# Applicable Geometric Invariant Theory

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- Hilbert's problem number 14 asks for a proof of the existance of $f_1, ..., f_d$.
- Nagata showed that there are examples where there is no finite $d$.
- Using the Hilbert basis theorem one can show that if $G$ acts completely reducibly on $V$ (if $W \subset V$ is a subspace with $GW \subset W$ then there exists a complementary subspace $W'$ such that $GW' \subset W'$). Hilbert's finiteness theorem.

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- Hilbert's proof gave no technique to find a generating set. Instead he set the stage to find the structure of the affine variety that comes from the SFT.
- His idea was to find a geometric way of describing the null cone. That is, consider $\mathcal{O}(V)_+^G = \{f \in \mathcal{O}(V)^G | f(0) = 0\}$. The null cone, $\mathcal{N}$, is the set of $x \in V$ such that $f(x) = 0$ for $f \in \mathcal{O}(V)_+^G$.

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- Let $\overline{G}$ denote the Zariski closure of $G$ in $GL(V)$. We note that $\overline{G}$ is an affine variety that is also a group and the group operations are morphisms. We also note that it also acts completely reducibly. Such a Z–closed subgroup of $GL(V)$ will be called reductive.

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- The Hilbert–Mumford theorem says that if $x \in V$ and if $G \subset GL(V)$ is closed and reductive then $x$ is in the null cone if and only if there exists $\phi : \mathbb{C}^\times \to G$ a group homorphism that is also a morphism such that $\lim_{z \to 0} \phi(z)x = 0$.

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- Here is an example

- Consider $G = SL(2,\mathbb{C}) \otimes SL(2,\mathbb{C}) \otimes SL(2,\mathbb{C}) \subset GL(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$. Then $G$ acts irreducibly. Let $e_1, e_2$ be the standard basis of $\mathbb{C}^2$.

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- Set $W = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1$. If

$$\phi(z) = \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} \otimes \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} \otimes \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$$

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Then $\phi(z)W = zW$.
- So $W$ is in the null cone. One checks that $\dim GW = 7$ and there exists a nontrivial invariant (next bullet) so its closure is an irreducible hypersurface. It is thus given as the zero set of a single irreducible polynomial. $f$. We can write down a polynomal that is $G$ invariant as follows:

- Recall that on $\mathbb{C}^2$ we have an $SL(2,\mathbb{C})$ invariant skew form $\omega(x,y)=\det[x,y]$. So $(...,...)=\omega\otimes\omega$ is a $SL(2,\mathbb{C})\otimes SL(2,\mathbb{C})$ invariant symmetric form on $\mathbb{C}^2\otimes\mathbb{C}^2$. If $v\in\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ then $v=e_1\otimes v_1+e_2\otimes v_2$. Set

$$h(v)=\det[(v_i,v_j)]$$

is $G$ invariant and $h(GHZ)=-1$ with $GHZ=e_1\otimes e_1\otimes e_1+e_2\otimes e_2\otimes e_2$. Thus $GHZ$ is not in the nullcone.

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- Upshot: the null cone is $\overline{GW}$ and the complement of the null cone is the open dense set $\mathbb{C}^\times(G \cdot GHZ)$ and $\mathcal{O}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)^G = \mathbb{C}[h]$.

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- These two vectors have an interesting history. To put them in context I will first look at $SL(2,\mathbb{C}) \otimes SL(2,\mathbb{C})$ acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$.

- In this case one sees that the product vectors: that is the ones of the form $v \otimes w$ with $v, w \in \mathbb{C}^2$ are in the null cone. Since this set is $(SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C})) \cdot e_1 \otimes e_1 \cup \{0\}$. We also note that this set is Z–closed in $\mathbb{C}^2 \otimes \mathbb{C}^2$ and of dimension 3.

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- One checks that $\varphi(v) = (v, v)$ has this set as part of its zero set. But $\varphi$ is an irreducible polynomial so the set of product vectors is the zero set of $\varphi$.
- Also if $Bell = e_1 \otimes e_1 + e_2 \otimes e_2$ (a Bell state) then $\varphi(Bell) = 2$ and $\mathbb{C}^\times (SL(2,\mathbb{C}) \otimes SL(2,\mathbb{C}))\, Bell$ is open and dense in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

- In physics a state is a unit vector in a Hilbert space $H$ and two such vectors are identified if they are multiples. Thus the states are elements of the projective space of $H$, $\mathbb{P}(H)$. If $H$ is a tensor product of several Hilbert spaces $H_1 \otimes H_2 \otimes \cdots \otimes H_m$ then we will say that a state is entangled if it is not a product state.

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- In the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$ if a state is entangled then it is up to the action of $SL(2,\mathbb{C}) \otimes SL(2,\mathbb{C})$ $(/\frac{gv}{\|gv)\|})$ in the orbit of the Bell state $\frac{1}{\sqrt{2}}B$.

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- It was observed that in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ that there were two types of highly entangled vectors $\frac{1}{\sqrt{2}}GHZ$ and $\frac{1}{\sqrt{3}}W$. These wierd notions will be somewhat more explained in my third lecture.

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- It is also equivalent to the condition that there exists a Hilbert space structure on $\mathbb{C}^n$, $\langle ..., ... \rangle$ such that $G$ is a Z–closed subgroup of $GL(n, \mathbb{C})$ such that $G$ is invariant under adjoint.
- Taking an orthonormal basis of $\mathbb{C}^n$ we may assume that the inner product is the standard one and the adjoint is conjugate transpose. If $K = G \cap U(n)$ then $G$ is the Zariski closure of $K$. Furthermore $K$ is a maximal compact subgroup and all maximal compact subgroups of $G$ are conjugate to $K$.

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- Obviously we have the ring homomorphism $\mathcal{O}(\mathbb{C}^n)^G \to \mathcal{O}(\mathbb{C}^n)$ which yields the surjective morphism of varieties $p : \mathbb{C}^n \to \mathbb{C}^n//G$. The null cone is a fiber of this map, $p^{-1}(p(0))$.

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- What about the other fibers, $p^{-1}(p(x))$? They can be described as the set of $y \in \mathbb{C}^n$ such that if $f \in \mathcal{O}(\mathbb{C}^n)^G$ then $f(y) = f(x)$.

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- If $Y$ and $Z$ are closed $G$–invariant disjoint subsets of $\mathbb{C}^n$ then there exists $f \in \mathcal{O}(\mathbb{C}^n)^G$ such that $f_{|X} = 1$ and $f_{|Y} = 0$.

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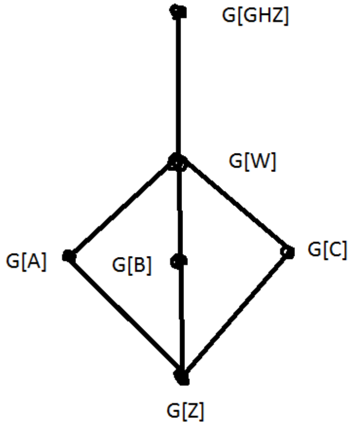
- Thus the variety $\mathbb{C}^n//G$ parametrizes the closed orbits.
- Consider the example $G = SL(2,\mathbb{C}) \otimes SL(2,\mathbb{C}) \otimes SL(2,\mathbb{C}) \subset GL(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$. We have seen that $\mathcal{O}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)^G = \mathbb{C}[f]$. So $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2//G$ is isomorphic with $\mathbb{C}$ as an affine variety. Also note that $h(zGHZ) = -z^4$.We have also seen that $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 - \mathbb{C}^\times GHZ$ is the null cone. We consider the orbits in projective space.




$Z = e_1 \otimes e_1 \otimes e_1$,
$A = e_1 \otimes Bell$, $B = e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_1 \otimes e_2$, $C = Bell \otimes e_1$.

- Note that $\mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)/G$ has 6 points but the projectivized categoric quotient has one point.

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- The extended Hilbert-Mumford theorem says that if $G$ is Z–closed and reductive in $GL(V)$ if $v \in V$ then there exists

$$\phi : \mathbb{C}^\times \to G$$

a morphic group homomorphism such that $\lim_{z \to 0} \phi(z)v = x$ with $Gx$ the unique closed orbit in $X_v$.

- Consider $V = e_2 \otimes e_2 \otimes e_2 + W$ how would we tell if the orbit is closed? Note $h(V) = -4$ thus a multiple of $v$ is in the orbit of $GHZ$. But suppose we don't know the invariants?

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- For example $Lie(SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C}))$ is the set of all $X \otimes I \otimes I + I \otimes Y \otimes I + I \otimes I \otimes Z$ with $trX = trY = trZ = 0$. If $X \in M_n(\mathbb{C})$ we use the notation $X^j$ for the operator on $\otimes^m \mathbb{C}^n$ that acts by $I \otimes I \otimes \cdots \otimes I \otimes X \otimes I \otimes \cdots$ that is the only tensor entry that isn't $I$ is $X$ in the $j$–th position.

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- We denote the standard inner product on $\mathbb{C}^n$ by $\langle ..., ... \rangle$.

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- Testing $GHZ = e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2$ then $E_{12}^1 GHZ = e_1 \otimes e_2 \otimes e_2$. Which is purpendicular to $GHZ$. In fact it is critical.

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- We note that using the invariant $h$ we know that there exists $g \in G$ so that $gV$ is critical.
- The Kempf-Ness theorem tells us in full generality that $Gv$ is closed if and only if there exists a critical element in $Gv$.

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- Reformulating Set $\mathrm{Crit}_G(\mathbb{C}^n)$ equal to the set of critical points then

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  for an appropriate element $w$ of the orbit.
- In our example suppose we decide to use $|h(v)|$ as a measure of the entanglement of a state $v$ in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

- We note that if $G = SL(2,\mathbb{C}) \otimes SL(2,\mathbb{C}) \otimes SL(2,\mathbb{C})$ then $h(gv) = h(v)$. So
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- $|h|$ is a standard measure of entanglement in quantum information theory (it is basically the tangle).

- We note that $\left|h(\frac{1}{\sqrt{2}}GHZ)\right| = \frac{1}{4}$ and $\left|h(\frac{1}{2}V)\right| = \frac{1}{4}$ thus $V \in K \cdot \frac{1}{\sqrt{2}}GHZ$.

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- We return to the general situation. We have a finite dimensional, complex Hilbert space $(V, \langle ..., ... \rangle)$ and $G \subset GL(V)$ Zariski closed subgroup that is invariant under complex conjugation. We have defined $V//G = \mathrm{Spec}_m \mathcal{O}(V)^G$. We will now use the Kempf-Ness theorem to make this more concrete.

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- Recall that $\mathrm{Crit}(V) = \mathrm{Crit}_G(V)$ is the real algebraic subvariety consisting of all those elements of $V$ such that $\langle Xv, v \rangle = 0$ for all $X \in Lie(G)$. We have seen that $v \in V$ has a closed $G$–orbit if and only if $Gv \cap \mathrm{Crit}(V) \neq \emptyset$. Also, if $w \in Gv \cap \mathrm{Crit}(V)$ then

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- We have also seen that $V//G$ parametrizes the closed orbits. Thus we have a map

$$\psi : V//G \to \mathrm{Crit}(V)/K.$$

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- This result was first oberved by Richardson and Slodowy.

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- More generally we might have an orthonormal basis $|0\rangle, |1\rangle, \ldots$ and a state $\sum a_i\,|i\rangle$ with $\sum |a_i|^2 = 1$. A measurement adapted to the basis will yield an attribute associated with the basis vector $|j\rangle$ with probability $|a_j|^2$ with the collapse to $|j\rangle$.

- If we have two particles with corresponding Hilbert spaces $H_1$ and $H_2$ then the Hilbert space for the joint states of the particles is $H_1 \otimes H_2$ and if we have two bases corresponing to measurements $|i\rangle$ and $|j\rangle$ we have a basis of the tensor product

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- If however we have $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ then a measurement in the first yielding say $|0\rangle$ with probablity $\frac{1}{2}$ causes the second electron to collapse to $|0\rangle$ . If the measurement yielded $|1\rangle$ in the first the second would collapes to $|1\rangle$ .

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- We consider $H_1, ..., H_r$ finite dimensional Hilbert spaces each of dimension at least 2 and $SL(H_1) \otimes SL(H_2) \otimes \cdots \otimes SL(H_r)$ acting on $H_1 \otimes H_2 \otimes \cdots \otimes H_r$.

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- A measure of entanglement is the value of a real polynomial invariant under $U(H_1) \otimes U(H_2) \otimes \cdots \otimes U(H_r)$ the that vanishes on the product states and satisfies a monotonicity condition under LOCC (Local Operations and Classical Communication).

- LOCC is a combination of a unitary operators in one tensor factor, a measurement in one factor and communication between devices that have measured the different factors. If one adds some probability to the outcome one gets SLOCC (S for stochastic). Invertible SLOCC is the action of $SL(H_1) \otimes SL(H_2) \otimes \cdots \otimes SL(H_r)$ as above ($v \mapsto \frac{gv}{\|gv\|}$).

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- When all of the Hilbert spaces are of dimension 2 then the states in the tensor product space are called qubits. This case is of particular interest in quantum computing and quantum information.

- Note that the standard orthonormal basis of $\mathbb{C}^2$, $|0\rangle, |1\rangle$ yields an orthonormal basis of $\otimes^n \mathbb{C}^2$,

$$|0...00\rangle, |0...01\rangle, |0...10\rangle, |0...11\rangle, ..., |1...11\rangle.$$

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- In the case of 2 qubits a very convenient orthonormal basis consisting of entangled states is the set of Bell states

$$u_0 = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), u_1 = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$

$$u_2 = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), u_3 = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

- Note that the standard orthonormal basis of $\mathbb{C}^2$, $|0\rangle, |1\rangle$ yields an orthonormal basis of $\otimes^n \mathbb{C}^2$,

$$|0...00\rangle, |0...01\rangle, |0...10\rangle, |0...11\rangle, ..., |1...11\rangle.$$

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- Each of these maximize the absolute value of the quadratic $G = SL(2,\mathbb{C}) \otimes SL(2,\mathbb{C})$ invariant $(\phi(v) = (\omega \otimes \omega)(v,v))$. Arguing as we did for the tangle the states are all critical. Furthermore the critical set is $\mathbb{C}Ku_0$ and the set $\mathbb{C}^\times Gu_0$ is Zariski dense.

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- Many physicists think that $W$ and $GHZ$ are of coequal entanglement. Many others (including the author of the Wikipedia article on it) feel that $\frac{1}{\sqrt{2}}(|00...0\rangle + |11...1\rangle)$ in $n$–tensor factors is maximally entangled. They are correct for $n = 2, 3$ but we shall see that for $n \geq 4$ its entanglement is mediocre

- In 4 qubits we have the orthonormal basis $u_i \otimes u_j$, $0 \leq i, j \leq 3$. Of particular implortance are the 4 vectors $v_i = u_i \otimes u_i$, $i = 0, 1, 2, 3$. In this case one can show that if $\mathfrak{a} = \mathbb{C}v_0 + \mathbb{C}v_1 + \mathbb{C}v_2 + \mathbb{C}v_3$ then with $G = SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C})$, $G\mathfrak{a}$ has non-empty Zariski interior. Thus the restriction map $\mathcal{O}(\otimes^4\mathbb{C}^2)^G \to \mathcal{O}(\mathfrak{a})$ is injective.

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- Note that the analogue of $GHZ$ is $\frac{1}{\sqrt{2}}(v_0 + v_1)$. A generic state in $\mathfrak{a}$ would be one given by $\sum x_i v_i$ with $x_i \neq \pm x_j$ for $i \neq j$. These are precisely the states on which the hyperdeteminant is non-zero. Furthermore the stabilzer of such a state in $G$ is finite. One can show that the stabilizer of the analogue of $GHZ$ is 3 dimensional.

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- Let $\mu_j(\sum x_i v_i) = \sum x_i^j$ and $\phi(\sum x_i v_i) = x_0 x_1 x_2 x_3$. Then $\mu_2, \mu_4, \phi, \mu_6$ generate the restriction of the invariants.

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- The set of critical points is $K\mathfrak{a}$. But in this case there is no $K$–orbit that gets singled out. Moreover in projective space there are 90 orbit types.
- Gilad Gour and I did an extensive study of the states in four qubits the title of the article is “All maximally entangled states in 4 qubits". You can find the paper in the archive by searching for me in quantphys.

- Consider $\otimes^m \mathbb{C}^n$ as the tensor product Hilbert space. We take $G = \otimes^m SL(n, \mathbb{C})$. If we choose $k$ of the $m$ factors then we can think of a state $v$ as a linear map, $T_{v,S}$, of $\otimes^{m-k}\mathbb{C}^n$ to $\otimes^k \mathbb{C}^n$ ($S$ is the set of $k$ factors).

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- A state is critical if and only if for every choice of one factor $TT^* = \frac{1}{2} I$.
- For a general such reduced trace we see that $\mathrm{tr}(TT^*) = 1$.

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- To write down the states we use the notation $\langle j_1 j_2 ..., j_5 \rangle$ is the sum of the five cycles. $|j_1 j_2 ... j_5\rangle + |j_5 j_2 ... j_4\rangle + ... + |j_2 j_3 ... j_1\rangle$. Then the orthonormal code space has orthonormal basis

$$t_1 = \frac{1}{4}\left(|0000\rangle + \langle 11000\rangle - \langle 10100\rangle - \langle 11110\rangle\right)$$

$$t_2 = \frac{1}{4}\left(|11111\rangle + \langle 00111\rangle - \langle 01011\rangle - \langle 00001\rangle\right)$$

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- One can show a necessary and sufficient condition for a state in $\otimes^n \mathbb{C}^2$ to be part of an (orthogonal) error correcting code is that if whenever we consider it to be bipartite as an element of

$$\otimes^{n-2}\mathbb{C}^2 \otimes \otimes^2\mathbb{C}^2$$

by choosing any of the two factors the Von Neumann entropy is the maximal value it could have (in this case $\log(4)$).

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- I will conclude with some results for trits. For example several independent particles of spin 1. Consider $G = SL(3,\mathbb{C})\otimes SL(3,\mathbb{C})\otimes SL(3,\mathbb{C})$ acting on $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$.

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- We write

$$u_0 = \frac{|000\rangle + |111\rangle + |222\rangle}{\sqrt{3}},$$

$$u_1 = \frac{|012\rangle + |201\rangle + |120\rangle}{\sqrt{3}},$$

$$u_2 = \frac{|210\rangle + |021\rangle + |102\rangle}{\sqrt{3}}.$$

- We set $\mathfrak{a} = Span\{u_0, u_1, u_2\}$. Then $K\mathfrak{a}$ is the set of critical elements. Furthermore, $\text{Crit}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)/K = \mathfrak{a}/W$ with $W$ a finite group. Vinberg's theory of $\theta$–groups (or Springer's theory of regular subgroups of reflection groups) which relatesw this example to $E_6$ implies that $W$ is a finite (complex) reflection group.

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- The theory of Shephard-Todd now implies that $\mathcal{O}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)^G$ is a polynomial algebra in three homogeneous generators. Springer's theory implies that we must take the degrees to be $6, 9, 12$.