ and the action is the right regular action. The corresponding unitary represntation is called the right regular represention of G.*

Example 39 *G a Lie group and H a closed subgroup of G. If we take the quotient measure on $H\backslash G$ relative to the left invariant measure then using the realization of the measures using left invariant forms one sees that the cocycle exists. A subexample of this is when $\Gamma \subset G$ is a discrete subgroup. Then we have the right regular representation of G on $L^2(\Gamma\backslash G)$.*

The example of $\Gamma\backslash G$ is the point of these lectures.

We end this section by using the same ideas to define parabolic induction. Let G be a reductive group over a locally compact non-discrete field, F. Let P be a parabolic sugroup defined over F. We will use the notation preceding Lemma 20.

Let (σ, H_σ) be a Hilbert representation of P. We assume that it is unitary when restricted to $K \cap P$ (this is no real assumption in light of Lemma 33). Let H_o^σ denote the space of all continuous functions

$$f : G \rightarrow H_\sigma$$

such that $f(pg) = \delta(p)^{\frac{1}{2}}\sigma(p)f(g)$ for $p \in P$ and $g \in G$. We note that if $f \in H_o^\sigma$ and $f_{|K} = 0$ then $f = 0$. We endow H_o^σ with a pre-Hilbert space structure by taking

$$\langle f_1, f_2 \rangle = \int_K \langle f_1(k), f_2(k) \rangle \, dk$$

for $f_1, f_2 \in H_o^\sigma$ (here the inner product inside the integral is that of H_σ). Let H^σ denote the Hilbert space completion of H_o^σ. If $g \in G$ then we define the operator $\pi_\sigma(g)$ on H_o^σ by $\pi_\sigma(g)f(x) = f(xg)$.

Lemma 40 *If $g \in G$ then $\pi(g)$ extends to a bounded operator on H^σ. Furthermore, (π_σ, H^σ) defines a Hilbert representation of G which is unitary if (σ, H_σ) is unitary.*

Proof. As in the proof of Lemma 20 , we will write $x = p(x)k(x)$. If Ω is a compact subset of G then since the ambiguity is in $K \cap P$ we see that $p(\Omega) \subset \Omega'$ a compact subset of P. Thus there exists a constant $C_\Omega < \infty$ such that $\|\sigma(p(x))\| \leq C_\Omega$ for all $x \in \Omega$. If $f \in H_o^\sigma$ then

$$\|\pi_\sigma(g)f\|^2 = \int_K \|f(kg)\|^2 \, dk = \int_K \delta(p(kg)) \, \|\sigma(p(kg))f(k(kg))\|^2 \, dk. \quad (1)$$

Now if $g \in \Omega$ then this last expression is less than or equal to

$$C_\Omega^2 \int_K \delta(p(kg)) \, \|f(k(kg))\|^2 \, dk = C_\Omega^2 \, \|f\|^2$$

in light of the previous lemma. Thus $\|\pi_\sigma(g)\| \leq C_\Omega$. We note that (1) above combined with the integral formula in Lemma 20 implies that $\pi_\sigma(g)$, for $g \in G$, is unitary if σ is unitary. We leave it to the reader to check that the matrix coefficients $g \longmapsto \langle \pi_\sigma(g)u, v\rangle$ are continuous for $u, v \in H_o^\sigma$. Thus Lemma 33 implies that π_σ defines a representation of G. ■

The representation (π_σ, H^σ) is usually denoted $Ind_P^G(\sigma)$ or $Ind_P^G(H_\sigma)$ and called an *parabolically induced* representation. If the parabolic subgroup is minimal and if the representation σ is one dimensional and is given by $p \longmapsto |p|^{i\nu}$ for $\nu \in \mathbb{R}$ then we say that $Ind_P^G(\sigma)$ is a spherical (or unramified) principal series representation.

Example 41 *These names were frst applied to the case of $G = SL(2, F)$ or $GL(2, F)$. We will dscribe them in this case. We take P to be the subgroup of upper triangular matrices in G and χ a continuous homomorphism of P into $\mathbb{C}^\times$. Then the corresponding representation will be denoted $I(\chi)$ and is called a principal series representation. If χ is unitary then the representation is unitary.*

2.2 Schur's Lemma.

There is a sens in which representation theory is just a series of applications of variants of Schur's lemma. In this section we will give several versions.

Definition 42 *Let (π, V) be a representation of G on a topological vector space V. A subspace W of V is said to be* invariant *if $\pi(g)W \subset W$. The representation is said to be irreducible if the only closed, invariant subspaces are $\{0\}$ and V.*

Definition 43 *If (π_i, V_i), $i = 1, 2$ are respectively representations of G on topological vector spaces V_1, V_2 then a continuous map $T : V_1 \to V_2$ will be called a G-*homomorphism *or G-*intertwining operator *if $T \circ \pi(g) = \pi(g) \circ T$ for all $g \in G$. We will use the notation $L_G(V_1, V_2)$ for the space of all G-homomorphisms from V_1 to V_2. The representations are said to be equivalent if there exists an element then is bijective with continuous inverse in $L(V_1, V_2)$.*

Remark 44 *In the literature the notation $Hom_G(V_1, V_2)$ is often used for what we are calling $L_G(V_1, V_2)$.*

Here is the first version of Schur's Lemma which is a direct consequence of the spectral theorem.

Proposition 45 *Let (π, H) be a unitary representation of G. Then it is irreducible if and only if $L_G(H, H) = \mathbb{C}I$.*

Proof. Suppose $L_G(H, H) = \mathbb{C}I$. Let V be a closed invariant subspace of H. Let P denote the orthogonal projection of H onto V. If $v \in H$ then $v = v_1 + v_2$ with $v_1 \in V$ and $v_2 \in V^\perp$. If $w \in V^\perp$ and if $v \in V$ then for each $g \in G$ we have $0 = \langle \pi(g)v, w \rangle = \langle v, \pi(g^{-1})w \rangle$ by the assumption of unitarity. But then $V^\perp$ is an invariant space. Thus $\pi(g)v_1 \in V$ and $\pi(g)v_2 \in V^\perp$. Hence $P\pi(g)v = \pi(g)v_1 = \pi(g)Pv$. Thus $P \in L_G(H, H)$. Thus $P = zI$, $z \in \mathbb{C}$. Since P is a projection $z = 0$ or I. Thus $V = \{0\}$ or $V = H$.

We now prove the converse. We first note that if $T \in L_G(H, H)$ then so is T^*. Since $T = \frac{T+T^*}{2} + i\frac{T-T^*}{2i}$ we must only show that if T is a self adjoint intertwining operator then T is a multiple of the identity. So we assume $T \in L_G(H, H)$ and $T^* = T$. To such an operator there is an associated family of spectral projections, P_S, for $S \subset \mathbb{R}$ a Borel set. (See Reed,M. and Simon,B., *Functional Analysis I,*Academic Press,1972., p.234.) The uniqueness of the spectral resolution and the fact that $\pi(g)T\pi(g)^{-1} = T$ implies that $\pi(g)P_S\pi(g)^{-1} = P_S$ for all S. Then $T = pI$ if and only if $P_{\{p\}} = I$. If the real interval $[a, b]$ contains the spectrum of T then $P_{[a,b]} = I$. Let $J_1 = [a, b]$. If we bisect J_1, then $J_1 = A \cup B$ and one of P_A or P_B is non-zero. Thus $P_A = I$ or $P_B = I$. Let J_2 be one of A, B such that $P_{J_2} = I$. We can bisect again and get J_3 one of the halves such that $P_{J_2} = I$. We this have a nested sequence of intervals $J_1 \supset J_2 \supset J_3 \supset ...$ such that J_i has length $2^{-i}(b - a)$ and $P_{J_i} = I$. We note that $\cap_i J_i = \{p\}$ for some $p \in \mathbb{R}$. The definition of spectral projections implies that the limit of the P_{J_i} in the strong operator topology is $P_{\{p\}}$. Thus $P_{\{p\}} = I$. Hence $T = pI$. ■

We will rephrase this result in the context of operator algebras. Let $A \subset L(H, H)$ be a subalgebra. Then it is called a **algebra* if whenever $T \in A$, $T^* \in A$. We say that A is an *irreducible subalgebra* if whenever $V \subset H$ is a closed subspace invariant under all the elements of A, $V = \{0\}$ or $V = H$.

Definition 46 *If A is a subset of $L(H, H)$ then we denote by A' the set $\{T \in L(H, H) | Ta = aT, a \in A\}$. A' is called the commutant of A.*

The above is a standard notation. It unfortunately conflicts with a standard notation for continuous dual space.

We will now restate Schur's lemma.

Corollary 47 *(to the proof)A* **algebra* $A \subset L(H,H)$ *is irreducible if and only if* $A' = \mathbb{C}I$.

Proof. We note that if $V \subset H$ is a closed subspace invariant under every element of A then so is $V^{\perp}$. Thus as above $P_V \in A'$. Thus if $A' = \mathbb{C}I$. Then A is irreducible (as above). To prove the converse, we note that if $T \in A'$ then so is T^*. If $a \in A$ is such that $a^* = a$ then the element

$$e^{ia} = \sum_{n=0}^{\infty} \frac{(ia)^n}{n!}$$

defines a unitary operator on H. Since

$$\frac{d}{dt}_{|t=0} e^{ita} T e^{-ita} = i(aT - Ta).$$

We see that $T \in A'$ if and only if $e^{ia} T e^{-ia} = T$ for all $a \in A$ such that $a = a^*$. We can now argue in exactly the same way as we did in the proof of the previous proposition to prove that $A' = \mathbb{C}I$. ■

We now come to the *Von Neumann density theorem*.

Theorem 48 *Let* $A \subset L(H,H)$ *be a* **subalgebra containing* I. *Let* $T \in (A')'$ *and let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* H *such that* $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$. *Then given* $\varepsilon > 0$ *there exists* $a \in A$ *such that* $\sum_{n=1}^{\infty} \|(T-a)x_n\|^2 < \varepsilon$.

Proof. Let V be a Hilbert space and let B be a *subalgebra of $L(V,V)$ containing I. Then

(1) If $v \in V$ then $(B')'v \subset \overline{Bv}$.

Indeed, $\overline{Bv}^{\perp}$ is B invariant since B is invariant under *. This implies that if P is the orthogonal projection of V onto $\overline{Bv}$ then $P \in B'$. Thus if $T \in (B')'$ then $TP = PT$. Hence $T(\overline{Bv}) \subset \overline{Bv}$. We therefore see that $(B')'(\overline{Bv}) \subset \overline{Bv}$. This proves the result since $v \in \overline{Bv}$.

We will apply this result to the Hilbert space V that consists of all sequences $\{x_n\}$ with $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$ and inner product $\langle \{x_n\}, \{y_n\} \rangle = \sum_n \langle x_n, y_n \rangle$. Let B be the algebra of operators $\sigma(a)$, $a \in A$ given by

$\sigma(a)\{x_n\} = \{ax_n\}$. Let $P_m\{x_n\} = x_m$ and let $Q_m x = \{\delta_{n,m}x\}_{n=1}^{\infty}$. Then $P_m^* = Q_m$.

Suppose that $T \in B'$ and $a \in A$. Then

$$P_m T\sigma(a)\{x_n\} = P_m\sigma(a)T\{x_n\} = aP_m T\{x_n\}.$$

Also

$$TQ_m ax = T\sigma(a)Q_m x = \sigma(a)TQ_m x.$$

This implies

(2) If $T \in B'$ then $P_n T Q_m \in A'$ for all $n, m \geq 1$.

This implies that if $S \in (A')'$ then the operator $\{x_n\} \longmapsto \{Sx_n\}$ is in $(B')'$. Hence if $T \in (A')'$ then

$$\{Tx_n\} \in \overline{B\{x_n\}}.$$

This implies that given $\varepsilon > 0$ there exists $a \in A$ such that $\sum_{n=1}^{\infty} \|(T-a)x_n\|^2 < \varepsilon$. ∎

The following result is also referred to as the Von Neumann density theorem in the literature.

Corollary 49 *Let A be a *subalgebra of $L(H,H)$ then if I is in the closure of A with respect to the strong operator topology then the algebra $(A')'$ is the closure of A in the strong operator topology.*

Proof. Let $C = A + \mathbb{C}I$. Then the above result implies that $(C')'$ is contained in the closure of C in the strong topology. Now the closure of C is the same as the closure of A by our hypothesis. Also it is clear that $A' = C'$. Thus $(A')'$ is contain in the closure of A. Since the reverse inclusion is clear, the result follows. ∎

This result yields an analog of Burnside's theorem.

Corollary 50 *Let A be a *subalgebra of $L(H,H)$ containing the identity in its closure in the strong operator topology and acting irreducibly on H then the closure of A in the strong topology is $L(H,H)$.*

Proof. $A' = \mathbb{C}I$. ∎

At this point we can introduce an important class of algebras for abstract representation theory.

Definition 51 *A* **subalgebra of* $L(H,H)$ *is called a* Von Neumann algebra *of it is closed in the strong operator topology and contains the identity.*

The above results imply

Proposition 52 *A* **subalgebra,* A*, of* $L(H,H)$ *is a Von Neumann algebra if and only if* $(A')' = A$*.*

We will use this result to give a useful variant of Schur's lemma.

Proposition 53 *Let* (π, H) *be an irreducible unitary representation of* G*. Let* D *be a dense subspace of* H *such that* $\pi(g)D \subset D$ *for all* $g \in G$ *and let* T *be a linear map of* D *to* H *such that* $T\pi(g)v = \pi(g)Tv$ *for all* $g \in G, v \in D$*. Assume that there exists a dense subspace* D' *in* H *and a linear map* S *from* D' *to* H *such that*

$$\langle Tv, w \rangle = \langle v, Sw \rangle$$

for all $v \in D$*,* $w \in D'$*. Then* $T = \lambda I$ *for some* $\lambda \in \mathbb{C}$*.*

Remark 54 *Notice that there is no topology assumed on* D *or* D' *and* T, S *are general subject to the assumptions in the proposition.*

Proof. Assume that $v \in D$ and v and Tv are linearly independent. Then there exists $B \in L(H,H)$ with $Bv = v$, $BTv = v$. Let A be the subalgebra of $L(H,H)$ spanned by $\{\pi(g)|g \in G\}$. Then A satisfies the hypothesis of Theorem 43. Schur's lemma implies that $(A')' = L(H,H)$. Hence there exists a sequence $a_j \in A$ such that

$$\lim_{j\to\infty} a_j v = v, \lim_{j\to\infty} a_j Tv = v.$$

On the other hand $a_j D \subset D$ and $Ta_j = a_j T$. Thus if $w \in D'$ then

$$\langle v, w \rangle = \lim_{j\to\infty} \langle a_j Tv, w \rangle = \lim_{j\to\infty} \langle Ta_j v, w \rangle = \lim_{j\to\infty} \langle a_j v, Sw \rangle = \langle v, Sw \rangle = \langle Tv, w \rangle .$$

Since D' is dense this yields the absurd conclusion that of v, Tv are linearly independent then $Tv = v$. Thus v, Tv are linearly dependent for all $v \in D$. This implies that T is a scalar multiple of the identity. ■

Exercise 55 *Show that if* V *is a vector space and* T *is a linear operator on* V *(no topology) and if for every* $v \in V$*,* v *and* Tv *are linearly dependent then* T *is a multiple of the identity.*

2.3 Square integrable representations I.The case of $SL(2,\mathbb{R})$.

Let G be a locally compact separable topological group. Then a unitary representation of G, (π, H), is said to be square integrable if it is irreducible and if one of the functions (called matrix entries), $c_{v,w}(g) = \langle \pi(g)v, w\rangle$ with v, w is square integrable with respect to left invariant measure, that is

$$\int_G |\langle \pi(g)v, w\rangle|^2 dg < \infty.$$

We will see in the next section that (π, H) is square integrable if and only if it is irreducible and very $c_{v,w}, v, w \in H$ is square intgrable. If G is compact then every irreducible unitary representation is square integrable. Some groups have no square integrable representations. For example, if $G = GL(1,\mathbb{R}) = \mathbb{R}^\times$ and if (π, H) is an irreducible unitary representation of G then since G is commutative, Schur's lemma implies that $\dim H = \dot{1}$. Sunce, π is unitary this implies that if v is an orthonormal basis for H then $|c_{v,v}(g)| = 1$ for all $g \in G$. Simlarly, if $G = GL(n,\mathbb{R})$ then the center $\mathbb{R}^\times I$ must act by scalars absolute value 1 on any irreducible unitary representation. Thus there are no square integrable representations in this case. It is more subtle, but still true that $SL(n,\mathbb{R})$ for $n \geq 3$ has no square integrable represenations. Never-the-less the square integrable rperesentations are the basic building blocks for representation theory of reductive groups. We will now describe (all) square integrable representations of $SL(2,\mathbb{R})$ in disguise as

$$SU(1,1) = \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} | a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \right\}.$$

Let $D = \{z \in \mathbb{C} | |z| < 1\}$. We use ordinary Lebesgue measure on D thinking of $z = x + iy$ as $(x, y) \in \mathbb{R}^2$. We write $dz = dx + idy$ and $d\bar{z} = dx - idy$ then $dx \bigwedge dy = \frac{1}{2i} d\bar{z} \bigwedge dz$. We let $G = SU(1,1)$.We define an action of G on D by

$$g \cdot z = \frac{az + b}{\bar{b}z + \bar{a}}$$

where

$$g = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}.$$

We will use the following formulas.

$$1 - \left| \frac{az + b}{\bar{b}z + \bar{a}} \right|^2 = \frac{1 - |z|^2}{|\bar{b}z + \bar{a}|^2}. \tag{2}$$

$$d(g\cdot z)=\frac{dz}{(\bar{b}z+\bar{a})^2}. \tag{3}$$

Using (2),(3) we have

$$\int_D \phi(g\cdot z)\frac{d\bar{z}dz}{(1-|z|^2)^2}=\int_D \phi(g\cdot z)\frac{d(\overline{g\cdot z})d(g\cdot z)}{(1-|g\cdot z|^2)^2}=\int_D \phi(z)\frac{d\bar{z}dz}{(1-|z|^2)^2}.$$

Thus

$$\mu(\phi)=\frac{1}{2i}\int_D \phi(z)\frac{d\bar{z}dz}{(1-|z|^2)^2} \tag{4}$$

defines a G-invariant measure on D.

Let H^k be the space of all holomorphic functions $f:D\to\mathbb{C}$ such that

$$\frac{1}{2i}\int_D |f(z)|^2(1-|z|^2)^k\frac{d\bar{z}dz}{(1-|z|^2)^2}<\infty.$$

If $f_1,f_2\in H^k$ then we set

$$\langle f_1,f_2\rangle_k=\frac{1}{2i}\int_D f_1(z)\overline{f_2(z)}(1-|z|^2)^k\frac{d\bar{z}dz}{(1-|z|^2)^2}. \tag{5}$$

We define for $f\in H^k$, and g as in (1)

$$\pi_k(g)f(z)=(-\bar{b}z+a)^{-k}f(g^{-1}\cdot z). \tag{6}$$

Then using formulas (2),(3) as we did in the proof of the invariance of μ. We find that

$$\langle \pi_k(g)f_1,\pi_k(g)f_2\rangle=\langle f_1,f_2\rangle\,, f_1,f_2\in H^k, g\in G.$$

Proposition 56 *If $k\geq 2$ then H^k is a Hilbert space and if $k\in\mathbb{Z}$, $k\geq 2,(\pi_k,H^k)$ is a square integrable representation of G.*

Proof. We first show that the space H^k is complete. For this we observe that if $z_o\in D$ and if $r=\frac{1-|z_o|}{2}$ then the set $\overline{D}_r=\{z\in\mathbb{C}||z-z_o|\leq r\}\subset D$. Then if $k\geq 2$ we have

$$(1-|z|^2)^{k-2}\geq(1-\frac{1}{4}(1+|z_o|)^2)^{k-2}\text{for all } z\in\overline{D}_r.$$

Thus we see that

$$\langle f, f\rangle_k \geq \frac{(1-\frac{1}{4}(1+|z_o|)^2)^{k-2}}{2i}\int_{\overline{D}_r}|f(z)|^2 d\bar{z}dz.$$

On $\overline{D}_r$ the holomorphic function f is given as a series

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_o)^n.$$

Then

$$\frac{1}{2i}\int_{\overline{D}_r}|f(z)|^2 d\bar{z}dz = \int_0^r\int_0^{2\pi}\sum_{n,m\geq 0} a_n\overline{a_m}s^n s^m e^{i(n-m)\theta}d\theta s ds =$$

$$2\pi\sum_{n\geq 0}|a_n|^2\int_0^r s^{2n+1}dr = 2\pi\sum_{n\geq 0}|a_n|^2\frac{r^{2n+2}}{2n+2}\geq 2\pi|a_0|^2r^2.$$

We therefore see that

$$\langle f, f\rangle_k \geq (1-\frac{1}{4}(1+|z_o|)^2)^{k-2}2\pi|f(z_o)|^2.$$

This implies the completeness, since if $\{f_j\}$ is Cauchy in H_k then it is Cauchy relative to the topology of uniform convergence on compacta. This implies that there is a continuous function on D, f, such that $\lim_{j\to\infty} f_j(z)=f(z)$ uniformly on compacta of D. But then f is holomorphic on D and it is easy to check that it is in H^k.

Notice that the function $f(z)\equiv 1$ is in H^k if $k\geq 2$. We calculate the matrix coefficient (g as in (1))

$$\begin{aligned}\langle \pi_k(g)1,1\rangle &= \frac{1}{2i}\int_D(-\bar{b}z+a)^{-k}(1-|z|^2)^{k-2}d\bar{z}dz\\ &=\int_0^1\int_0^{2\pi}(-\bar{b}re^{i\theta}+a)^{-k}(1-r^2)^{k-2}d\theta r dr\\ &= a^{-k}\int_0^1 r(1-r^2)^{k-2}\int_0^{2\pi}(-\frac{\bar{b}}{a}re^{i\theta}+1)^{-k}d\theta dr.\end{aligned}$$

We observe that since $|a|^2-|b|^2=1$, $|\frac{\bar{b}}{a}|\leq 1$. Thus if $0\leq r<1$ then the function

$$\phi(z)=(-\frac{\bar{b}}{a}rz+1)^{-k}$$

is holomorphic in z for $|z| < \frac{1}{r}$. This implies that

$$\int_0^{2\pi} \phi(e^{i\theta}) d\theta = 2\pi\phi(1) = 2\pi.$$

We therefore see that

$$\langle \pi_k(g)1, 1\rangle_k = a^{-k} 2\pi \int_0^1 r(1-r^2)^{k-2} dr = c_k a^{-k}.$$

Notice that this is a continuous function of g. Let $f(g) = a^{-k}$ we will show that

$$\int_G |f(g)|^2 dg < \infty.$$

For this we need a formula for the Haar integral analogous to the formula for polar coordinates. Set

$$K = \{k(\theta) | k(\theta) = \left[\begin{array}{cc} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{array}\right], \theta \in \mathbb{R}\}$$

and

$$A^+ = \{a_t | a_t = \left[\begin{array}{cc} \cosh t & \sinh t \\ \sinh t & \cosh t \end{array}\right], t \in \mathbb{R}, t \geq 0\}.$$

Then $G = KA^+K$ (exercise we will see it in general later). Furthermore, if ϕ is summable on G then up to constants of normalization

$$\int_G \phi(g) dg = \int_0^{2\pi} \int_0^{\infty} \int_0^{2\pi} \phi(k(\theta_1) a_t k(\theta_2)) \sinh(2t) d\theta_1 dt d\theta_2.$$

This can be proved by observing that $K = \{g \in G | g \cdot 0 = 0\}$. Thus if $f \in C_c(G)$ then

$$\bar{f}(gK) = \int_0^{2\pi} f(gk(\theta)) d\theta$$

Defines a function on $D = G \cdot 0$. If we write out the invariant measure given in formula (4) above in polar coordinates and consider the change of variables $r \longmapsto \tanh t$,$t > 0$ the formula follows. Now

$$f(k(\theta_1) a_t k(\theta_2)) = (e^{i\theta_1} \cosh t e^{i\theta_2})^{-k}.$$

Thus

$$\int_G |f(g)|^2 dg = (2\pi)^2 \int_0^\infty (\cosh t)^{-2k} \sinh(2t) dt$$
$$= 4\pi^2 \int_0^\infty (\cosh t)^{-2k} \cosh(t) \sinh(t) dt = \frac{2\pi^2}{k-1}.$$

This shows that $c_{1,1}$ is square integrable. The exercise below proves that π_k is a representation. We will prove the irreducibility later. ■

For later reference we note that $c_k = \langle 1, 1 \rangle_k$ thus we have (up to normalization of measures)

$$\int_G |\langle \pi(g)1, 1 \rangle_k|^2 dg = \frac{2\pi^2}{k-1} \langle 1, 1 \rangle_k^2 . \tag{7}$$

Exercise 57 *Calculate $\langle \pi_k(g) z^l, z^m \rangle$ for $l, m = 0, 1, 2, \ldots$ as above and show that it is a continuous function of g. Show that the span of the functions $1, z, z^2, \ldots$ is dense in H^k for $k \geq 2$. Now use an appropriate extension of Lemma 73 (allowing the element w in 2. to be taken from a dense subspace) to show that (π_k, H^k) is a representation for $k \geq 2$.*

Exercise 58 *Give the details of the proof of the integration formula for Haar measure on G as sketched in the above proof.*

The series of representations (π_k, H^k) $k > 1$ is called the holomorphic discrete series of representations of G. If we take the space of anti-holomorphic functions instead and denote the space by H^{-k} then one has representations (π_k, H^k) for $|k| > 1$. This series of representations is called the discrete series of G. There are also two representations corresponding to $|k| = 1$ that are not quite discrete series. As the name suggests there are other (continuous) series of representations of G. In fact the continuous series are just the principal series of Example 41 and the complementary series which we will describe (and the π_k for $k = \pm 1$) after an interlude on general square integrable representations and representations of compact groups.

2.4 Square integrable representations II

In this setion we will collect generalities on square integrable representations. In the next we will apply this theory to representations of compact groups.

We will consider $L^2(G)$ as a unitary representation under the *right* regular action. Here we write $R_g f(x) = f(xg)$.

Proposition 59 *Let (π, H) be a square integrable representation of G. Then every matrix entry ($c_{v,w}$, $v, w \in H$) is square integrable. Furthermore, there exists an element $T \in L_G(H, L^2(G))$ with closed range consisting of continuous functions that is a unitary bijection onto its range. The map T can be implemented as follows: fix v_o in H a unit vector then $T(w) = c_{w,v_o}$.*

Proof. Fix v_o a unit vector in H such that c_{v_o,v_o} is in $L^2(G)$. Let D denote the space of all $v \in H$ such that $c_{v,v_o} \in L^2(G)$. We note that

$$c_{\pi(g)v,w} = R_g c_{v,w}.$$

Thus D is an invariant non-zero subspace. Since $v_o \in D$ the irreducibility implies that D is a dense subspace. On D we put the pre-Hilbert space structure

$$(v, w) = \langle v, w \rangle + \langle c_{v,v_o}, c_{w,v_o} \rangle .$$

The last inner product is the L^2-inner product the first one on the right hand side is the inner product on H.

We now come to the key point.

(*) D is complete with respect to $(..., ...)$.

Indeed, if $\{v_j\}$ is a Cauchy sequence in D then it is Cauchy in H and $\{c_{v_j,v_o}\}$ is Cauchy in $L^2(G)$. Since H is complete there exists $v \in H$ such that $\lim_{j\to\infty} v_j = v$. Since $L^2(G)$ is complete by definition there exists $f \in L^2(G)$ such that $\lim_{j\to\infty} c_{v_j,v_o} = f$ in L^2. We note that

$$|c_{v_j,v_o}(g) - c_{v,v_o}(g)| \leq \|v_j - v\| , g \in G.$$

Let U be an open subset of G such that $\bar{U}$ is compact and let $\phi \in C_c(G)$ be such that $\phi(x) \geq 0$ for all $x \in G$ and $\phi(x) = 1$ if $x \in \bar{U}$. We note that the operator of multiplication by ϕ on $C_c(G)$ extends to a bounded operator $T_\phi : L^2(G) \to L^2(G)$. Now we have $\lim_{j\to\infty} \phi c_{v_j,v_o} = \phi c_{v,v_o}$ in $L^2(G)$ by the above uniform convergence. We also have

$$\lim_{j\to\infty} \phi c_{v_j,v_o} = T_\phi f$$

in $L^2(G)$. Hence we have $\phi c_{v,v_o} = T_\phi f$. This implies that f is represented by the continuous function c_{v,v_o}. But then $v \in D$.

We note that if $g \in G$ and $v, w \in D$ then

$$(\pi(g)v, \pi(g)w) = (v, w).$$

Thus the operators $\pi(g)_{|D}$ define unitary operators $\rho(g)$ on D with respect to $(...,...)$. Let $S(v) = v$ for $v \in D$ but looked upon as a map of the Hilbert space D into H. Then

$$\langle S(v), S(v)\rangle \leq (v, v)$$

for all $v \in D$. This implies that S extends to a bounded operator from the Hilbert space completion, D, into H. Furthermore, $S \circ \rho(g) = \pi(g) \circ S$. Let $S^* : H \to D$ denote the adjoint operator. Then $S^* \circ \pi(g) = \rho(g) \circ S^*$ for all $g \in G$. We therefore see that $SS^* \in L_G(H, H)$. Schur's lemma implies that $SS^* = \lambda I$ and it is clear that λ is real and $\lambda > 0$. Now if $v \in H$ then $S^*(v) \in D$ so $\lambda v = S(S^* v) = S^* v$. But then $v \in D$. Hence $D = H$. We also note that this implies that

$$(v, v) \leq \frac{1}{\lambda^2} \langle v, v\rangle$$

for all $v \in H$. Thus $\|c_{v,v_o}\|_2^2 \leq \frac{1-\lambda^2}{\lambda^2} \|v\|^2$. Define $T(v) = c_{v,v_o}$. To complete the proof we note that all we used about v_o in the proof above was that the set $\{w \in H | c_{v,v_o} \in L^2(G)\}$ is non-zero. By the above this is true for every $v \in H$ since v_o is in the corresponding set. ■

The next theorem is a general form of the Schur orthogonality relations.

Theorem 60 *Let (π, H) and (ρ, V) be square integrable representations of G. If π and ρ are inequivalent then their matrix coefficients are orthogonal. There exists a positive real number $d(\pi)$ (which depends only on π and the normalization of Haar measure) such that if $v_1, v_2, w_1, w_2 \in H$ then*

$$\int_G \langle \pi(g)v_1, w_1\rangle \overline{\langle \pi(g)v_2, w_2\rangle} dg = \frac{1}{d(\pi)} \langle v_1, v_2\rangle \langle w_2, w_1\rangle .$$

Proof. Assume that $h_o \in H$ and $v_o \in V$ are unit vectors and that there exists $h \in H$, $v \in V$ such that

$$\int_G \langle \pi(g)h, h_o\rangle \overline{\langle \rho(g)v, v_o\rangle} dg \neq 0.$$

Let $T : H \to L^2(G)$ and $S : V \to L^2(G)$ be as in the proof of the preceding proposition. That is $T(x) = c_{x,h_o}$ and $S(y) = c_{y,v_o}$. Then we showed that

T and S respectively define injective intertwining operators from H and V into $L^2(H)$ with closed range. Consider

$$(x, y) = \langle T(x), S(y) \rangle .$$

Then $(h, v) \neq 0$ and $(\pi(g)x, \rho(g)y) = (x, y)$. Finally the pairing is continuous in x, y. Thus the Riesz representation theorem, for H, implies that $(x, y) = \langle x, A(y) \rangle$ with $A : V \to H$ a bounded operator. It is easy to see that $A \in L_G(V, H)$. Since $A \neq 0$. We see that $\ker A = 0$. We also see that $\operatorname{Im} A$ is dense in H. We also observe that $A^*A \in L_G(V, V)$ and $AA^* \in L_G(H, H)$. Thus each is a scalar by Schur's lemma. We conclude that there is a scalar $s > 0$ such that sA is a unitary bijection, We therefore conclude that π and ρ are unitarily equivalent.

To prove the last part we see that

$$\int_G \langle \pi(g)v_1, w_1 \rangle \overline{\langle \pi(g)v_2, w_2 \rangle} dg = a(w_2, w_1) \langle v_1, v_2 \rangle$$

and

$$\int_G \langle \pi(g)v_1, w_1 \rangle \overline{\langle \pi(g)v_2, w_2 \rangle} dg = b(v_1, v_2) \langle w_2, w_1 \rangle$$

for v_1, v_2, w_1, w_2. This implies that $a(w_2, w_1)$ is a positive multiple of $\langle w_2, w_1 \rangle$. We call the multiple $\frac{1}{d(\pi)}$. ■

Definition 61 *We call the number $d(\pi)$ the* formal degree *of π.*

Example 62 *If (π_k, H^k) is as above for $SU(1, 1)$ then $d(\pi_k) = \frac{k-1}{2\pi^2}$.*

2.5 Representations of compact groups.

In this section we will show how the results of the preceding section apply to compact groups. In this section G will denote a compact group unless otherwise specified.

Clearly, an irreducible unitary representation of G is square integrable. We have

Theorem 63 *Let (π, H) be an irreducible Hilbert representation of G. Then $\dim H < \infty$. If (π, H) is unitary and we normalize the Haar measure, μ, on G such that $\mu(1) = 1$ then $d(\pi) = \dim H$ (recall $d(\pi)$ is the formal degree).*

Proof. We may assume that (π, H) is unitary. Then it is square integrable. There is therefore an injective intertwining operator $T : H \to L^2(G)$ with closed image contained in $C(G)$. We look upon $C(G)$ as a Banach space under the sup-norm, $p_G(f) = \max_{x\in G} |f(x)|$. Let V denote the closure of $T(H)$ in $C(G)$. Then if we normalize the Haar measure as in the statement of the theorem it is clear that

$$\|f\|_2 \leq p_G(f).$$

Thus the map $f \longmapsto f$ of $T(H)$ to $T(H)$ extends to a continuous linear map of V to $T(H)$ (since $T(H)$ is closed in $L^2(G)$). The closed graph theorem (Yoshida, *Functional Analysis*,p. 79, Theorem 1) implies that this map is continuous. Hence there exists $C < \infty$ such that if $f \in T(H)$ then

$$p_G(f) \leq C \|f\|_2 .$$

We will show that this implies that $T(H)$, hence H, is finite dimensional.

Let $f_1, ..., f_d$ be orthonormal in $T(H)$ then if $\lambda_i \in \mathbb{C}$ we have

$$|\sum_i \lambda_i f_i(x)| \leq p_G(\sum_i \lambda_i f_i) \leq C \left\|\sum_i \lambda_i f_i\right\|_2 = C\left(\sum_i |\lambda_i|^2\right)^{\frac{1}{2}}.$$

We apply this with $\lambda_i = \overline{f_i(x)}$. We conclude that

$$\sum_i |f_i(x)|^2 \leq C\left(\sum_i |f_i(x)|^2\right)^{\frac{1}{2}}.$$

Hence

$$\sum_i |f_i(x)|^2 \leq C^2.$$

Integrating both sides of the equation over G yields $d \leq C^2$.

We now calculate the formal degree. Let $v_1, ..., v_n$ be an orthonormal basis of H then the matrix $\left[c_{v_i,v_j}(g)\right]$ is unitary for all $g \in G$. Hence

$$\sum_{i,j} |c_{v_i,v_j}(g)|^2 = n$$

for all $g \in G$. If we integrate both sides of this equation and take into account the Schur orthogonality relations we have

$$\frac{1}{d(\pi)} n^2 = n.$$

■

If $\{H_n\}_{1 \leq n < N}$ with $N \leq \infty$ is a sequence of Hilbert spaces then we write $\widehat{\bigoplus}_{n<N} H_n$ for the space of all sequences $\{x_n\}_{n<N}$ such that $\sum_n \|x_n\|^2 < \infty$ we define

$$\langle \{x_n\}, \{y_n\} \rangle = \sum \langle x_n, y_n \rangle .$$

This endows $\widehat{\bigoplus}_{n<N} H_n$ with a Hilbert space structure. This construction defines the Hilbert space direct sum. Notice that it is a completion of the algebraic direct sum.

Definition 64 *Let B a locally compact topological group and for each $n, 1 \leq n < N$ we have a unitary representation (π_n, H_n) of B then the unitary direct sum of these representations is the representation $(\widehat{\bigoplus}_{n<N} \pi_n, \widehat{\bigoplus}_{n<N} H_n)$ of B with*

$$\left(\widehat{\bigoplus}_{n<N} \pi_n(g) \{x_n\} \right) = \{\pi_n(g) x_n\}.$$

Recall that if $T : H_1 \to H_2$ is a continuous linear map of Hilbert spaces then T is said to be completely continuous (or compact) if the image of a bounded set has compact closure. If H is a Hilbert space then we denote by $CC(H)$ the space of all completely continuous operators from H to H. The following result is completely standard the simplest proof of it that we know is in N.Wallach, *RRGI* p.326, 8,A.1.2.

Lemma 65 *Let $T \in CC(H)$ be such that $T = T^*$. Then there exists an orthonormal basis, $\{v_n\}$, of $\ker T^{\perp}$ and $\lambda_j \in \mathbb{R}$ such that $T v_n = \lambda_n v_n$ and the dimension of $T^{\perp}$ is infinite then $\lim_{n \to \infty} \lambda_n = 0$.*

Definition 66 *Let B be a locally compact, separable topological group then a unitary representation (π, H) of B is said to be of* class CC *if $\pi(f)$ is completely continuous for all $f \in C_c(B)$. We say that B is a* CCR group *if every irreducible unitary representation of B is of class CC.*

One of Harish-Chandra's basic theorems is that all real reductive groups are CCR groups.

The next result is a generalization of the Peter-Weyl theorem and is basic to the theory of automorphic forms (it applies to the so-called cuspidal spectrum). In the course of the proof of the result we will be using the fact that if B is unimodular than

$$\int_B f(b)db = \int_B f(b^{-1})db.$$

Theorem 67 *Let B be a locally compact, separable topological group and let (π, H) be a unitary representation of B of class CC. Then (π, H) is unitarily equivalent with a unitary direct sum of irreducible representations of B.*

Example 68 *Before we prove the theorem we will describe the general form of the main application. Assume that B is unimodular. Let $X = B/C$ with C a closed unimodular subgroup of B and assume that X is compact and that there exists a B-invariant measure on X, λ (one can show that this is not really an assumption under our hypotheses on B and C). We will write the measure as*

$$\lambda(f) = \int_X f(x)dx$$

as usual. Let $\phi \in C_c(B)$ then we can choose Haar measure on B and C such that if we set $\bar{\phi}(bC) = \int_C \phi(bc)dc$ (the integration with respect to Haar measure on C) then

$$\int_B \phi(b)db = \int_X \bar{\phi}(x)dx.$$

Let $H = L^2(X)$ and $\pi(b) = L_b$. We calculate

$$\pi(\phi)f(x) = \int_B \phi(b)f(b^{-1}x)db = \int_B \phi(b)f(b^{-1}gC)db =$$

$$\int_B \phi(b)f((g^{-1}b)^{-1}C)db = \int_B \phi(gb)f(b^{-1}C)db =$$

$$\int_B \phi(gb^{-1})f(bC)db = \int_X \int_C \phi(gcb^{-1})f(bC)dcd(bC).$$

Let

$$k_\phi(gC, bC) = \int_C \phi(gcb^{-1})dc.$$

The function $k_\phi \in C(X \times X)$ *and*

$$\pi(\phi)f(x) = \int_X k_\phi(x,y)f(y)dy.$$

on $L^2(X)$*. The lemma below implies that* $(\pi, L^2(X))$ *is of class CC.*

Lemma 69 *Let* Y *be a locally compact, separable, topological space and let* λ *be a regular measure on* Y*. Let* $k \in L^2(Y \times Y)$ *(with respect to the product measure). If we define* $T : L^2(Y) \to L^2(Y)$ *by* $T(f)(x) = \lambda(k(x,\cdot)f)$*.then* T *defines a compact operator.*

Proof. Let $\{\phi_n\}$ be an orthonormal basis of $L^2(Y)$ consisting of continuous functions ($L^2(Y)$ is separable since Y is separable). Define $u_{n,m}(x,y) = \phi_n(x)\overline{\phi_m(y)}$. Then $\{u_{n,m}\}$ is an orthonormal basis of $L^2(Y \times Y)$. Now

$$k = \sum_{n,m} a_{n.m} u_{n,m}$$

in $L^2(Y \times Y)$. Set $k_N = \sum_{n,m \le N} a_{n.m} u_{n,m}$. Then the operator

$$T_N(f)(x) = \int_Y k_N(x,y)f(y)dy$$

is of finite rank hence compact. Also

$$(T - T_N)f = \sum_{m,n>N} a_{n,m}\phi_n \langle f, \phi_m \rangle .$$

An application of the Schwarz inequality yields

$$\|T - T_N\|^2 \le \sum_{m,n>N} |a_{n,m}|^2.$$

Thus T is in the norm closure of the finite rank operators. Hence it is compact. ■

We will now prove the theorem. Let $\mathcal{S}$ denote the set of all closed invariant subspaces, V, of H such that V is a Hilbert space direct sum of irreducible subrepresentations ordered by inclusion. If $\{V_\alpha\}$ is a linearly ordered subset of $\mathcal{S}$ then the closure of $\bigcup_\alpha V_\alpha$ is in $\mathcal{S}$ (exercise). Hence Zorn's lemma implies that there is a maximal element V in $\mathcal{S}$. We will now prove that $V = H$ and thereby prove the Theorem. Let $W = V^\perp$. If $u \in C_c(B)$ then $\pi(u)W = W$. Let $w \in W$ be a unit vector. Let $\{u_n\}$ be a delta sequence such that $u_n(x^{-1}) = u_n(x)$ for all $x \in B$. Then $\pi(u_n)^* = \pi(u_n)$ (exercise)for all n. Now $\lim_{n\to\infty} \pi(u_n)w = w$. Hence there exists n such that $\pi(u_n)w \neq 0$. Fix $T = \pi(u_n)_{|W}$. Then T is a compact, non-zero self-adjoint operator on W. Lemma 65 implies that T has a nonzero eigenvalue on W. Let Z denote the corresponding eigenspace. Then Lemma 65 also implies that $\dim Z < \infty$. Let $m > 0$ denote the positive minimal dimension of an intersection of a closed B-invariant subspace with Z. Fix M an intersection of this type with $\dim M = m$. Let U denote the intersection of all closed invariant spaces Y such than $Y \cap Z = M$. Then U is closed and invariant. If N is a closed invariant subspace of U then both N and $N^\perp$ are T invariant. Thus $M = M \cap N \bigoplus M \cap N^\perp$. But then $M \cap N = M$ or $M \cap N^\perp = M$. If $M \cap N = N$ (resp.$M \cap N^\perp = M$) then $N = U$ (resp. $N^\perp = U$) by definition of U. Thus U is a closed, invariant, irreducible subspace of W and thus $V \bigoplus U$ is in $\mathcal{S}$. This contradicts the definition of V. Hence $W = 0$ and the result is proved.

Let $\widehat{G}$ denote the set of equivalence classes of irreducible finite dimensional representations of G. For each $\gamma \in \widehat{G}$ we fix $(\tau_\gamma, V_\gamma) \in \gamma$ which we assume is unitary. If (π, V) is a representation of G then we set $V(\gamma)$ equal to the sum of the closed, G-invariant, irreducible subspaces in the class of γ.

Definition 70 *The space $V(\gamma)$ is called the* γ-isotypic component *of V.*

We will now concentrate on $L^2(G)$ we first note that since G is compact the discussion in Example 68 implies that the right (or the left) regular representation is of class CC. If $\gamma \in \widehat{G}$ then we define a map $A_\gamma : V_\gamma^* \bigotimes V_\gamma \to L^2(G)$ by

$$A_\gamma(\lambda \bigotimes v)(g) = \lambda(\pi(g)v).$$

Set $d(\gamma) = \dim V_\gamma$. If $\lambda \in V_\gamma^*$ then there exists $v_\lambda \in V_\gamma$ such that $\lambda(v) = \langle v, v_\lambda \rangle$ for all v. We define $\langle \lambda_1, \lambda_2 \rangle = \langle v_{\lambda_2}, v_{\lambda_1} \rangle$. Then the Schur orthogonality relations imply that $\sqrt{d(\gamma)}A_\gamma$ is a unitary operator from $V_\gamma^*\bigotimes V_\gamma$ onto its image. We also observe that $A_\gamma(\lambda \circ \tau_\gamma(g)^{-1}\bigotimes v) = L_g A_\gamma(\lambda\bigotimes v)$.

The next result is the Peter-Weyl theorem.

Theorem 71 *The γ-isotypic component of $L^2(G)$ is the image of A_γ. Furthermore, $L^2(G)$ is the Hilbert space direct sum of the spaces $L^2(G)(\gamma)$.*

Proof. Let V be a closed, invariant, irreducible subspace of $L^2(G)$. Then in particular it is an irreducible unitary representation hence Theorem 63 implies that $\dim V < \infty$. If $u \in C(G)$ then $\pi(u)V \subset V$. We have seen that the span of the elements $\pi(u)v$ with $u \in C(G)$, $v \in V$ is dense in V. Hence it is equal to V. We leave it to the reader to check that this implies that $V \subset C(G)$. Define $\lambda(f) = f(1)$ for $f \in V$. Then $\lambda \in V^*$ and $\lambda(\pi(g)f) = f(g)$. Let $T : V_\gamma \to V$ be a bijective intertwining operator. Let $\xi = \lambda \circ T^{-1}$. Then if $T(v) = f$, $f = A_\gamma(\xi\bigotimes v)$. The last assertion now follows from Theorem 67. ∎

Definition 72 *If (τ, V) is a finite dimensional representation of G then its* character *is defined to be the function $\chi_V(g) = tr(\tau(g))$.*

We note that $\chi_V \in C(G)$ and that $\chi_V(xgx^{-1}) = \chi(g)$ for all $x, g \in G$. We also observe that if (τ_1, V_1) and (τ_2, V_2) are equivalent then $\chi_{V_1} = \chi_{V_2}$. We will now show that the converse is also true. We first observe that this implies that if $V_1, V_2 \in \gamma \in \widehat{G}$ then $\chi_{V_1} = \chi_{V_2}$. This common value will be denoted χ_γ. We also set $\alpha_\gamma = d(\gamma)\overline{\chi}_\gamma$(complex conjugate). We note that the Schur orthogonality relations imply that

$$\alpha_\gamma * \alpha_\tau = \delta_{\gamma,\tau}\alpha_\gamma$$

for $\gamma, \tau \in \widehat{G}$. Also, since $\alpha_\gamma(xgx^{-1}) = \alpha_\gamma(g)$ for $x, g \in G$ we have

$$\pi(\alpha_\gamma)\pi(g) = \pi(g)\pi(\alpha_\gamma)$$

for all $\gamma \in \widehat{G}$.

Lemma 73 *The orthogonal projection of $L^2(G)$ onto $L^2(G)(\gamma)$ is the operator $P_\gamma = \pi(\alpha_\gamma)$.*

Proof. Let $v_1, ..., v_d$ be an orthonormal basis of V_γ. Then $\alpha_\gamma = d(\gamma) \sum \overline{c_{v_i,v_i}}$. We thus have

$$\pi(\alpha_\gamma)f(x) = d(\gamma) \sum \int_G \overline{c_{v_i,v_i}}(g)f(xg)dg = d(\gamma) \int_G \overline{c_{v_i,v_i}}(g)L_{x^{-1}}f(g)dg.$$

If $\mu \in \widehat{G}$, $\mu \neq \gamma$ and f is in the image of A_μ then the above integral is 0 by the Schur orthogonality relations and the observation preceding Theorem 71. If $\mu = \gamma$ then assuming that $f(g) = \langle \tau_\gamma(g)v, w \rangle$ for some v, w in V_γ we have $f(xg) = \langle \tau_\gamma(g)v, \tau_\gamma(x)^{-1}w \rangle$. The Schur orthogonality relations yield

$$d(\gamma) \int_G \overline{c_{v_i,v_i}}(g)L_{x^{-1}}f(g)dg = d(\gamma) \sum \int_G \overline{\langle \tau_\gamma(g)v_i, v_i \rangle} \left\langle \tau_\gamma(g)v, \tau_\gamma(x)^{-1}w \right\rangle dg =$$

$$\sum \langle v, v_i \rangle \left\langle v_i, \tau_\gamma(x)^{-1}w \right\rangle = \left\langle v, \tau_\gamma(x)^{-1}w \right\rangle = \langle \tau_\gamma(x)v, w \rangle = f(x).$$

■

Let (π, H) be a Hilbert representation of G. By Lemma 37 we may assume that the representation is unitary. If $\gamma \in \widehat{G}$ then we set $E_\gamma = \pi(\alpha_\gamma)$. Then if $v, w \in H$ we have

$$\langle E_\gamma \pi(g)v, w \rangle = (P_\gamma c_{v,w})(g).$$

Thus if $E_\gamma v = 0$ for all $\gamma \in \widehat{G}$ then $c_{v,w} = 0$ for all $v \in H$. Hence $v = 0$. If $v \in H(\gamma)$ then $c_{v,w} \in L^2(G)(\gamma)$. Thus we see that E_γ is the orthogonal projection of H onto $H(\gamma)$.

We conclude

Proposition 74 *Let (π, H) be a Hilbert representation of G then the algebraic sum of the spaces $H(\gamma)$, $\gamma \in \widehat{G}$ is dense in H. Furthermore, if (π, H) is unitary then H is the Hilbert space direct sum of the spaces $H(\gamma), \gamma \in \widehat{G}$.*

Suppose that M is a closed subgroup of G and (σ, H_σ) is a unitary representation of M. We set H_0^σ equal to the space of all continous maps

$$f : G \to H_\sigma$$

such that

$$f(mg) = \sigma(m)f(g), m \in M, g \in G.$$

We define an inner product on H_0^σ by

$$\langle f_1, f_2 \rangle = \int_G (f_1(g), f_2(g))dg$$

where $(...,...)$ is the Hilbert space inner product on H_σ. We define

$$\pi_\sigma(g)f(x) = f(xg), x, g \in G.$$

Then it is a direct consequence of the unimodularity of G that

$$\langle \pi_\sigma(g)f_1, \pi_\sigma(g)f_2 \rangle = \langle f_1, f_2 \rangle$$

for all $g \in G$ and $f_1, f_2 \in H_0^\sigma$. Thus the operators $\pi_\sigma(g), g \in G$ extend to unitary operators on the Hilbert space completion of H_0^σ. We will denote this Hilbert space by $Ind_M^G(\sigma)$ (or $Ind_M^G(H_\sigma)$ if module notation is more convenient). We leave it to the reader to check that $(\pi_\sigma, Ind_M^G(\sigma))$ defines a representation of G.

As a representation of M, H_σ breaks up into a direct sum of isotypic components

$$H_\sigma = \bigoplus_{\mu \in \widehat{M}} H_\sigma(\mu).$$

It is not hard to see that as a representation of G

$$Ind_M^G(\sigma) \cong \bigoplus_{\mu \in \widehat{M}} Ind_M^G(H_\sigma(\mu)).$$

As a representation of M, $H_\sigma(\mu) \cong W_\mu \bigotimes V_\mu$ with (σ_μ, V_μ) a (unitary) representative of μ and $W_\mu = Hom_M(V_\mu, H_\sigma(\mu))$ with an appropriate Hilbert space structure and M acts trivially on the first factor and via σ_μ on the second. Indeed we have

Lemma 75 *Let B be a compact group and let (ν, V) be a unitary representation of B such that there exists $\mu \in \widehat{M}$ such that $V = V(\mu)$. Fix a unitary representative of μ, (ξ, W). Then the map*

$$\Psi(T \bigotimes w) = T(w)$$

of $Hom_B(W, V) \bigotimes W$ to V is a unitary bijection if we define $(T, S) = tr(S^*T)/\dim W$.

Proof. The map is surjective by the definition of isotypic component. If $T, S \in Hom_B(W, V)$. Then $T^*S \in Hom_B(W, W) = \mathbb{C}I$. Thus $T^*S = \frac{1}{\dim V}\mathrm{tr}(T^*S)I$. If $\sum_{i=1}^{d} T_i \bigotimes v_i \longmapsto 0$ and $v_1, ..., v_d$ are linearly independent then if $S \in Hom_B(W, V)$ we have $0 = \sum S^*T_i(v_i) = \frac{1}{\dim W}\sum(T_i, S)v_i$. Hence $(T_i, S) = 0$ for all S and i. Hence $T_i = 0$ for all i. ■

We now put it all together

$$Ind_M^G(\sigma) \cong \bigoplus_{\mu \in \widehat{M}} Hom(W_\mu, H_\sigma) \bigotimes Ind_M^G(\sigma_\mu)$$

(note that we can replace H_σ by $H_\sigma(\mu)$ in the first factor). We will now analyse the second factor. Here we have the decomposition

$$Ind_M^G(\sigma_\mu) = \bigoplus_{\gamma \in \widehat{G}} Ind_M^G(\sigma_\mu)(\gamma)$$

into isotypic components. Applying the lemma above we have a unitary isomorphism

$$Hom_G(V_\gamma, Ind_M^G(\sigma_\mu)) \bigotimes V_\gamma \to Ind_M^G(\sigma_\mu)(\gamma)$$

given by $T \bigotimes v \longmapsto T(v)$ here (τ_γ, V_γ) is a fixed unitary representative of γ. As a ast step in this general abstract nonsense we have Frobenius reciprocity

Lemma 76 *The map $F : Hom_G(V_\gamma, Ind_M^G(\sigma_\mu)) \to Hom_M(V_\gamma, W_\mu)$ given by $F(T)(v) = T(v)(1)$ is a positive scalar multiple of a unitary bijection.*

Proof. If $S \in Hom_M(V_\gamma, W_\mu)$ then define $(L(S)(v))(k) = S(\tau_\gamma(k)v)$. Then $L(S)(v) \in Ind_M^G(\sigma_\mu)$ and $F(L(S)) = S, L(F(T)) = T$. We will leave it to the reader to unwind the scalars. ■

Definition 77 *If B is a compact group and if (π, H) is a unitary represen- ation of B then we say that it is admissible if $\dim Hom_B(W, H) < \infty$ for all finite dimensional representations W of B. Equivalently, $\dim H(\mu) < \infty$ for all $\mu \in \widehat{B}$.*

Proposition 78 *A necessary and sufficient condition that $Ind_M^G(\sigma)$ be admissible for G is that σ be admissible for M.*

2.6 C^∞-vectors and $(\mathfrak{g}, K)$-modules.

In this section G will denote a Lie group with a finite number of connected components.

Definition 79 *Let (π, V) be a representation of G on a locally convex topological space. Then a vector $v \in V$ is said to be a C^∞-vector if the map $g \longmapsto \pi(g)v$ is a C^∞ map of G into V.*

The following observation is due to Gårding.

Lemma 80 *Let (π, H) be a Hilbert representation of G. If $f \in C_c^\infty(G)(= C^\infty(G) \cap C_c(G))$ and if $v \in H$ then $\pi(f)v$ is a C^∞ vector.*

Proof. Let U be a relatively compact subset of G containing 1. Let $L^1(U)$ denote the subspace of all L^1-functions on G that are limits of elements of $C_c(G)$ with support in U. Let V be an open subset of U such that it is invariant under inverse and such that $VV \subset U$. Then we have
1. If $f \in C_c^\infty(V)$ then the map of V to $L^1(U)$ given by $x \longmapsto F(x) = L(x)f$ is of class C^∞. Indeed, if $X \in Lie(G)$ then we set $L(X)f(g) = \frac{d}{dt}_{|t=0} f(\exp(-tX)g)$ for $g \in G$. Taylor's theorem with remainder implies that there exists $\epsilon > 0$ and a function, E, of t, g for $|t| \leq \epsilon$ such that $|E(t,g)| \leq \phi(g)$ with $\phi \in C_c(G)$ for $|t| \leq \varepsilon$ and

$$f(\exp(-tX)g) = f(g) + tL(X)f(g) + t^2E(t,g)$$

for $|t| \leq \epsilon$ and $g \in V$. This implies that

$$\|L(x)L(X)f - (1/t)(L(x\exp(tX))f - L(x)f)\|_1 =$$

$$\|L(X)f - (1/t)(L(x\exp(tX))f - L(x)f)\|_1 \leq |t|C$$

for $|t| \leq \epsilon$ for $C > 0$ and appropriate constant. Hence the function F is of class C^1. This argument can be iterated to prove the result. We have seen (at least implicitly) that the map $f \longmapsto \pi(f)$ is continuous from $L^1(U)$ to H. Thus, since linear continuous maps are smooth, we see that if $v \in H$ then the map from V to H given by $x \longmapsto \pi(L(x)f)v$ is of class C^∞. Now $\pi(L(x)f)v = \pi(x)\pi(f)v$. The lemma now follows using a partition of unity argument. ■

We denote by V^∞ the space of all C^∞ vectors in V. Now arguing as in the proof of Lemma 33 we have

Theorem 81 *Let (π, H) be a Hilbert representation of G. Then the space of C^∞ vectors in H is dense.*

Proof. There exists a Delta sequence $\{u_j\}$ in G with each $u_j \in C_c^\infty(G)$. We have shown

$$\lim_{j\to\infty} \langle \pi(u_j)v, w\rangle = \langle v, w\rangle$$

for all $v, w \in H$. Now suppose that $w \in (H^\infty)^\perp$ then since $\pi(u_j)w \in H^\infty$ for all j we have

$$0 = \lim_{j\to\infty} \langle \pi(u_j)w, w\rangle = \langle w, w\rangle .$$

■

If (π, H) is a Hilbert representation of G and if $v \in H^\infty$ then we define for $X \in Lie(G)$

$$d\pi(X)v = \frac{d}{dt}\pi(\exp tX)v_{|t=0}.$$

We have

- $d\pi(X)H^\infty \subset H^\infty$ for all $X \in Lie(G)$,$\pi(g)H^\infty \subset H^\infty$ for $g \in G$.
- $d\pi(aX + bY) = ad\pi(X) + bd\pi(Y)$, $a, b \in \mathbb{R}$, $X, Y \in Lie(G)$.
- $d\pi([X, Y]) = d\pi(X)d\pi(Y) - d\pi(Y)d\pi(X)$, for all $X, Y \in Lie(G)$.
- If $g \in G$, $X \in Lie(G)$ then $\pi(g)\pi(X)v = \pi(Ad(g)X)\pi(g)v$.

We will simplify notation and write $\pi(X)$ for $d\pi(X)$. The first assertion is clear from the definition of C^∞. The second follows from

$$\exp(tX)\exp(tY) = \exp(t(X + Y) + O(t^2)).$$

The third follows from

$$\exp(tX)\exp(Y)\exp(-tX) = \exp(Y + t[X, Y] + O(t^2)).$$

The fourth follows from

$$\exp(tAd(g)X) = g(\exp tX)g^{-1}.$$

The three bullet items imply that (π, H^∞) defines a representation of $Lie(G)$. The fourth is a compatibility condition that will play a role later. We will also consider this to be a representation of the complexification of $Lie(G)$, that is $Lie(G)_\mathbb{C}$. Set $\mathfrak{g}_\mathbb{C} = Lie(G)_\mathbb{C}$. Then (π, H^∞) extends to a representation of the universal enveloping algebra, $U(\mathfrak{g}_\mathbb{C})$. We define $Z_G(\mathfrak{g}_\mathbb{C})$ to be the subalgebra if $U(\mathfrak{g}_\mathbb{C})$ consisting of those z such that $Ad(g)z = z$ for all $g \in G$. We define an involution denoted * on $U(\mathfrak{g}_\mathbb{C})$ by the following rules

- $(z1)^* = \bar{z}1$.
- $X^* = -X$ for $X \in Lie(G)$.
- $(xy)^* = y^* x^*$ for $x, y \in U(\mathfrak{g}_\mathbb{C})$.

We note that the anti-homomorphism $x \longmapsto x^*$ exist by the universal problem solved by the universal enveloping algebra and also the naturality implies that if $g \in G$ then $(Ad(g)x)^* = Ad(g)(x^*)$.

Lemma 82 *If (π, H) is a unitary representation of G and if $v, w \in H^\infty$ and $x \in U(\mathfrak{g}_\mathbb{C})$ then*

$$\langle \pi(x)v, w \rangle = \langle v, \pi(x^*)w \rangle .$$

Proof. We note that if $X \in Lie(G)$ and $v, w \in H$ then

$$\langle \pi(\exp tX)v, w \rangle = \langle v, \pi(\exp(-tX))w \rangle$$

for all $t \in \mathbb{R}$. If $v, w \in H^\infty$ then both sides of the equation are differentiable in t. Taking the derivative at $t = 0$ yields

$$\langle \pi(X)v, w \rangle = \langle v, \pi(-X)w \rangle = \langle \pi(x)v, w \rangle = \langle v, \pi(X^*)w \rangle .$$

Now use the fact the $Lie(G)$ generates $U(\mathfrak{g}_\mathbb{C})$ over $\mathbb{C}$. ■

The next result is an application of the variant of Schur's lemma in Proposition 53.

Theorem 83 *Let (π, H) be an irreducible unitary representation of G then there exists an algebra homomorphism $\eta_\pi : Z_G(\mathfrak{g}_\mathbb{C}) \to \mathbb{C}$ such that $\pi(z)v = \eta_\pi(z)v$ for all $v \in H^\infty$.*

Proof. In Proposition 88 take $D = D'$ to be H^∞. If $z \in Z_G(\mathfrak{g}_\mathbb{C})$ then take $T = \pi(z)$. We note that if $v \in H^\infty$ then

$$\pi(g)T\pi(g)^{-1}v = \pi(g)\pi(z)\pi(g)^{-1}v$$

$$= \pi(Ad(g)z)v = \pi(z)v = Tv.$$

Also take $S = \pi(z^*)$. Then the previous lemma implies that the hypotheses of Proposition 53 are satisfied. Thus $\pi(z)$ acts as a scalar on H^∞. Denote this scalar by $\eta_\pi(z)$. ∎

Definition 84 *The homomorphism η_π is called the* infinitesimal character *of (π, H).*

If (π, H) is a Hilbert representation of G and if $x \in U(\mathfrak{g}_\mathbb{C})$ then we denote by p_x the semi-norm on H^∞defined by $p_x(v) = \|\pi(x)v\|$. We give H^∞ the corresponding locally convex topology. Notice that if $\{x_i\}$ is a basis of $U(\mathfrak{g}_\mathbb{C})$ then the semi-norms $\{p_{x_i}\}$ suffice to define the topology. The following result uses basic calculus in its proof. The interested reader can refer to *RRG I*, Lemma 1.6.2.

Lemma 85 *The space H^∞ is a Fréchet space with respect to the locally convex topology given above. Furthermore, (π, H^∞), is a smooth Fréchet representation (i.e. if $v \in H^\infty$ then the map $g \longmapsto \pi(g)v$ defines a C^∞ map from G to H^∞.*

Let K be a compact subgroup of G. Set $\mathfrak{g} = Lie(G)$ and $\mathfrak{k} = Lie(K)$. The most important special case is when K is a maximal compact subgroup of G.

Definition 86 *A $(\mathfrak{g}, K)$ module is a vector space, V, over $\mathbb{C}$ that is a module for the Lie algebra $\mathfrak{g}$ and a module for K (as an abstract group) such that*

1. $k \cdot X \cdot v = (Ad(k)X) \cdot k \cdot v$ for $k \in K$, $X \in \mathfrak{g}$, $v \in V$.

2. If $v \in V$ then $W_v = span_\mathbb{C}\{k \cdot v | k \in K\}$ is a finite dimensional vector space such that the map $k \to k \cdot w$, is C^∞ as a map from K to W_v for all $w \in W_v$.

3. If $Y \in \mathfrak{k}$ and $v \in V$ then $\frac{d}{dt}_{|t=0} \exp(tY) \cdot v = Yv$ (here the differentiation is as a map into W_v).

Our main class of example of $(\mathfrak{g}, K)$ modules are given as follows. Let (π, H) be a Hilbert representation of G. Let H^∞ be the space of C^∞ vectors of H. We have seen that this space is dense in H. Then the material after Theorem 93 implies that condition 1. is satisfied. As is condition 3. (but as a map into H). We set $H^\infty_{(K)}$ equal to the space of all $v \in H^\infty$ that satisfy 2. Then if $v \in H^\infty_{(K)}$ it satisfies 3. The only condition missing is that $\mathfrak{g}$ still acts.

Lemma 87 *If $X \in \mathfrak{g}$ then $XH^\infty_{(K)} \subset H^\infty_{(K)}$.*

Proof. Let $v \in H^\infty_{(K)}$ then we have a map of $\mathfrak{g} \bigotimes W_v \to H^\infty$ given by $X \bigotimes w \longmapsto Xw$. The compatibility condition 1. implies that the image of this map is a K-invariant finite dimensional space. It clearly contains Xv. Thus $Xv \in H^\infty_{(K)}$. ∎

Definition 88 *The $(\mathfrak{g}, K)$ module $H^\infty_{(K)}$ is called the underlying $(\mathfrak{g}, K)$-module of (π, H).*

The $(\mathfrak{g}, K)$-modules form a full subcategory $\mathcal{C}(\mathfrak{g}, K)$ of the category of $\mathfrak{g}$ and K modules. That is $\mathrm{Hom}_{(\mathfrak{g},K)}(V, W) = \mathrm{Hom}_{\mathfrak{g}}(V, W) \cap \mathrm{Hom}_K(V, W)$. We say that a $(\mathfrak{g}, K)$-module V is irreducible if the only $\mathfrak{g}$ and K invariant subspaces are V and 0.

If V is a $(\mathfrak{g}, K)$-module and if $\gamma \in \widehat{K}$ then we set $V(\gamma)$ equal to the span of all v such that the representation $W_v \in \gamma$.

Definition 89 *A $(\mathfrak{g}, K)$-module V is said to be admissible if $\dim V(\gamma) < \infty$ for all $\gamma \in \widehat{K}$.*

The following result is useful in proving irreducibility of represntations of G.

Theorem 90 *Let (π, H) be a Hilbert representation of G such that the underlying $(\mathfrak{g}, K)$-module is admissible and irreducible. Then (π, H) is irreducible.*

Proof. Let $V = (H^\infty)_{(K)}$. Suppose that V is reducible. Let W be a closed invariant subspace of H. Then $\pi(\alpha_\gamma)W \subset W$ for all $\gamma \in \widehat{K}$. Since H^∞ is dense in H and $\pi(\alpha_\gamma)H^\infty \subset H^\infty$ this implies that $H^\infty \cap H(\gamma) =$

$H(\gamma)$ since π is admissibel. Now if $W(\gamma) = H(\gamma)$ for all $\gamma \in \widehat{K}$ then $W = H$. Also as a subrepresentation of H we have $W^\infty = W \cap H^\infty$. This implies that $(W^\infty)_{(K)} \subset (H^\infty)_{(K)}$. If the two spaces are equal then the above considerations imply that $(W^\infty)_{(K)} = (H^\infty)_{(K)}$. Assume $W \neq H$. Since $(W^\infty)_{(K)} \subset (H^\infty)_{(K)}$ is a $\mathfrak{g}$ and a K-invariant subspace and $(H^\infty)_{(K)}$ is an irreducible $(\mathfrak{g}, K)$-module this implies that $(W^\infty)_{(K)} = (0)$. Hence $W(\gamma) = 0$ for all $\gamma \in \widehat{K}$. But than $W = 0$. Hence π is irreducible as asserted. ■

This result allows us to finish the discussion of the holomorphic discrete series of $SU(1,1)$. Here we take K and much of our notation as in 2.3. Since the map $T \to K$, $e^{i\theta} \longmapsto k(\theta)$ defines an isomorphism of $T = \{z \in \mathbb{C} | |z| = 1\}$ with K. One sees easily that if $\eta_n(k(\theta)) = e^{in\theta}$ then $\widehat{K} = \{\eta_n | n \in \mathbb{Z}\}$. From the definition of (π_k, H^k) we have $\pi_k(k)z^l = \eta_{-k-2l}(k)z^l$. It is easily seen that if $j \neq \{-k-2l | l \in \mathbb{Z}, l \geq 0\}$ then $H^k(\eta_j) = 0$ and that $H^k(\eta_{-k-2l}) = \mathbb{C}z^l$ otherwise. Thus $V = (H^k)^\infty_{(K)}$ is just the space of all polynomials in one complex variable. We will now prove the irreducibility. Set

$$h = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, u = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

Then both are elements of $Lie(SU(1,1))$. Set $Z^+ = \frac{h-iu}{2}$ and $Z^- = \frac{h+iu}{2}$ then

$$Z^+ z^l = -l z^{l-1}, Z^- z^l = (k+l) z^{l+1}, l \geq 0.$$

Now suppose that W is an invariant non-zero subspace of V. Then $W(\gamma) \neq 0$ for some $\gamma \in \widehat{K}$. This implies that $z^l \in W$ for some $l \geq 0$. Now $(Z^+)^l z^l = l!1$. This $1 \in W$. But $(Z^-)^m 1 = k\,(k+1) \cdots (k+m-1) z^m$. Hence $z^m \in W$ for all $m \geq 0$ so $W = V$.

We will now look at the principal series. We assume that G is a reductive group over R, that P is a parabolic subgroup and that K is a maximal compact subgroup. Then $G = PK$. Let (σ, H_σ) be a Hilbert representation of P that is unitary when restricted to $K \cap P$. Then as a representation of $K \cap P$, $H_\sigma = \bigoplus_{\mu \in \widehat{K \cap P}} H_\sigma(\mu)$ a direct sum of isotypic components. We will use the notation at the end of section 2.1. If $f \in H_0^\sigma$ then f is completely determined by its restiction to K since $f(pk) = \sigma(p) f(k)$. Also, $f_{|K} \in Ind_{K\cap P}^K(H_\sigma)$. Furthermore, if $\phi \in Ind_{K \cap P}^K(H_\sigma)$ is continuous then we define $f(pk) = \sigma(p)\phi(k)$ and note that the ambiguity in the definition is irrelevant and that $f \in H_o^\sigma$. This implies that as a represntation of K, $Ind_P^G(\sigma)$ is equivalent with $Ind_{P\cap K}^K(\sigma_{|K\cap P})$. Now Frobenius reciprocity (and the entire discussion at the end of 2.5) implies

Proposition 91 *The representation $Ind_P^G(\sigma)$ is admissible for K if and only if σ is admissible for $K\cap P$. Furthermore if $\gamma\in\widehat{K}$ and if $V_\gamma\in\gamma$ and for each $\mu\in\widehat{K\cap P}$, $W_\mu\in\mu$ then*

$$\dim Ind_P^G(\sigma)(\gamma)=\sum_{\mu\in\widehat{K\cap P}}\dim Hom_{K\cap P}(V_\gamma,W_\mu)\dim Hom_{K\cap P}(W_\mu,H_\sigma).$$

We will now apply all of this materal to the case of $G=SL(2,\mathbb{R})$ or $SL(2,\mathbb{C})$. As usual we take B to be the subgroup of upper triangular elements of G. We take K to be the unitary elements of G (i.e. $SO(2)$ or $SU(2)$). Then in the two cases we have respectively, $K\cap B$ is $\{\pm I\}$ for $\mathbb{R}$ and for $\mathbb{C}$

$$\left\{\begin{bmatrix} z & 0\\ 0 & z^{-1}\end{bmatrix}|z\in\mathbb{C},|z|=1\right\}.$$

We assume that $\chi:B\to\mathbb{C}^\times$ is a one dimensional representation of B. We denote by $I(\chi)$ the corresponding induced representation of G. We note that $\chi_{|B\cap K}$ is a one dimensional unitary representation of $K\cap M$. If $K=SU(2)$ we will parametrize the irreducible representations by their dimensions, we choose a representative (τ_k,V^k). Also we note that

$$\chi\left(\begin{bmatrix} e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{bmatrix}\right)=e^{in_\chi\theta}$$

with $n_\chi\in\mathbb{Z}$. If $K=SO(2)$ then the irreducible unitary representations will be paramatrized by integers

$$\begin{bmatrix} \cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\longmapsto e^{ik\theta}$$

denoted χ_k.

We have

Lemma 92 *Let $G=SL(2,\mathbb{R})$ then $\dim I(\chi)(\chi_k)=0$ or 1 and it is 1 if and only if $\chi_k(-I)=\chi(-I)$. If $G=SL(2,\mathbb{C})$ then $\dim Hom_K(V^k,I(\chi))=0$ or 1 and it equals 1 if and only if $n_\chi=k-1-2r$ for some $0\le r\le k-1$.*

We will now parametrize the characters χ. First if $G = SL(2,\mathbb{R})$ then the characters are of the form

$$\chi\left(\left[\begin{array}{cc} a & x \\ 0 & a^{-1} \end{array}\right]\right) = \left(\frac{a}{|a|}\right)^{\varepsilon} |a|^{\nu}$$

with $\varepsilon = 0$ or 1 and $\nu \in \mathbb{C}$. We will write $\chi = \chi_{\varepsilon,\nu}$ in this case. If $G = SL(2,C)$ then

$$\chi\left(\left[\begin{array}{cc} a & x \\ 0 & a^{-1} \end{array}\right]\right) = \left(\frac{a}{|a|}\right)^{n} |a|^{\nu}$$

with $n \in \mathbb{Z}$ and $\nu \in \mathbb{C}$ we write $\chi_{n,\nu}$ in this case. Here is the main theorem relative to the principal series the proof is similar to the one given for the discrete series.

Theorem 93 *Let $G = SL(2,\mathbb{R})$ then the representations $I(\chi_{0,i\nu})$ are irreducible for $\nu \in \mathbb{R}$. The representations $I(\chi_{1,i\nu})$ are irreducible for $\nu \in \mathbb{R}^{\times}$. Furthermore, $I(\chi_{1,0})$ splits into the direct sum of 2 irreducible representations of G, which we will denote by H^1 and H^{-1} these representations are described in the $SU(1,1)$ picture in the same way as the holomorphic and anti-holomorphic discrete series for parameter 1 and the square integrability condition replaced by square integrable boundary values on the circle. If $G = SL(2,\mathbb{C})$ then the representation $I(\chi_{n,i\nu})$ is irreducible for all $\nu \in \mathbb{R}$.*

In both cases there exists one more continuous series of representations the complementary series. This series is described as follows.

Theorem 94 *Let I_ν be the underlying $(\mathfrak{g}, K)$ module of $I(\chi_{0,\nu})$ for $0 < \nu < 1$. Then there exists an irreducible unitary representation of G whose underlying $(\mathfrak{g}, K)$-module is equivalent with I_ν.*

We have at this point described all irreducible unitary representations for these two groups except for the "trivial", $g \longmapsto 1$.

2.7 Characters.

Here we will confine our attention to the case of G the F points of a reductive group over a locally compact, non-discrete field F. We denote by $C_c^\infty(G)$ the infinitely differentiable functions compactly supported on G if $F = \mathbf{R}$ or

$\mathbb{C}$ and the locally constant complex valued functions with compact support if F is non-Archimedian.

We have the following basic theorem of Harish-Chandra (F of characteristic 0) and Bernstein (general non-archimedian).

Theorem 95 *If (π, H) is an irreducible unitary representation of G then the operators $\pi(f)$ are trace class for $f \in C_c^\infty(G)$. Furthermore, if F is $\mathbb{R}$ or $\mathbb{C}$ then the linear functional $f \longmapsto tr(\pi(f))$ is a distribution (that is continuous in the topology defined in section 1.5).*

The linear functional $f \longmapsto \mathrm{tr}(\pi(f))$ is called the distribution character of π (we will usually just call it the character). In the case when G is compact it is given by integrating against the usual character.

A Hilbert representation, (π, H), of G is said to be essentially unitary if there exists a continuous homomprhism, χ, of G to $\mathbb{C}^\times$ such that $\chi^{-1}\pi$ is uintary. If P is a parabolic subgroup and if $P = MN$ is a Levi decomposition let (σ, H_σ) be an essentially unitary, irreducible reperesenation of M we extend it to a representation of P by $\sigma(mn) = \sigma(m)$ for $n \in N$, $m \in M$. We will use the notation $I_{P,\sigma}$ for the induced representation from this extesion of σ to P. In the case of $\mathbb{R}$ or $\mathbb{C}$ we have seen that there is a refined decomposition $M = {}^oMA$ and we may assume in this case we consider σ to be an irreducible unitary representation of ${}^oM$ and ν a continuous homomorphism of A to $\mathbb{C}^\times$. We set $\sigma_\nu(man) = a^\nu\sigma(m)$ (notice how we are writing characters). In this case we will denote the induced representation by $I_{P,\sigma,\nu}$ it is unitary when ν is a unitary character. One can show that the representations $I_{P,\sigma}$ are such that the operators $\pi_\sigma(f)$ are trace class for $f \in C_c^\infty(G)$.

Harish-Chandra has calculated the characters.

Theorem 96 *Let the notation be as in section 1.5.*

$$tr(\pi_\sigma(f)) = tr(\sigma(\overline{f}^P)).$$

We will write $\Theta_{P,\sigma}$ (or $\Theta_{P,\sigma,\nu}$ in the more refined case) for this character. In the case of $\mathbb{R}$ or $\mathbb{C}$ this formula combined with the material in section 1.6 and Harish-Chandra's theory of discrete series characters relates the character theory directly with the orbital integrals. The Langlands conjectures would do the same for the non-Archimedian case.

We will just touch on this difficult and important part of the theory. References are Varadarajan, *Harmonic analysis on reductive groups*, *RRGI* and, of course, Harish-Chandra's original papers.

The first main theorem is that in the notation of 1.4, square integrable representations exist precisely when $K'' \neq \emptyset$. Thus assuming (for simplicity) that G is connected this implies that there is a Cartan subgroup T in G that is contained in K. Harish-Chandra showed that the square integrable representations have a natural parametrization by regular characters on T (this is not quite accurate one may need to go to a 2-fold covering but we will pretend that this is unnecessary). A unitary character $t \longmapsto t^{\lambda}$ is said to be regular if $d\lambda$ is not perpendicular to any root of $\mathfrak{g}_{\mathbb{C}}$ relative to $Lie(T)_C$. For $G = SU(1,1)$ the Harish-Chandra parametrization is our π_k is his π_{k-1} for $k \geq 2$ and our π_{-k} is Harish-Chandra's $\pi_{-k+1.}$ Thus for every $k \neq 0$ there is a square integrable representation.

The following result is usually attributed to Rossmann. The result is implicit in *RRGI* 8.7.3 (1) (unfortunately, unattributed).

Theorem 97 *Let Θ_λ denote the character of the discrete series representation parametrized by λ. If f is an element of $\mathcal{C}(G)$ with $L_g f^P = 0$ for all parabolic subgroups $P \neq G$ and all $g \in G$ then F_f^T is a smooth function on T and furthermore up to constants of normalization and a sign*

$$\Theta_\lambda(f) = \int_T F_f^T(t) t^{\lambda} dt.$$

The theorem needs a few words of explanation since if G is not compact there are no elements of $C_c^\infty(G)$ satisfying these conditions. One proves that in fact the characters, Θ_λ, extend to continuous functionals on $\mathcal{C}(G)$. The subspace of functions in $\mathcal{C}(G)$ satisfying the conditions of the theorem are called cusp forms for G and Harish-Chandra proves that this space is the closure of the linear span of the matrix coefficients of the discrete series.

2.8 Intertwining operators.

In this secton we will describe a few points on intertwining operators. This material is difficult. We will describe just enough for our discussion of Eisenstein series.

Let G be reductive over F a locally compact non-discrete field. If $P, Q$are two parabolic subgroups of G then they are said to be *associate* parabolic subgroups if they have a common Levi factor. Let P, Q be associate parabolic subgroups of G with common Levi factor M. We will now describe a method of relating $I_{P,\sigma}$ and $I_{Q,\sigma}$ Formally, if N_P is the unipotent radical of P and N_Q that of Q then we write for $f \in I_{P,\sigma}$

$$A_{Q|P}(\sigma)f(g) = \int_{(N_P \cap N_Q)\backslash N_Q} f(ng)d(N_P \cap N_Q n).$$

Notice that we are taking right quotient measure rather than left (all modular functions are 1 so there no problems here). Also that at least formally that

$$A_{Q|P}(\sigma)f(nmg) = \sigma(m)A_{Q|P}(\sigma)f(g), m \in M, n \in N_Q, g \in G.$$

Also

$$A_{Q|P}(\sigma)\pi_{P,\sigma}(g) = \pi_{Q,\sigma}(g)A_{Q|P}(\sigma), g \in G$$

at least formally. The real problem is that the integral has no reason to converge and in general it doesn't. The point of this section is to find useful conditions for the convergence of these integrals and to show that there is a method of analytic continuation that will lead to intertwining operators for the type of representations that we want to study.

For the case when F is $\mathbb{R}$ or $\mathbb{C}$ Theorem 27 is usually used to find a best possible range of convergence when the representation σ of ${}^oM$ is tempered (in particular square integrable). We will be dealing with more general choices of σ so we will content ourself with a relatively weak but adequate range of convergence based on the next lemma. To state it we need the notion of opposite parabolic subgroup. If $P = MN$ is a parabolic subgroup with given Levi decomposition then we may look at the set of all associate parabolic subgroups with Levi factor M. Then among them there is exactly one, $\overline{P}$, such that $\overline{P} \cap P = M$. The existance of this associate parabolic is usually proved using the absolute root system. Suffice for our purposes it exists (in the case of the classical examples as given in 1.2 this parabolic subgrup is just the image of P under transpose. We will call $\overline{P}$ the opposite parabolic subgroup, we write $\overline{P} = M\overline{N}$. We note that it depends on the Levi decomposition. If $g \in G$ the write $g = pk$ with the ambiguity noted earlier.

Lemma 98 *Let $P = MN$ be a parabolic subgroup of G and let $\overline{N}$ be the nilradical of the opposite parabolic subgroup then*

$$\int_{\overline{N}} \delta_P(p(\overline{n}))d\overline{n} < \infty.$$

This result of Harish-Chandra is usually derived by the techniques of section 1.2 using integration formulas associated with the fact that $NM\overline{N}$ is open and dense in G so Proposition 11 applies with $A = NM$ and $B =.\overline{N}$.

Using this result one can get a crude range of convergence for the intertwining integrals. We will give the result in the Archimedean case a completely analogous result is true for the non-Archimedean fields. Let $M = {}^oMA$, as usual. If $\lambda \in Lie(A)^*$ then we set $Lie(N)_\lambda = \{X \in Lie(N)|[h,X] = \lambda(h)X, h \in Lie(A)\}$. Let $\Phi(P,A) = \{\lambda|Lie(N)_\lambda \neq \{0\}, \lambda \neq 0\}$.

Proposition 99 *There exists a constant $C_P > 0$ such that if $\nu \in Lie(A)^*_C$ and $\mathrm{Re}(\nu,\lambda) > C_P$ for all $\lambda \in \Phi(P,A) \cap \Phi(\overline{Q},A)$ then if (σ, H_σ) is a unitary representation of ${}^oM$ then the integrals defining $A_{Q|P}(\sigma_\nu)f$ for $f \in H_0^{\sigma_\nu}$ converge absolutely and uniformly in compacta of $Lie(A)^*_C$.*

A proof of this result can be found in *RRGII* Lemma 10.1.10. We will consider σ to be fixed and write $A_{Q|P}(\nu)$. The result above implies that if $f \in H_0^{\sigma_\nu}$ then the map

$$f \longmapsto A_{Q|P}(\nu)f$$

is a holomorpic map on the tube defined in the previous lemma (here one must observe that the Hilbert space of the induced representations is $Ind_{K\cap P}^K(\sigma_{|K\cap P})$ so independent of ν.

The names of many authors are associated with the meromorphic cnontinuation of these integrals (Kunze,Stein,Harish-Chandra, Knapp,...) the most general result in this direction is due to Vogan and the author. Let $I_{P,\sigma}^\infty$ denote the space of all C^∞ elements of $Ind_{K\cap P}^K(\sigma_{|K\cap P})$. The following result is contained in Theorem 10.1.6 in *RRGII*.

Theorem 100 *If $f \in I_{P,\sigma}^\infty$ then the map $\nu \longmapsto A_{Q|P}(\nu)f$ has a meromorphic continuation to $Lie(A)^*_C$. Furthermore, there exists a non-zero holomorphic function, β, (depending on σ) such that $\nu \longmapsto \beta(\nu)(A_{Q|P}(\nu)f$ is holomorphic.*

There is an industry of finding normalizing factors such as that in the above theorem. They are intimately connected with the Langlands ε factors.

Let $N(A) = \{g \in G|gAg^{-1} = A\}$. Set $W(A) = N(A)/M$. We look at $W(A)$ as a group of automorphisms of A. That is, $s(a) = gag^{-1}$ for $g \in s$. If ν is a character of A then we define $s\nu$ by $a^{s\nu} = (s^{-1}a)^\nu$. Also, we define

$s\sigma(m) = \sigma((s^*)^{-1}ms^*)$ for a fixed choice of $s^* \in s$. We note that up to equivalence this is well defined but the actual representation depends on the choice of s^*. We also set $sP = s^*P(s^*)^{-1}$. Then sP has M as a Levi factor. We can define an intertwining operator

$$A_{sP|P}(s,\nu) : I_{P,\sigma.\nu} \to I_{sP,s\sigma,s\nu}$$

by $A_{sP|P}(s,\nu)f(g) = f((s^*)^{-1}g)$.

We finally have the most general intertwining operators that we will be needing

$$A_{Q|P}(s,\nu) = A_{Q|sP}(\nu)A_{sP,P}(s,\nu).$$

Note that our original operators are now $A_{Q|P}(1,\nu)$.

The operators $A_{P|P}(s,\nu)$ are usually called the Kunze-Stein operators for P minimal and the Knapp-Stein for general parabolic P and σ in the discrete series of ${}^oM$.

In the special case of $G = SL(2,F)$ with $F = \mathbb{R}$ or $\mathbb{C}$. These operators remove the ambiguity in the classification. One has $I_{P,\sigma,\nu} \cong I_{P,s\sigma,-\nu}$ for the parameters indicated in the classification.

3 Automorphic forms

We are now ready to begin the main topic of these lectures.

3.1 The case of compact quotient.

We first look at a relatively simple but geometrically useful example. Let G be a semi-simple Lie group and let Γ be a discrete subgroup of G such that $\Gamma\backslash G$ is compact. We fix an invariant measure on G and put the counting measure on Γ (that is $\mu(\gamma) = 1$ for all $\gamma \in \Gamma$). Then we have a measure on $\Gamma\backslash G$ that is invariant under the right regular action of G and such that

$$\int_{\Gamma\backslash G} \left(\sum_{\gamma\in\Gamma} f(\gamma g) \right) d(\Gamma g).$$

We define a representation of G on $L^2(\Gamma\backslash G)$ by $\pi_\Gamma(g)f(\Gamma x) = f(\Gamma xg)$. Then $(\pi_\Gamma, L^2(\Gamma\backslash G))$ is a unitary representation of G(see example 39). Since the quotient $\Gamma\backslash G$ is compact we see that this representation is of class CC (see Theorem 67). We therefore see that $L^2(\Gamma\backslash G)$ splits into an unitary direct sum of irreducible representations. We will use the notation $\widehat{G}$ for the set of equivalence classes of irreducible unitary representations. If $\omega \in \widehat{G}$ then we fix a choice $(\pi_\omega, H^\omega) \in \omega$. Then we can write (as in the case of isotypic components) the decomposition that we have asserted as

$$L^2(\Gamma\backslash G) \cong \bigoplus_{\omega\in\widehat{G}} Hom_G(H_\omega, L^2(\Gamma\backslash G)) \bigotimes H_\omega$$

since

$$m_\Gamma(\omega) = \dim Hom_G(H_\omega, L^2(\Gamma\backslash G)) < \infty$$

We call $m_\Gamma(\omega)$ the multiplicity of ω in $L^2(\Gamma\backslash G)$.

Before we move on to the trace formula in this case we will describe one of the more important geometric applications of this decompositions. We first recall a bit of relative cohomology theory (details can be found in A.Borel and N.Wallach, *Continuous cohomology,...*). Let V be a $(\mathfrak{g}, K)$-module. Then one has the relative Lie algebra cohomology complex

$$C^k(\mathfrak{g}, K; V) \overset{d}{\rightarrow} C^{k+1}(\mathfrak{g}, K; V).$$

With

$$C^k(\mathfrak{g}, K; V) = Hom_K(\bigwedge{}^k(\mathfrak{g}/\mathfrak{k}), V)$$

and d is given by a formula that we will not be using (see the theorem below). Suffice to say that it is a complex. The cohomology of this complex is usually denoted $H^k(\mathfrak{g}, K; V)$. If W is the underlying $(\mathfrak{g}, K)$-module of a unitary represenation such that the infinitesimal character is the same as that of a finite dimensional module $F$and if $V = W\bigotimes F^*$ then one can show that $d = 0$. We therefore have

Theorem 101 *If V is the underlying $(\mathfrak{g}, K)$-module of a unitary representation that has infinitesimal character equal to that of the finite dimensional represenation F then*

$$H^k(\mathfrak{g}, K; V\bigotimes F^*) \cong Hom_K(\bigwedge{}^k(\mathfrak{g}/\mathfrak{k}), V\bigotimes F^*).$$

This result combined with some ideas of Matsushima and Kuga implies .

Theorem 102 *If F is an irreducible finite dimensional representation of G then let $\widehat{G}_F$ denote the elements of $\widehat{G}$ whose representatives have infinitesimal equal to that of F. Then one has*

$$H^k(\Gamma, F) = \bigoplus_{\omega \in \widehat{G}_F} Hom_G(H_\omega, L^2(\Gamma\backslash G)) \bigotimes Hom_K(\bigwedge{}^k(\mathfrak{g}/\mathfrak{k}), H_\omega \bigotimes F^*).$$

Here $H^k(\Gamma, F)$ is the usual Eilenberg-Maclane cohomology of Γ with coefficients in F. This is one reason for the work done by many authors (Hotta, Parthasarathy, Enright, Kumaresan, Zuckerman, Vogan,...) to determine the elements of $\widehat{G}_F$. Before all this was completed Borel and I (in the book mentioned above proved that if $0 < k < rank_R(G)$ and if G has on compact normal subgroups and the representation H_ω has finite kernel then if V is the underlying $(\mathfrak{g}, K)$ module of H_ω then

$$H^k(\mathfrak{g}, K; V \bigotimes F^*) = 0.$$

This vanshing theorem is derivable from the classification but it is not completely trivial.

One should consult Borel-Wallach for some implications of this line of reasoning.

The point so far is that we have been able to skirt the obvious question: How do we compute the $m_\Gamma(\omega)$? Since 0 times anything is 0. The answer to the question is that one has no real method. However, one can deduce results about the distribution of the multiplicities. These results generally involve the trace formula or elliptic operator theory (or both).

3.2 The trace formula in the case of compact quotient.

We retain the notation of the previous section. We will write π for π_Γ. The material in example 68 applies to the case when $X = \Gamma\backslash G$. One finds using those calcuations that if $f \in C_c(G)$ and if

$$k_f(\Gamma x, \Gamma y) = \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y)$$

then (notice that we have interchanged right and left)

$$\pi(f)\phi(x) = \int_{\Gamma\backslash G} k_f(x,y)\phi(y)dy.$$

The argument in the proof of Lemma 69 proves that the operator $\pi(f)$ is of Hilbert-Scmidt class. This means that if $\{v_j\}$ is an orthonormal basis of $L^2(\Gamma\backslash G)$ then $\sum_j \|\pi(f)v_j\|^2$ converges. One can take as a definition of trace class operator a product of two Hilbert-Schmidt class operators. It is a theorem of Dixmier-Malliavan that if $f \in C_c^\infty(G)$ then f can be written as a finite sum of convolutions $\sum f_i * g_i$ with $f_i, g_i \in C_c^\infty(G)$. Since

$$\pi(f_i * g_i) = \pi(f_i)\pi(g_i)$$

we see that if $f \in C_c^\infty(G)$ then $\pi(f)$ is of trace class. One can consult, say, *RRGI* for the basics of trace class operators. All we need here is that if T is a trace class operator on a Hilbert space H and if $\{v_i\}$ is an orthonormal basis of H then

$$\sum \langle Tv_i, v_i\rangle$$

converges absolutely and the sum, which we denote $\mathrm{tr}T$, is independent of the basis. For the kernel operators we are studying the trace is just integration on the diagonal. We have

Theorem 103 *If $f \in C_c^\infty(G)$ then $\pi_\Gamma(f)$ is of trace class. Furthermore,*

$$tr(\pi_\Gamma(f)) = \int_{\Gamma\backslash G} \left(\sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)\right) d(\Gamma x).$$

Notice that the inner sum is a function on $\Gamma\backslash G$. Gelfand,Graev, Piattetsky-Shapiro (*Generalized Functions 6*) give a refinement of this formula which I will describe presently. First we will complete this one. If $\omega \in \widehat{G}$ then $\pi_\omega(f)$ is trace class and so we have another formula for the trace (the so called spectral side).

Theorem 104 *Let $f \in C_c^\infty(G)$ then*

$$\sum_{\omega\in\widehat{G}} m_\Gamma(\omega)tr(\pi_\omega(f)) = \int_{\Gamma\backslash G} \left(\sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)\right) d(\Gamma x).$$

For our first application of the trace formula we will assume that Γ contains no elements of finite order (this can be accomplished by going to a normal subgroup of finite index, Selberg, Mostow). Then we have a smooth manifold $\Gamma\backslash G/K$. On G/K we have a Riemannian structure corresponding to the invariant form that we have chosen (we must make sure that the form is negative definite on $Lie(K)$ it will then be positive definite on $Lie(K)^{\perp}$. This we identify with the tangent space, $T(G/K)_{1K}$, at the identity coset. We then move this inner product to every tangent space by left translation. Since the Riemannian structure is invariant under left translation by elements of G (by construction) it pushes down to a Riemannian structure on the quotient manifold $\Gamma\backslash G/K$. Let

$$p : G/K \to \Gamma\backslash G/K$$

be the natural projection. Then the Riemannian (geodesic) exponential map is given by

$$Exp_{p(gK)}(v) = p(g\exp(tX)K)$$

for $v \in T(\Gamma\backslash G/K)_{p(gK)}$ being the image under the differential of p of the vector gotten by translation by g if the tanget vector at $1K$ corresponding to $X \in Lie(K)^{\perp}$.

The general theory of Riemannian manifolds implies that if $x \in \Gamma\backslash G/K$ then there is a ball of radius $r > 0$ on which the exponential map is injecive. Let $r(x)$ be the suppremum of all such r. Since $\Gamma\backslash G/K$ is compact $r_\Gamma = \inf_{x\in\Gamma\backslash G/K} r(x)$ is strictly positive. Let

$$U_\Gamma = \{\exp(X)k | X \in Lie(K)^{\perp}, B(X,X) < r_\Gamma^2, k \in K\}.$$

Note that if $x \in U$ then $x^{-1} \in U$.

Lemma 105 *If $x \in G$ then $x^{-1}\Gamma x \cap U = \{1\}$.*

Proof. By definition $p(xu) \neq p(x)$for all $u \in U_\Gamma, u \notin K$. Thus if $xu = \gamma x$ for some $\gamma \in \Gamma$ and $u \in U_\Gamma$ then $u \in K$. This implies that $x^{-1}\gamma x \in K$. Since Γ is discrete this implies that γ has finite order and hance $\gamma = 1$. ■

We conclude that if $\mathrm{supp}(f) \subset U_\Gamma$ then

$$\mathrm{tr}\,(\pi_\Gamma(f)) = \int_{\Gamma\backslash G} f(1)d(\Gamma x) = f(1)vol(\Gamma\backslash G).$$

David DeGeorge and the author applied this formula to prove limit formulas for the multiplicities of representations. We will just sketch how this can be done. Let

$$V_r = \{\exp(X)k | X \in Lie(K)^{\perp}, B(X,X) < r^2, k \in K\}.$$

Then if supp$f \subset V_r$ for $r \le r_\Gamma/2$ then supp$f * f \subset U_\Gamma$. Let ϕ be the characteristic function of $V_{r_\Gamma/(2+\varepsilon)}$. Let $\omega \in \widehat{G}$ and let $f(g) = \phi(g)\overline{\langle \pi(g)v, v\rangle}$. Then the observation above implies that $\pi_\Gamma(f)$ is of Hilbert-Schmidt class. Thus we can apply the trace formula to $f * f$. That is

$$\mathrm{tr}(\pi_\Gamma(f * f)) = f * f(1) vol(\Gamma\backslash G)$$

$$= vol(\Gamma\backslash G)\int_G f(x)f(x^{-1})dx = vol(\Gamma\backslash G)\int_{V_{r_\Gamma/(2+\varepsilon)}} |\langle \pi_\omega(g)v, v\rangle|^2 dg.$$

We now use the left hand side of the formula.

$$\mathrm{tr}(\pi_\Gamma(f * f)) = \sum_{\mu \in \widehat{G}} m_\Gamma(\mu)\mathrm{tr}(\pi_\mu(f * f)).$$

We observe that all of the terms are non-negative so in particular

$$m_\Gamma(\omega)\mathrm{tr}(\pi_\omega(f * f)) \le \mathrm{tr}(\pi_\Gamma(f * f)).$$

Now we assume that v is a unit vector. Thus if $\{v_j\}$ is an orthonomal basis of H^ω with $v = v_1$. Then we have

$$\mathrm{tr}(\pi_\omega(f * f)) = \mathrm{tr}(\pi_\omega(f)\pi_\omega(f)^*).$$

Now

$$\langle \pi_\omega(f)v_i, v_j\rangle = \int_G \phi(g)\overline{\langle \pi_\omega(g)v, v\rangle}\,\langle \pi_\omega(g)v_i, v_j\rangle\, dg.$$

Thus

$$\mathrm{tr}(\pi_\omega(f * f)) = \sum_{i,j}\left|\int_G \phi(g)\overline{\langle \pi_\omega(g)v, v\rangle}\,\langle \pi_\omega(g)v_i, v_j\rangle\, dg\right|^2.$$

Again all of the terms in the sum are positive so any one term is dominated by the left hand side. We look at the term with $i = j = 1$. We have

$$\left| \int_{V_{r_\Gamma/(2+\varepsilon)}} |\langle \pi_\omega(g)v, v\rangle|^2 dg \right|^2 \leq \mathrm{tr}(\pi_\omega(f * f)).$$

Putting all of this together we have (after taking the limit as $\varepsilon \to 0$)

Proposition 106 *Let $\omega \in \widehat{G}$ and let $v \in H^\omega$ be a unit vector then*

$$\frac{m_\Gamma(\omega)}{vol(\Gamma\backslash G)} \leq \frac{1}{\int_{V_{r_\Gamma}/2} |\langle \pi_\omega(g)v, v\rangle|^2 dg}.$$

In our paper with DeGeorge we showed that for each such Γ there exists a tower of normal subgroups of finite index in Γ, $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset ... \supset \Gamma_n \supset ...$ such that

$$\bigcap_{i=1}^{\infty} \Gamma_i = \{1\}$$

and (hence) $\lim_{n\to\infty} r_{\Gamma_n} = \infty$.

We therefore have

Theorem 107 *Let the notation be as above. If $\omega \in \widehat{G}$ and ω is not square integrable then*

$$\lim_{n\to\infty} \frac{m_{\Gamma_n}(\omega)}{vol(\Gamma_n\backslash G)} = 0.$$

Using this and a proportionality principle we also proved

Theorem 108 *If $\omega \in \widehat{G}$ and ω is square integrable then*

$$\lim_{n\to\infty} \frac{m_{\Gamma_n}(\omega)}{vol(\Gamma_n\backslash G)} = d(\omega).$$

Notice that the above results indicate that this is an upper bound.

We will now describe the promised refinement of the formula and then give an application due to Langlands. If $x \in G$ we set $G_x = \{g \in G | gx = xg\}$. If $\gamma \in \Gamma$ then we set $\Gamma_\gamma = \Gamma \cap G_\gamma$. We normalize measures such that

$$\int_G f(g)dg = \int_{G_\gamma \backslash G} \left(\int_{G_\gamma} f(xg)dx \right) dg$$

and

$$\int_{G_\gamma} h(g)dg = \int_{\Gamma_\gamma \backslash G_\gamma} \left(\sum_{\gamma \in \Gamma} h(\gamma x) \right) d(\Gamma_\gamma x).$$

We now show that $\Gamma_\gamma \backslash G_\gamma$ is compact. We first consider the right action of G on $X = \Gamma \backslash G$. Let $\gamma \in \Gamma$ we consider the set $H_\gamma = \{g \in G | g\gamma g^{-1} \in \Gamma\}$. Note that $\Gamma H_\gamma = H_\gamma$. Then the set $X^\gamma = \{x \in X | x\gamma = x\}$. One has $X^\gamma = \Gamma \backslash H_\gamma$. We denote by H_γ^o the identity component of H_γ. Since Γ is discrete we see that if $g \in H_\gamma^o$ we must have $g\gamma g^{-1} = \gamma$. Thus the identity component of H_γ is the same as that if G_γ. This implies that the identity component of X^γ is $\Gamma \backslash (\Gamma H_\gamma^o)$. This implies that $\Gamma \backslash (\Gamma G_\gamma)$ is a union of connected components of X^γ. Since X is compact this implies that $\Gamma \backslash (\Gamma G_\gamma) = \Gamma_\gamma \backslash G_\gamma$ is compact.

We denote by $[\gamma]$ the conjugacy class of γ in Γ. We write $[\Gamma]$ for the set of conjuagacy classes.

Theorem 109 *If $f \in C_c^\infty(G)$ then*

$$tr\pi_\Gamma(f) = \sum_{[\gamma] \in [\Gamma]} vol(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(g^{-1}\gamma g) d(G_\gamma g).$$

The point here is that the formula is made up completely of the orbital integrals. This formula is derived from the original as follows we first note that the sum

$$\sum_{\tau \in [\gamma]} f(x^{-1}\tau x) = \sum_{\tau \in \Gamma_\gamma \backslash \Gamma} f(x^{-1}\tau^{-1}\gamma\tau x)$$

is a function on $\Gamma \backslash G$. We now take the raw formula and note that it is given

$$\int_{\Gamma \backslash G} \left(\sum_{\gamma \in \Gamma} f(x^{-1}\gamma x) \right) d(\Gamma x) = \int_{\Gamma \backslash G} \left(\sum_{[\gamma] \in [\Gamma]} \sum_{\tau \in [\gamma]} f(x^{-1}\tau x) \right) d(\Gamma x).$$

The sum over the conjugacy class coverges absolutly and uniformly in compacta in G so the expression is

$$\sum_{[\gamma]\in[\Gamma]}\int_{\Gamma\backslash G}\left(\sum_{\tau\in[\gamma]} f(x^{-1}\tau x)\right)d(\Gamma x)=\sum_{[\gamma]\in[\Gamma]}\int_{\Gamma\backslash G}\left(\sum_{\tau\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\tau^{-1}\gamma\tau x)\right)d(\Gamma x)$$

$$=\sum_{[\gamma]\in[\Gamma]}\int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)d(\Gamma x).$$

Now we apply the integration formulae above and have

$$\sum_{[\gamma]\in[\Gamma]}\int_{G_\gamma\backslash G}\int_{\Gamma_\gamma\backslash G_\gamma} f((xy)^{-1}\gamma xy)d(\Gamma_\gamma x)d(G_\gamma y).$$

The function being integrated is constant in y so the integral over $\Gamma_\gamma\backslash G_\gamma$ gives the volume term.

Now for Langlands' application. We assume that Γ has no elements of finite order. We first note that the trace formula is true for functions in $\mathcal{C}^1(G)$. Let ω be the class of a discrete series representation of G such that one matrix coefficient $f(g) = c_{v,v}(g) = \langle \pi_\omega(g)v, v\rangle$ with $\|v\| = 1$ is in $L^1(G)$(we call such a square integrable represenation an L^1-discrete series Then one can show that it is in $\mathcal{C}^1(G)$. One can show that the characters of irreducible unitary representations extend to continuous functions on $\mathcal{C}^1(G)$. An extension exactly the same argument used in the proof of the orthogonality part of the Schur orthogonality relations proves that if $\omega_1 \neq \omega$ then $\pi_{\omega_1}(f) = 0$ (even if ω_1 is not square integrable). Finally, a direct calculation (completely analogous to the ones above) yields

$$\mathrm{tr}\pi_\omega(f) = \frac{1}{d(\omega)}.$$

Also, since there are no elements of finite order in Γother than 1 this implies that all of the orbital integrals in the formula factor trhough the Harish-Chandra transform for appropriate parabolic subgroups. Since, but the Harish-Chandra transforms vanish for matrix entries of discrete series (see the discussion at the end of section 2.7. The upshot is that we again have exactly one term in the geometric side of the trace formula. Thus

$$\frac{m_\Gamma(\omega)}{d(\omega)} = vol(\Gamma\backslash G).$$

Thus for L^1-discrete series the limit formula can be replaced by an equality.

3.3 The constant term.

We now assume that $\mathbf{G} \subset GL(n, \mathbb{C})$ is a reductive algebraic group defined over $\mathbb{Q}$ (see the material after Proposition 11) and that G is the group of real points $(\mathbf{G} \cap GL(n, \mathbb{R}))$ which we assume has the property ${}^o\mathbf{G} = \mathbf{G}$ and Γ is arithmetic that is if $\mathbf{G}_{\mathbb{Z}} = \mathbf{G} \cap GL(n, \mathbb{Z})$ then $\Gamma \cap \mathbf{G}_{\mathbb{Z}}$ is of finite index in both $G_{\mathbb{Z}}$ and in Γ. The basic reference for the theory of arithmetic groups is A. Borel, *Introduction aux Groupes Arithmetiques*. Assume that $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ defined over $\mathbb{Q}$ then then we write P for the $\mathbb{R}$-rational points. Let $\mathbf{N}$ be the unipotent radical of $\mathbf{P}$ then N is defined over $\mathbb{Q}$. Let N be the group of $\mathbb{R}$-rational points. We note that $\mathbf{M} = \mathbf{P}/\mathbf{N}$ is also a reductive group defined over $\mathbb{Q}$. Thus $\mathbf{M} = {}^o\mathbf{M}\mathbf{A}$ with $\mathbf{A}$ a maximal $\mathbb{Q}$ split torus (i.e. commected group whose $\mathbb{Q}$ points are diagonalizable over $\mathbb{Q}$). Let $M, {}^oM$ and A denote the $\mathbb{R}$ rational points of $\mathbf{M}, {}^o\mathbf{M}$, and $\mathbf{A}$ respectively.

Theorem 110 *The space* $(\Gamma \cap N)\backslash N$ *is compact.*

We will always normalize the measure on $(\Gamma \cap N)\backslash N$ so that the total measure is 1.

The standard example is $\mathbf{G} = SL(2, \mathbb{C})$, $G = SL(2, \mathbb{R})$ defined by $\det -1 = 0$, $\mathbf{P}$ the group of upper triangular matrices in $\mathbf{G}$, $\Gamma = SL(2, \mathbb{Z})$. Then

$$N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} | x \in \mathbb{R} \right\}$$

and

$$N \cap \Gamma = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} | n \in \mathbb{Z} \right\}.$$

Then $N \cap \Gamma \backslash N$ is isomorphic with the the unit circle in $\mathbb{C}$.

The condition of compact quotient is that there are no parabolic subgroups of $\mathbf{G}$ defined over $\mathbb{Q}$ other than $\mathbf{G}$. We will now assume that this is not the case. It is still true that $\Gamma\backslash G$ has finite total volume (Borel,Harish-Chandra, Annals of Mathematics,1962). We define a class of functions on G that will be our main objects of study. We fix a realization of G in $GL(n, \mathbb{R})$ and assume, as we may, that $G \cap O(n)$ is a maximal compact subgroup. Let $\|g\|$ be the Hilbert-Schmidt norm of the matrix $g \in GL(n, \mathbb{R})$. We denote by $\mathcal{A}^\infty(\Gamma\backslash G)$ the space of all smooth functions, f, from $\Gamma\backslash G$ to $\mathbb{C}$ such that (here $\mathfrak{g} = Lie(\mathbf{G})$ and $Z_G(\mathfrak{g})$ is defined in the material before Lemma 82)

1. The space $Z_G(\mathfrak{g})f$ is finite dimensional.

2. If $x \in U(Lie(G))$ then there exist k and C_x such that

$$|xf(g)| \leq C_x \|g\|^k .$$

The critical point here is that k is independent of x This condition is A. Borel's notion of uniform moderate growth. We will call f an automorphic function. If we impose in addition the condition

3. The span over $\mathbb{C}$ of $R_K f$ is finite dimensional.

Then we say that f is a K-finite automorphic function and denote the space of all such f by $\mathcal{A}(\Gamma\backslash G)$. We note that since the right action of K on $\mathcal{A}(\Gamma\backslash G)$ splits into irreducible K subrepresentations and the Lie algebra of G acts by the usual action of left invariant vector fields $\mathcal{A}(\Gamma\backslash G)$ is a $(\mathfrak{g}, K)$-module.

On $\mathcal{A}^\infty(\Gamma\backslash G)$ we put the union topology relative to the decomposition

$$\mathcal{A}^\infty(\Gamma\backslash G) = \cup_k \mathcal{A}_k^\infty(\Gamma\backslash G)$$

with $\mathcal{A}_k^\infty(\Gamma\backslash G)$ the subspace of elements that satisfy 1. and 2. above (i.e. with that k). We use the seminorms

$$p_{k,x}(f) = \sup_g \|g\|^k |xf(g)|$$

for $x \in U(\mathfrak{g})$.

If P is the group of real points of a parabolic subgroup of $\mathbf{G}$ defined over $\mathbb{Q}$ and let N be the unipotent radical then we define for $f \in \mathcal{A}^\infty(\Gamma\backslash G)$

$$f_P(g) = \int_{(\Gamma\cap N)\backslash N} f(ng)d\overline{n}.$$

The function f_P is called the constant term of f along P. We note that $f_P(ng) = f_P(g)$ for all $n \in N$, $g \in G$. We also note that since N is normal in P, $N \cap \Gamma$ is normal in $P \cap \Gamma$ Set $\Gamma_M = (\Gamma \cap P)/(\Gamma \cap N)$. Then $\Gamma_M \cap A$ is finite and this implies that if $\gamma \in \Gamma_M$ then

$$f_P(\gamma g) = f_P(g).$$

With these observations in mind we set

$$\mathbf{f}_P(g)(p) = f_P(pg), p \in P, g \in G.$$

We note that $\mathbf{f}_P(p_o g)(p) = f_P(pp_o g) = \mathbf{f}_P(g)(pp_o) = (R(p_o)(\mathbf{f}_P(g)))(p)$. That is

$$\mathbf{f}_P(p_o g) = R(p_o)\mathbf{f}_P(g).$$

We also observe that $\mathbf{f}_P(g)(\gamma p) = f(\gamma pg) = f(pg) = \mathbf{f}(g)(p)$. For the moment we will fix g and condsider the function $u(p) = \mathbf{f}(g)(p)$ for $p \in P$. Then from the above we see that $u(np) = u(p)$ for $n \in N$ and thus we may look upon u as being a function on $N \backslash P = M$. Also since $u(\gamma p) = u(p)$ for $\gamma \in \Gamma \cap P$ we have, as a function on M, $u(\gamma m) = u(m)$ for $m \in M$ and $\gamma \in \Gamma_M = (\Gamma \cap P)/(\Gamma \cap N)$.

Relative to the action of A, $\mathfrak{n} = Lie(N)$ decomposes into root spaces. That is

$$\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{n}_\alpha.$$

We set $\overline{\mathfrak{n}} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{n}_{-\alpha}$ then $Lie(G) = \overline{\mathfrak{n}} \bigoplus Lie(M) \bigoplus \mathfrak{n}$. This implies (using the Poincare-Birkhoff-Witt theorem) that

$$U(\mathfrak{g}) = U(\mathfrak{m}) \bigoplus (\overline{\mathfrak{n}}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n})$$

where $\mathfrak{g} = Lie(G) \bigotimes \mathbb{C}$ and $\mathfrak{m} = Lie(M) \bigotimes \mathbb{C}$. Let p denote the projection relative to this direct sum decomposition. One can show that

$$p : U(\mathfrak{g})^A \to U(\mathfrak{m})$$

is an algebra homomorphism ($U(\mathfrak{g})^A$ is the algebra of $Ad(A)$-invariants in $U(\mathfrak{g})$).

We note that $z \in U(\mathfrak{g})^A \cap (\overline{\mathfrak{n}}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n})$ if and only if $z \in U(\mathfrak{g})^A \cap U(\mathfrak{g})\mathfrak{n}$. We now consider $p(Z_G(\mathfrak{g}))$. One can prove (c.f. *RRGI*)

Theorem 111 *The algebra $Z_M(\mathfrak{m})$ is integral over $p(Z_G(\mathfrak{g}))$ (that is there exist $u_1, ..., u_d \in Z_M(\mathfrak{m})$ such that $Z_M(\mathfrak{m}) = \sum p(Z_G(\mathfrak{g}))u_i$).*

Now f_P as above satsifies ($L(x)$ corresponds to the action of $U(\mathfrak{g})$ as right invariant operators, that is by differentiation on the left)

$$L(z)f_P = p(z)f_P.$$

and

$$L(z)(f_P) = (L(z)f)_P$$

for $z \in Z_G(\mathfrak{g})$. This implies that the function $u = \mathbf{f}_P(g)$ (for fixed $g \in G$) satsifies $\dim Z_M(\mathfrak{m})u < \infty$. In particular we have that $\dim U(\mathfrak{a})u < \infty$ (here $\mathfrak{a} = Lie(A)$). We fix coordinates $x_1 ...., x_l$. We therefore see that there exist elements $\nu \in \mathfrak{a}_C^*$ (the complex valued real linear forms on A) and smooth functions $u_{\nu,I}$ on ${}^oM$ for $I \in (\mathbb{Z}_{\geq 0})^l$ of which only a finite number are non-zero such that

$$u(nma) = \sum a^\nu (\log a)^I u_{\nu,I}(m)$$

for $m \in {}^oM$, $n \in N$ and $a \in A$ here $x^I = x_1^{i_1} \cdots x_l^{i_l}$. We note that

$$\dim Z_{{}^oM}({}^o\mathfrak{m})u_{\nu,I} < \infty.$$

We also note that the growth condition satisfied by f implies the analogous growth condition for $u_{\nu,I}$. We will write $\mathbf{f}_{P,\nu,I}(g) = u_{\nu,I}$. We have

Proposition 112 *With the notation above* $\mathbf{f}_{P,\nu,I} : G \to \mathcal{A}^\infty(\Gamma_{{}^oM} \backslash {}^oM)$.

For the sake of simplicity we will assume that there exists a homomorphism, χ, of $Z_G(\mathfrak{g})$ to $\mathbb{C}$ such that $zf = \chi(z)f$ for $z \in Z_G(\mathfrak{g})$. The space of such f will be denoted $\mathcal{A}^\infty(\Gamma\backslash G, \chi)$. We will use the notation $\mathcal{A}(\Gamma\backslash G, \chi)$ for the elements of $\mathcal{A}^\infty(\Gamma\backslash G, \chi)$ that satisfy condition 2. above.

Using the integrality assertion above we can prove that given χ the set of $\nu \in \mathfrak{a}_C^*$ such that there exists $f \in \mathcal{A}^\infty(\Gamma\backslash G, \chi)$ and some I with $\mathbf{f}_{P,\nu,I} \neq 0$ is finite. Further, for each ν the set of all I such that there is some $f \in \mathcal{A}^\infty(\Gamma\backslash G, \chi)$ with $\mathbf{f}_{P,\nu,I} \neq 0$ is a finite set which we denote by $\mathcal{P}(\nu)$. We can order $\mathcal{P}(\nu)$ so that if for $f \in \mathcal{A}^\infty(\Gamma\backslash G, \chi)$ we form the vector, $\mathbf{f}_{P,\nu}$ with components $\mathbf{f}_{P,\nu,I}$ in the order then we have

$$\mathbf{f}_{P,\nu}(ag) = \mu_\nu(a)\mathbf{f}_{P,\nu}(g)$$

with

$$\mu_\nu(a) = \begin{bmatrix} a^\nu & * & * & * \\ 0 & a^\nu & * & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a^\nu \end{bmatrix}.$$

We can think of $\mathbf{f}_{P,\nu}$ as being an element of $\mathcal{A}^\infty(\Gamma_{{}^oM}\backslash {}^oM) \bigotimes V(\Gamma, \chi, \nu)$ with $V = V(\Gamma, \chi, \nu)$ a vector space of dimension equal to the cardinality of $\mathcal{P}(\nu)$. We look upon μ_ν as an action of A on V making it into a representation of A.

In general for a smooth manifold, X, we endow $C^\infty(X)$ with the topology of uniform convergence with all derivatives on compacta. With this topology $\mathcal{A}^\infty(\Gamma_{{}^oM}\backslash {}^oM)$ becomes a representation of ${}^oM$ and we interpret our discussion in the following result.

Proposition 113 *Set $T_{P,\nu}(f) = \mathbf{f}_{P,\nu}$. Then $T_{P,\nu}$ defines an intertwining operator*

$$T_{P,\nu} : \mathcal{A}^\infty(\Gamma\backslash G, \chi) \to Ind_P^G(\mathcal{A}^\infty(\Gamma_{{}^oM}\backslash {}^oM) \bigotimes V(\Gamma, \chi, \nu))$$

from the right regular action to the representation induced from the representation of ${}^oMAN$ geiven by $man \longmapsto R(m) \bigotimes \mu_\nu(a)$. If the infinities are removed we have the same result with $(\mathfrak{g}, K)$ induction.

The point of Eisenstein series is to give an approximate inverse to this mapping. Before we dig into that subject we will look at a key property of the constant term. For this we must recall the notion of Siegal sets associated with P. Let $\omega \subset {}^oMN$ be a compact set. Let Σ^+ be as above and let $t > 1$ be given set $A_t^+ = \{a \in A | a^\alpha \geq t\}$. We will write ${}^oP = {}^oMN$.

Definition 114 *A set of the form $\omega A_t^+ K$ is called a Siegel set and denoted $\mathcal{S}_{P,\omega,t}$.*

Let $P_1, ..., P_r$ be a complete set of representatives for the Γ conjugacy classes of minimal parabolics defined over Q (which is finite, Borel based on the methods developed by Borel and Harish-Chandra) then there are compact subsets of ${}^oP_i$ and $t_i > 1$, $i = 1, ..., r$ such that

$$G = \cup \Gamma S_{P_i \omega_i, t_i}.$$

See B-HC for this also.

We return to the P that we have been studying. Set $A^+ = \cup_{t>1} A_t^+$. We define

$$\beta(a) = \min_{\alpha \in \Sigma^+} a^\alpha.$$

We have the following lemma which was inspired by a technique of Langlands (The Functional Equations Satisfied by Eisenstein Series, section 3; we will refer to this book as *Langlands*).

Lemma 115 *Fix $t > 1$ and $\omega \subset {}^oP$. Let $f \in \mathcal{A}^\infty(\Gamma \backslash G)$ and let k be such that $|xf(g)| \leq C_x \|g\|^k$ for $x \in U(g)$ and $g \in S_{P,\omega,t}$. Then*

$$|f(g) - f_p(g)| \leq \beta(a(g))^{-1} C \|g\|^k$$

Here $g = n(g)m(g)a(g)k(g)$ with $n(g) \in N$, $m(g) \in {}^oM$, $a(g) \in A$ and $k(g) \in K$ as usual.

We will prove this result for $SL(2,\mathbb{R})$ assumed defined over $\mathbb{Q}$ and such that the usual upper triangular parablic, P, subgroup is defined over $\mathbb{Q}$. This will give the main idea of the proof. We note that $\Gamma \cap N$ is a discrete subgroup of N which is isomorphic with the additive group $\mathbb{R}$. Thus $\Gamma \cap N$ must be of the form

$$\left\{ \begin{bmatrix} 1 & r^2 n \\ 0 & 1 \end{bmatrix} | n \in \mathbb{Z} \right\}$$

with $r > 0$. After conjugating Γ by

$$a = \begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix}$$

we can assume that $r = 1$. We set

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We can write

$$f(g) - f_P(g) = \int_0^1 (f(g) - f(\exp(xX)g))dg.$$

We write $g = nak$ with a as above and $r > t$. We may assume that $n = \exp(yX)$ with $0 \leq y \leq 1$. We have (recall $L_X u(g) = \frac{d}{dt}u(\exp(-tX)g)_{|t=0}, R_X u(g) = \frac{d}{dt}u(g\exp(tX))_{|t=0}$

$$\int_0^1 (f(g) - f(\exp(xX)g))dx = -\int_0^1 \int_0^x \frac{d}{ds} f(\exp((s+y)X)ak)dsdx =$$

$$= \int_0^1 \int_0^x L_X f(\exp((s+y)X)ak)dsdx$$

$$= -r^{-2} \int_0^1 \int_0^x R_{Ad(k)^{-1}X} f(\exp((s+y)X)ak)dsdx.$$

We now use the moderate growth property and see that the integrand in absolute value is dominated by a constant times $\|a\|^k \leq C\|g\|^k$ with C another constant Since $0 < x < 1$ the integration makes things no bigger and we wind up with the desired estimate since $r^2 = a(g)^\alpha$ for α the positive root

$$|f(g) - f_P(g)| \leq C_1 a(g)^{-\alpha} \|g\|^k$$

for $g \in S_{P,\omega,t}$ since $r^2 = a(g)^\alpha$ for α the positive root.

We will now make a few observations about the Lemma. First is that on $S_{P,\omega,t}$ there is are constants C_1, C_2, k_1, k_2 all positive real numbers such that

$$C_2 \|g\|^{k_2} \leq \beta(a(g)) \leq C_1 \|g\|^{k_1}.$$

Next we note that the proof only uses the $N \cap \Gamma$ invariance of f. We can replace $S_{P,\omega,t}$ by $\Gamma \cap N \backslash (NS_{P,\omega,t})$. And do the same argument for $f - f_P$. If we continue to iterate we find that

$$|f(g) - f_P(g)| \leq C_m \|g\|^{-m}$$

for all m for $g \in S_{P,\omega,t}$.

We define $\mathcal{A}^\infty_{cusp}(\Gamma\backslash G)$ to be the space of all $f \in \mathcal{A}^\infty(\Gamma\backslash G)$ such that $f_P = 0$ for all proper parabolic subgroups defined over $\mathbb{Q}$. The results and observations above imply

Theorem 116 *If $f \in \mathcal{A}^\infty_{cusp}(\Gamma\backslash G)$ then for each P, ω, t , m we have $|f(g)| \leq C_{P,\omega,t,m} \|g\|^{-m}$ for $g \in S_{P,\omega,t}$.*

Definition 117 *An element $f \in \mathcal{A}^{\infty}_{cusp}(\Gamma\backslash G)$ is called a cusp form.*

In the literature the condition $f \in \mathcal{A}(\Gamma\backslash G)$ is assumed for cusp forms. One can prove easily from the above theorem that $\mathcal{A}^{\infty}_{cusp}(\Gamma\backslash G) \subset L^2(\Gamma\backslash G)$. It is also clear that $\mathcal{A}^{\infty}_{cusp}(\Gamma\backslash G)$ is invariant under the operators $\pi_\Gamma(g)$ for $g \in G$. This implies that we have a G-invariant subspace $L^2_{cusp}(\Gamma\backslash G)$.

Langlands has shown (using an argument very similar to the proof of the lemma above see the Corolary on p.41 of Langlands) that if $f \in C^1_c(\Gamma\backslash G)$ then $\pi_\Gamma(f)_{|L^2_{cusp}(\Gamma\backslash G)}$ is of Hilbert-Schmidt class. This combined with the theorem of Dixmier-Malliavan mentioned earlier imples

Theorem 118 *If $f \in C^\infty_c(G)$ then $\pi_\Gamma(f)_{|L^2_{cusp}(\Gamma\backslash G)}$ is trace class.*

In particular we know that if $\omega \in \widehat{G}$ then $\dim Hom_G(H^\omega, L^2_{cusp}(\Gamma\backslash G)) < \infty$.

Corollary 119 *We have the decomposition*

$$L^2_{cusp}(\Gamma\backslash G) = \bigoplus_{\omega \in \widehat{G}} Hom_G(H^\omega, L^2_{cusp}(\Gamma\backslash G)) \bigotimes H^\omega.$$

Put in the simplest terms the "philosophy of cusp forms" says that for each Γ-conjugacy classes of $\mathbb{Q}$-rational parabolic subgroups one should construct automorphic functions (from objects from spaces of lower dimensions) whose constant terms are zero for other conjugacy classes and the constant terms for and element of the given class give all constant terms for this parabolic subgroup. This is almost possible and leads to a discription of all automorphic forms in terms of these constructs and cusp forms. The construction that does this is the Eisenstein series.

3.4 Eisenstein Series.

We retain the notation of the previous section. Fix P a parabolic subgroup of G defined over $\mathbb{Q}$. Let $P/N = {}^oMA$ as above. We take $K_M = \pi(K \cap P)$ where $\pi : P \to M = P/N$ is the natural projection. Let q be the projection of M onto ${}^oM = M/A$. We note that $K_M \cap A$ is finite and so we set $K_{{}^oM} = q(K_M)$. This notation becomes very heavy and so we will just assume that $K_M \subset {}^oM$. Before we define the Eisenstein series we will need to define our notion of Schwartz space. Let $P_1, ..., P_r$ be a complete set of Γ-conjugacy classes of parabolic sugroups defined over $\mathbb{Q}$. Let S_{P_i,ω_i,t_i} be Siegel sets such that

$$\bigcup_i \Gamma S_{P_i,\omega_i,t_i} = G.$$

We denote by $\mathcal{C}(\Gamma\backslash G)$ the space of all $f \in C^\infty(\Gamma\backslash G)$ satisfying

$$p_{i,k,x}(f) = \sup_{g=pk\in S_{P_i,\omega_i,t_i}} (1+\log\|g\|)^k \delta_P(p)^{\frac{1}{2}} |xf(g)| < \infty$$

(the expression $g = pk$ corresponds to the decomposition $G = P_i K$) for all i, k and $x \in U(\mathfrak{g})$. We endow $\mathcal{C}(\Gamma\backslash G)$ with the topology induced by the above seminorms. Then $\mathcal{C}(\Gamma\backslash G)$ is invariant under $\pi_\Gamma(g)$ (which you should recall is right translation) and defines a smooth Fréchet representation.

We are now ready to define Eisenstein series. Let

$$\varphi : K \to \mathcal{C}(\Gamma_M\backslash {}^oM)$$

be a continous mapping such that

$$\varphi(mk) = R_{\pi(m)}\varphi(k)$$

for $m \in K \cap P$. We define for each $\nu \in \mathfrak{a}_C^*$

$$\varphi_\nu(pk) = a(\pi(p))^{\nu+\rho}\varphi(k)(q \circ \pi(p)), p \in P, k \in K.$$

Here $a(ma) = a$ for $m \in {}^oM$ and $a \in A$ and we note that $\delta_P(N) = \{1\}$ so $\delta_P(a)$ makes sense for $a \in A$. We write $\delta_P(a)^{\frac{1}{2}} = a^\rho$. The transformation rule for φ implies that this formula does, indeed define a function on G. We note that φ_ν extended in this way is an element of $C((\Gamma \cap P)N\backslash G)$. We also note that if we take a Jordan-Hölder series for the action of P on $Lie(N)$. Then on the irreducible subquotients N acts trivially. Thus we have an action of M on the semisimplification (the direct sum of the irrecucibles), V. In particular, the characters of A that appear will be denotes by $\Sigma(P, A)$ and if m_α is the multiplicity of the character in V then $\rho = \frac{1}{2}\sum m_\alpha\alpha$.

We write

$$E(P, \varphi, \nu)(g) = \sum_{\gamma\in(\Gamma\cap P)\backslash\Gamma} \varphi_\nu(\gamma g).$$

One can show that the series in the above expression converges for $\mathrm{Re}(\nu,\alpha) > (\rho,\alpha)$ for all $\alpha \in \Sigma(P,A)$. These series are the Eisenstein series introduced in somewhat greater generality by Langlands. The simplest prof of the convergence of the series follows from a method of Godement explained by A.Borel in his article in *Proceedings of Symposia in Pure Mathematics*, 9. Introduction to Automorphic Forms, 199-210, Lemma 11.1.

The first step in their study is to give a meromorphic continuation of the Eisenstein series to all of $\mathfrak{a}_C^*$. Unfortunately, in this generality there is no such continuation known. In fact, the continuation is really only needed for the series that are constructed from φ that take $\mathcal{C}(\Gamma_M\backslash {}^oM) \cap \mathcal{A}^\infty(\Gamma_M\backslash {}^oM)$. In fact it is known even for values in $\mathcal{C}(\Gamma_M\backslash {}^oM) \cap \mathcal{A}(\Gamma_M\backslash {}^oM)$ this fact is comes at the end of the story (Chapter 7 in *Langlands*).

The first step is the continuation if the values are taken in $\mathcal{A}_{cusp}(\Gamma_M\backslash {}^oM)$. This is very difficult and very indirect in *Langlands* (a similar method can be found in Harish-Chandra's note Springer Lecture Notes, 62).

The next step is to examine the poles of the meromorphic continuation. Some of these poles have residues that are elements of $\mathcal{C}(\Gamma_M\backslash {}^oM) \cap \mathcal{A}(\Gamma_M\backslash {}^oM)$. The difficulty is to determine which ones have this property. This is the reason why Osborne and Warner developed their Eisenstein systems and Langlands developed his theory of "admissible subspaces" associated with parabolic subgroups defined over $\mathbb{Q}$. With almost infinite care (since we have no idea which poles are the important ones) one can identify certain residues with the analytic continuation of Eisenstein series constructed from enough elements of $\mathcal{C}(\Gamma_M\backslash {}^oM) \cap \mathcal{A}(\Gamma_M\backslash {}^oM)$. The theorem is

Theorem 120 *If $\varphi : K \to \mathcal{C}(\Gamma_M\backslash {}^oM) \cap \mathcal{A}(\Gamma_M\backslash {}^oM)$ is as above then the map*

$$(\nu, g) \longmapsto E(P,\varphi,\nu)(g)$$

extends to a meropmorphic function in ν in $\mathfrak{a}_C^*$ that is jointly smooth in G. Furthermore, the map

$$\nu \longmapsto E(P,\varphi,\nu)$$

is meromorphic as a map to the Fréchet space $\mathcal{A}^\infty(\Gamma\backslash G)$. Furthermore it is holomorphic in a neigborhood of $i\mathfrak{a}^*$.

3.5 The decomposition of $L^2(\Gamma\backslash G)$.

In this section we will give an short exposition of Langlands' decomposition of $L^2(\Gamma\backslash G)$. A somewhat more detailed account can be found in J.Arthur's paper in the *Proceedings of Symposia in Pure Mathematics*,33 part 1, 253-274 however he explains the results over the Adeles and so his exposition is of a less general case (the congruence subgroups). Let P be the real points of a parabolic subgroup defined over $\mathbb{Q}$. We will retain the notation of the previous section. If $\alpha \in C_c^\infty(\mathfrak{a}^*)$ and φ is as in the last section with values in $\mathcal{C}(\Gamma_M\backslash{}^oM) \cap \mathcal{A}(\Gamma_M\backslash{}^oM)$ then we set

$$E(P,\varphi,\alpha) = \int_{\mathfrak{a}^*} \alpha(\nu)E(P,\varphi,i\nu)d\nu.$$

We will call such a function a wave packet associated with P.

Theorem 121 *The function $E(P,\varphi,\alpha)$ is square integrable, indeed, there exists a constant c_P depending only on P and the normalizations of invariant measures such that*

$$\left\|E(P,\varphi,\alpha)\right\|_2^2 = c_P \left\|\alpha\right\|_2^2 \int_{K\times(\Gamma_M\backslash{}^oM)} |\varphi(k)(m)|^2 dk d(\Gamma_M m)$$

Here all of the norms indicated by $\|...\|_2$ are the corresponding L^2-norms.

We set $L^2_{cont}(\Gamma\backslash G)$ equal to the closure of the span of the $E(P,\varphi,\alpha)$ for all P,φ and α as above. Set $L^2_{disc}(\Gamma\backslash G)$ equal to the closure of the sum of the irreducible closed subspaces of $L^2(\Gamma\backslash G)$. Langlands' spectral decomposition can be stated as follows.

Theorem 122 *We have $L^2(\Gamma\backslash G) = L^2_{disc}(\Gamma\backslash G) \bigoplus L^2_{cont}(\Gamma\backslash G)$ orthogonal direct sum.*

We will end with one further decomposition due to Langlands. If P,Q are parabolic subgroups defined over $\mathbb{Q}$ then we say that they are associate if there is some Levi factor over $\mathbb{Q}$ of P that is conjugate via an element of $\mathbf{G}_\mathbb{Q}$ with a Levi factor over $\mathbb{Q}$ of Q.

We have

Theorem 123 *If P and Q are non-associate parabolics defined over $\mathbb{Q}$ then*

$$\langle E(P,\varphi,\alpha), E(Q,\psi,\beta)\rangle = 0$$

with $E(Q,\psi,\beta)$ a wave packet associated with Q.

We look at the elements of $L^2_{disc}(\Gamma\backslash G)$ as wave packets associated with G. We will use the notation $\{P\}$ for the set of parabolic subgroups associate to P. Let $L^2_{\{P\}}(\Gamma\backslash G)$ denote the closure of the span of all wave packets associated with the elements of $\{P\}$. Pick a set of representatives $P_1, ..., P_d$ for the associativity classes of parabolic subgroups defined over $\mathbb{Q}$. As a corrolary to the preceding two theorems we have.

Theorem 124 *The space*

$$L^2(\Gamma\backslash G) = \bigoplus_{\{P\}} L^2_{\{P\}}(\Gamma\backslash G)$$

a Hilbert space direct sum of irreducible spaces.

One can also prove that $\mathcal{C}(\Gamma\backslash G) = \bigoplus_{\{P\}} L^2_{\{P\}}(\Gamma\backslash G) \cap \mathcal{C}(\Gamma\backslash G$.