1 Some algebraic geometry

1.1 Basic definitions.

1.1.1 Zariski and standard topology.

Let F be a field (almost always $\mathbb{R}$ or $\mathbb{C}$). Then we say that a subset $X \subset F^n$ is *Zariski closed* if there is a set of polynomials $S \subset F[x_1, ..., x_n]$ such that $X = F^n(S) = \{x \in F^n | s(x) = 0, s \in S\}$. One checks that the set of Zariski closed sets, Z, in F^n satisfies the axioms for the closed sets in a topology. That is

- F^n and $\emptyset \in Z$.
- If $T \subset Z$ then $\cap_{Y \in T} Y \in Z$.
- If $Y_1, ..., Y_m \in Z$ then $\cup_j Y_j \in Z$.

The Hilbert basis theorem (e.g. A.1.2 [GW]) implies that we can take the sets S to be finite.

If X is a subset of F^n then the *Z-topology* on X is the subspace topology corresponding to the Zariski topology.

If $F = \mathbb{R}$ or $\mathbb{C}$ then we can also endow F^n with the *standard metric topology* corresponding to $d(x, y) = \|x - y\|$ where $\|u\|^2 = \sum_{i=1}^n |u_i|^2$ for $u = (u_1, ..., u_n)$. We will call this topology the standard topology (in the literature it is also called the classical, Hausdorff or Euclidean topology). Our first task will be to study relations be tween these very different topologies.
Examples. $n = 1$. Then if $Y \in Z$ then $Y = F$ or Y is finite.

$n = 2$.A Zariski closed subset is a finite union of plane curves or all of F^2. Thus the Zariski topology on F^2 is not the product topology.

1.1.2 Noetherian topologies.

The Zariski topology on a closed subset, X, of F^n is an example of a Noetherian topology. That is if $Y_1 \supset Y_2 \supset ... \supset Y_m \supset ...$ is a decreasing sequence of closed subsets of X then there exists N such that if $i, j \geq N$ then $Y_i = Y_j$ (this is a direct interpretation of the Hilbert basis theorem).

If X is a topological space that be cannot written $X = Y \cup Z$ with Y and Z closed and both are proper the X is said to be irreducible. Clearly an irreducible space is connected but the converse is not true.

Exercise. What are the irreducible Hausdorff topological spaces?

Lemma 1 *Let X be a Noetherian topological space. Then X is a finite union of irreducible closed subspaces. If $X = X_1 \cup X_2 \cup \cdots \cup X_m$ with X_i closed and irreducible and if $X_i \not\subseteq X_j$ for $i \neq j$ then the X_i are unique up to order.*

For a proof see A.1.12 [GW].The decomposition $X = X_1 \cup X_2 \cup \cdots \cup X_m$ with X_i closed and irreducible and $X_i \not\subseteq X_j$ for $i \neq j$ is called the *irredundant decomposition* of X into irreducible components. Each of the X_i is called an *irreducible component.*

1.1.3 Nullstellensatz.

If X is a subset of F^n then we set $\mathcal{I}_X = \{f \in F[x_1, ..., x_n] | f_{|X} = 0\}$. If $\mathcal{I} \subset F[x_1, ..., x_n]$ is an ideal then we set $F^n(\mathcal{I}) = \{x \in F^n | f(x) = 0, f \in \mathcal{I}\}$.

Lemma 2 *A closed subset, X, of F^n is irreducible if and only if $\mathcal{I}_X$ is a prime ideal (i.e. $F[x_1, ..., x_n]_{|X}$ is an integral domain).*

We now recall the Hilbert Nullstellensatz.

Theorem 3 *Let F be algebraically closed and let $\mathcal{I}$ be an ideal in $F[x_1, ..., x_n]$. Then*

1. *$F^n(\mathcal{I}) = \emptyset$ if and only if $\mathcal{I} = F[x_1, ..., x_n]$.*
2. *$I_{X(I)} = \sqrt{\mathcal{I}}$.*

We will give a proof of 1. in this theorem for the case of $F = \mathbb{C}$ since the method of proof will be consistent with the techniques of the later material to be presented. That 1. implies 2. is the trick of Rabinowitz which we defer to the references.

We prove that if I is a proper ideal then $\mathbb{C}^n(\mathcal{I}) \neq \emptyset$. Let $\mathcal{M}$ be a maximal proper ideal with $\mathcal{I} \subset \mathcal{M}$. Then $k = \mathbb{C}[x_1, ..., x_n]/\mathcal{M}$ is a field containing $\mathbb{C}$. Since the image of the monomials $x_1^{m_1} \cdots x_n^{m_n}$ span k as a vector space over $\mathbb{C}$ we see that $\dim_{\mathbb{C}} k$ is at most countable. Since $\mathbb{C}$ is algebraically closed, if $k \neq \mathbb{C}$ there must be $t \in k$ that is transcendental over $\mathbb{C}$. We assert that the elements $\{\frac{1}{t-a} | a \in \mathbb{C}\}$ are independent over $\mathbb{C}$. Indeed, if

$$\sum_{i=1}^{m} \frac{c_i}{t - a_i} = 0$$

with $a_1, ..., a_m \in \mathbb{C}$ and distinct and $c_1, ..., c_m$ in $\mathbb{C}$ then multiplying by

$$\prod_{i=1}^{m}(t - a_i)$$

we find that

$$\sum_{j=1}^{m} c_j \prod_{i \neq j}(t - a_i) = 0.$$

This defines a polynomial satisfied by t that is trivial if and only if all of the c_j are 0 (evaluate at a_i and get $c_i \prod_{j \neq i}(a_i - a_j)$) Since $\mathbb{C}$ is not countable this is a contradiction. We have proved that k is one dimensional over $\mathbb{C}$ with basis $1 + \mathcal{M}$. Thus $x_i + \mathcal{M} = z_i 1 + \mathcal{M}$ with $z_i \in \mathbb{C}$. We have shown that $x_i - z_i \in \mathcal{M}$ for all $i = 1, ..., n$. Since the ideal $\mathcal{I}_{\{z\}}$ is maximal we see that $\mathcal{M} = \mathcal{I}_{\{z\}}$ and since $\mathcal{I} \subset \mathcal{M}$ we have $z \in \mathbb{C}^n(\mathcal{I})$.

1.2 Affine varieties and dimension.

1.2.1 Affine varieties.

Let X, Y be sets and let $f : X \to Y$ be a mapping. If g is a function from Y to F then we set $f^*g = g \circ f$ which is a function on X.

In this section we will assume that F is algebraically closed (you can assume it is $\mathbb{C}$). An *affine variety* over F is a pair (X, R) with X a topological space and R and algebra over F of continuous functions from X to F (F is endowed with the Zariski topology) such that there exists a Z-closed subset Z in F^n for some n and a homeomorphism, f, of X onto Z such that $f^* : F[x_1, ..., x_n]_{||Z} \to R$ is an algebra isomorphism. A morphism affine varieties (X, R) and (Y, S) is a map $f : X \to Y$ such that $f^*S \subset R$. An isomorphism is a morphism that is one to one and onto and f^* is an isomorphism of algebras.

We note that if (X, R) is an affine variety then R must be a finitely generated algebra over F and if $a \in R$ is such that $a^k = 0$ for some $k > 0$ then $a = 0$. That is, the only nilpotent element in R is 0. The converse is also true. Let R be a finitely generated algebra over F without nilpotents. Let $r_1, ..., r_m$ generate R as an algebra over F. Then we have an algebra homomorphism $\mu : F[x_1, ..., x_m] \to R$ defined by $\mu(x_i) = r_i$. Let $\mathcal{I} = \ker \mu$ and set $Z = F^n(\mathcal{I})$. Then since $R \cong F[x_1, ..., x_m]/\mathcal{I}$ and R has no nilpotents we must

have $\sqrt{\mathcal{I}} = \mathcal{I}$. Thus $\mathcal{I} = \mathcal{I}_Z$ (by the nullstellensatz). Thus R is isomorphic with $\mathcal{O}(Z) = F[x_1, ..., x_m]_{|Z}$. Actually more is true. The topological space X is also completely determined by R. Indeed, the nullstellensatz implies that the maximal ideals in $\mathcal{O}(Z)$ are the deals $\mathcal{I}_{\{z\}} + \mathcal{I}$ for $z \in Z$. Thus the set Z can be identified with the set of maximal ideals in $\mathcal{O}(Z)$. The topology is determined as follows. If Y is a closed subset of Z and $\mathcal{J}$ is the ideal of elements in $\mathcal{O}(Z)$ such that $Y = \{z \in Z | g(z) = 0, g \in \mathcal{J}\}$ then $Y + \{z \in Z | \mathcal{J} \subset \mathcal{I}_{\{z\}}\}$ for $\mathcal{I}$ an ideal of R. Returning to R we may define $\mathcal{M}_R$ to be the set of maximal proper ideals endowed with the topology that has as its closed sets the sets $\mathcal{M}_R(\mathcal{I}) = \{\mathcal{M} |$ maximal with $\mathcal{I} \subset \mathcal{M}\}$. We have just seen that if we use the fact that R is isomorphic with $\mathcal{O}(Z)$ and the maximal ideals in $\mathcal{O}(Z)$ are the images of the ideals $\mathcal{I}_{\{z\}}$ for $z \in Z$. This implies that if $\mathcal{M} \in \mathcal{M}_R$ then $R/\mathcal{M} = F1 + \mathcal{M}$. Thus if $r \in R$ then we can define $r(\mathcal{M}) = c$ if $r = c1 + \mathcal{M}$. We therefore have made R into an algebra of F valued functions on $\mathcal{M}_R$. The bottom line is that all of the information is in the algebra R.

Recall that if $F = \mathbb{C}$ and if $Y \subset \mathbb{C}^n$ is Z-closed then we also have the standard (metric) topology on Y. If (X, R) is an affine variety and if (X, R) is isomorphic with $(Y, \mathbb{C}[x_1, ..., x_n]_{|Y})$ and $(V, \mathbb{C}[x_1, ..., x_m]_{|V})$ with Y Z-closed in $\mathbb{C}^n$ and V Z-closed in $\mathbb{C}^m$. Then we have $f : X \to Y$ and $g : X \to V$ homeomorphisms in the Z-topology such that $f^* : \mathbb{C}[x_1, ..., x_n]_{|Y} \to R$ and $g^* : \mathbb{C}[x_1, ..., x_m]_{|V} \to R$ algebra isomorphisms. This $g \circ f^{-1} : Y \to V$ is given by the map $\Phi = (\phi_1, ..., \phi_m)$ with $\phi_j = (g \circ f^{-1})(x_{j|V}) \in \mathbb{C}[x_1, ..., x_n]_{|Y}$. Thus $g \circ f^{-1}$ is continuous in the standard metric topology. This implies that endowed with the standard topology Y and V are homeomorphic. We therefore see that this implies that (X, R) also has a natural metric topology which we will also call standard.

We will say that an irreducible affine variety (X, R) is *normal* if R is integrally closed in $K(X)$ the quotient field of R.

Theorem 4 *If (X, R) is an irreducible affine variety then if S is the integral closure of R in $K(X)$ then S is finitely generated over F and $(\mathcal{M}_S, S)$ is a normal affine variety.*

For a proof see [S] Theorem II.5.4. We will call the variety $(\mathcal{M}_S, S)$ with the map $M \to z$ with $z \in X$ such that $M \cap R = M_z$ the *normalization* of (X, R) one can show that this map is finite (in particular finite to one).

1.2.2 Dimension.

Let (X, R) be an irreducible affine variety over an algebraically closed field F. Then R is an integral domain. We denote by $K(X)$ the quotient field of R. Then $K(X)$ is a field extension of F.If (X, R) is isomorphic with a Z-closed subset of F^n then the field $K(X)$ is isomorphic is generated as a field by at most n generators. We define the *dimension* of X to be the transcendence degree of $K(X)$ over F.

Example. $(F^n, F[x_1, ..., x_n])$ has dimension n since $K(F^n) = F(x_1, ..., x_n)$ the field of rational functions in n variables. If $X \subset F^n$ is Z-closed then $\dim X \leq n$.

This is a precise definition but is hard to compute. We will now discuss an effective way to calculate the dimension of an affine variety.

Let R be an algebra over F then a *filtration* of R is an increasing sequence $\mathcal{F}_0 R \subset \mathcal{F}_1 R \subset ... \subset \mathcal{F}_m R \subset ...$ of subspaces of R such that $\mathcal{F}_i R \cdot \mathcal{F}_j R \subset \mathcal{F}_{i+j} R$ If $\mathcal{F}$ is a filtration of R then we define $Gr\mathcal{F}R = \bigoplus_{j \geq 0} \mathcal{F}_j R / \mathcal{F}_{j-1} R$ $(\mathcal{F}_{-1} R = \{0\})$ with multiplication

$$(a + \mathcal{F}_{i-1} R)(b + \mathcal{F}_{j-1} R) = ab + \mathcal{F}_{i+j-1} R$$

for $a \in \mathcal{F}_i R$ and $b \in \mathcal{F}_j R$. If we set $Gr\mathcal{F}^j R = \mathcal{F}_j R / \mathcal{F}_{j-1} R$ then $Gr\mathcal{F}R$ is a graded algebra over F. Here, a graded algebra over F is an algebra S such that as a vector space over F, $S = \bigoplus_{j \geq 0} S^j$ such that $S^i S^j \subset S^{i+j}$.

We will say that a filtration, $\mathcal{F}$, of R is *good* if $\dim Gr\mathcal{F}^1 R < \infty$ and it generates $Gr\mathcal{F}R$ as an algebra over F.

Example. Let $X \subset F^n$ be Z-closed with $R = F[x_1, ..., x_n]_{|X}$. Then we set $\mathcal{F}_j R$ equal to the span of the restrictions $x^a_{|X}$ with $a = (a_1, ..., a_n) \in \mathbb{N}^n$and $x^a = x_1^{a_1} \cdots x_n^{a_n}$. Then $Gr\mathcal{F}R$ is generated by the images of the x_i in $Gr\mathcal{F}^1 R$.

Theorem 5 *Let R be an algebra over F with a good filtration $\mathcal{F}$ then there exists, $N \in \mathbb{N}$ and a polynomial $h_{\mathcal{F}}(q)$ such that $\dim \mathcal{F}_j R = h_{\mathcal{F}}(j)$ for $j \geq N$. Furthermore, if $\mathcal{F}$ and $\mathcal{G}$ are good filtrations of R then $\deg h_{\mathcal{F}} = \deg h_{\mathcal{G}}$.*

Examples.

1. Let $X = F^n$ with $R = F[x_1, ..., x_n]$ with the filtration, FT, given by degree as in the example above. Then $\dim F_j R = \binom{n+j}{n} = \frac{j^n}{n!} +$ lower in j.

2. Let $f \in F[x_1, ..., x_n]$ be a non-constant polynomial. Let

$$R = F[x_1, ..., x_n] / fF[x_1, ..., x_n]$$

on R we filter by degree in the x_i. Then if $\deg f = m > 0$ we have

$$0 \to \mathcal{F}_{k-j}F[x_1, ..., x_n] \to \mathcal{F}_k F[x_1, ..., x_n] \to \mathcal{F}_k R \to 0$$

is exact with the map from filtered degree $k-j$ to k given by multiplication by f. Thus

$$\dim \mathcal{F}_k R = \binom{n+k}{n} - \binom{n+k-j}{n} = j\frac{k^{n-1}}{(n-1)!} + \text{lower degree in } k.$$

Theorem 6 *Let (X, R) be an affine variety over F then if $\mathcal{F}$ is a good filtration of R then $\deg h_{\mathcal{F}}$ is the maximum of the dimensions of the irreducible components of X.*

We will denote by $\mathrm{Dim}R$ the degree of $h_{\mathcal{F}}$ for $\mathcal{F}$ a good filtration of R.

More important for computations is

Theorem 7 *Let I be an ideal in $F[x_1, ..., x_n]$ then*

$$DimF[x_1, ..., x_n]/I = DimF[x_1, ..., x_n]/\sqrt{I}.$$

This implies that we can compute the dimension of a variety from a defining set of equations.

We also note that Theorem 6 tells us that we should (and do) define $\dim X = \mathrm{Dim}R = \max \dim Z$ for Z an irreducible component of X.

We record here two results that we will need later.

Theorem 8 *If X is an irreducible affine variety and if Y is a closed subvariety then $\dim Y \leq \dim X$ with equality if an only if $Y = X$.*

Theorem 9 *Let (X, R) be an irreducible affine variety of dimension n. Then if f is a non-constant element of R then every irreducible component of $X(f) = \{x \in X | f(x) = 0\}$ is of dimension $n-1$.*

For a proof see Theorem I.6.5 p. 74 in [S].

1.2.3 Tangent space.

If $f \in F[x_1, ..., x_n]$ then we define for

$$p = (p_1, ..., p_n) \in F^n, df_p : F^n \to F$$

by

$$df_p(z_1, ..., z_n) = \sum \frac{\partial f}{\partial x_i}(p) z_i.$$

Here $\frac{\partial f}{\partial x_i}(p)$ is the formal derivative. We have the Taylor formula

$$f(p+x) = f(p) + df_p(x) + \sum_{|\alpha| \geq 2} c_\alpha x^\alpha =$$

$$f(p) + df_p(x) + \sum_i x_i g_i(x)$$

here for $\alpha = (\alpha_1, ..., \alpha_n)$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and g_i is an appropriate choice of polynomial which can choose so that $g_i(0) = 0$. If $F = \mathbb{R}$ or $\mathbb{C}$ this coincides with the usual derivative. We set $T_p(X) = \{v \in F^n | df_p(v) = 0, f \in \mathcal{I}_X\}$. We will now define this space in terms of the algebra $R = F[x_1, ..., x_n]_{|X}$. If $p \in X$ then we set $\mathcal{M}_p = \{f \in R | f(p) = 0\}$. Then the *Zariski cotangent space* to X at p is the vector space $\mathcal{M}_p/\mathcal{M}_p^2$. We will now define a natural pairing between this space and $T_p(X)$. If $f \in R$ and $f(p) = 0$ then we choose $g \in F[x_1, ..., x_n]$ such that $g_{|X} = f$. If $z \in T_p(X)$ then we assert that $dg_p(z)$ depends only on f and not on the choice of g. Indeed, if $h_{|X} = f$ then $g - h \in I_X$. Thus $d(g-h)_p(z) = 0$. Now if $f \in \mathcal{M}_p^2$ then f is a sum of elements of the form $u_1 u_2$ with $u_i \in \mathcal{M}_p$. If $h_{i_{|X}} = u_i$ then using

$$d(h_1 h_2)_p = h_1(p) d(h_2)_p + h_2(p) d(h_1)_p = 0$$

we see that we do indeed have a natural (independent of choices) pairing of $\mathcal{M}_p/\mathcal{M}_p^2$ and $T_p(X)$. Furthermore, if $f(p) = 0$ and $df_p = 0$ then $f + \mathcal{I}_X \in \mathcal{M}_p^2$ (see Taylor's formula above).

We thus have an intrinsic definition of $T_p(X) = (\mathcal{M}_p/\mathcal{M}_p^2)^*$.

Theorem 10 *Let X be an irreducible affine variety. If $p \in X$ then* $\dim T_p(X) \geq \dim X$. *Furthermore,* $\dim X = \min_{p \in X} \dim T_p(X)$.

For this c.f. [GW] A.3.2. We have

Theorem 11 *Let X be an irreducible affine variety then the set of $p \in X$ such that $\dim T_p(X) = \dim X$ is Z-open.*

We can see this as follows. We assume that X is Z-closed in F^n. Let $d = \dim X$. Then if $f = (f_1, ..., f_{n-d}) \in \mathcal{I}_X^{n-d}$ then we set U_f equal to the subset of X consisting of those points, p, such that some $(n-d) \times (n-d)$ minor of

$$\left[\frac{\partial f_i}{\partial x_j}(p)\right]$$

is non-zero. One sees that each U_f is Z-open in X (perhaps empty) and the union of the U_f is the set of smooth elements.

Let X be an irreducible affine variety. We will call a point *smooth* if $\dim T_p(X) = \dim X$.

In our application of these ideas we will use (the case $n = 1$) of the following theorem.

Theorem 12 *If (X, R) is a normal affine variety then the set of singular points (i.e. not smooth points) is Z-closed and of dimension at most $n-2$.*

For a proof of this see [S], Theorem II.5.3 p.117. This result implies that since $\dim \emptyset = -1$ we have the normalization of a curve is smooth (all points are smooth).

We will need to extend the concept of tangent space to non-irreducible varieties, For this we will need some new concepts.

1.2.4 Affine varieties revisited.

We take F to be algebraically closed. Let (X, R) be an affine variety. Let $\gamma \in R$ be non-constant. Let $X_{\{\gamma\}} = \{x \in X | \gamma(x) \neq 0\}$. If (X, R) is isomorphic with the affine algebraic set $Y = F^n(S)$ then R is isomorphic with $\mathcal{O}(Y) = F[x_1, ..., x_n]_{|Y}$ under the map $f : X \to Y$ (i.e. f is a Z-homeomorphism and $f^* : \mathcal{O}(Y) \to R$ is an algebra isomorphism). Let $\phi \in F[x_1, ..., x_n]$ be such that if $\overline{\phi} = \phi_{|Y}$ then $f^*\overline{\phi} = \gamma$. We consider the subset

$$Z = \{(x, \frac{1}{\phi(x)}) | x \in Y_{\{\overline{\phi}\}}\}$$

of F^{n+1} . We look upon $F[x_1, ..., x_n]$ as the subalgebra of $F[x_1, ..., x_{n+1}]$ spanned by the monomials that don't involve x_{n+1}. We note that if $h(x_1, ..., x_{n+1}) =$

$x_{n+1}\phi(x_1, ..., x_n) - 1$ then

$$Z = F^{n+1}(S \cup \{h\}).$$

One checks that $\mathcal{O}(Z)$ is the localization of $\mathcal{O}(Y)$ by the multiplicative set $\{1, \overline{\phi}, \overline{\phi}^2, ...\}$. Thus if $R_{(\gamma)}$ is the corresponding localization of R then we have $(X_{\{\gamma\}}, R_{(\gamma)})$ is an affine variety.

Now let U be Z-open in $\dot{Y}$. Then $U = Y - W \cap Y$ with W Z-closed in F^n. That is $W = F^n(S)$ with Sa finite set of polynomials. This implies that $U = \cup_{\phi \in S} Y_{\{\overline{\phi}\}}$. This implies that the affine subvarieties $(X_{\{\gamma\}}, R_{(\gamma)})$ define a basis for the open sets in the Z-topology.

If $U \subset X$ is open then we define $\mathcal{O}_X(U)$ to be the functions α from U to F such that for each $p \in U$ there is a $\gamma \in R$ such that $\gamma(p) \neq 0$, $X_{\{\gamma\}} \subset U$ and $\alpha_{|X_{\gamma\}}} \in R_{(\gamma)}$. To make sure that this makes sense we need to know

Theorem 13 *Let (X, R) be an affine variety. Then $\mathcal{O}_X(X) = R$.*

1.2.5 The tangent space revisited.

Let (X, R) be an affine variety and let for each Z-open subset $U \subset X$, $\mathcal{O}_X(U)$ be as in the previous section. If $p \in X$ and if U and V are Z-open in X and $p \in U \cap V$ then we say that $\alpha \in \mathcal{O}_X(U)$ and $\beta \in \mathcal{O}_X(V)$are equivalent if there exists W a Z-open subset with $p \in W \subset U \cap V$ such that $\alpha_{|W} = \beta_{|W}$. We define $\mathcal{O}_{X,p}$ to be the set of equivalence classes under this relation.

If $\alpha \in \mathcal{O}_X(U)$ for some Z-open U with $p \in U$ let $[\alpha]_p$ be the corresponding equivalence class. Then one checks that we can define an algebra structure on $\mathcal{O}_{X,p}$ by defining $[\alpha]_p [\beta]_p = [\alpha_{|W}]_p [\beta_{|W}]_p$ for $\alpha \in \mathcal{O}_X(U)$ and $\beta \in \mathcal{O}_X(V)$ and $p \in W \subset U \cap V$.

This algebra is called the local ring of X at p. One can define this ring to be the localization of R at $\mathcal{M}_p$ (see section 1.4). If $[\alpha]_p \in \mathcal{O}_{X,p}$ then the value of $\alpha(p)$ depends only on $[\alpha]_p$ and we use the notation $\tau(p)$ for the value at p of $\tau \in \mathcal{O}_{X,p}$. This defines a homomorphism of $\mathcal{O}_{X,p}$ onto F. We will use the notation $\mathfrak{m}_{X,p}$ for the kernel of this homomorphism. Then it is easily seen that $\mathcal{O}_{X,p}$ is a local ring with maximal ideal $\mathfrak{m}_{X,p}$(see section 1.4). We consider the map $\iota_p : \mathcal{M}_p/\mathcal{M}_p^2 \to \mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2$by $\alpha \to [\alpha]_p + \mathfrak{m}_{X,p}^2$.

Theorem 14 *If X is irreducible and if $p \in X$ then ι_p is a bijection.*

This leads to a definition of Zariski cotangent space. $T_p^*(X) = \mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2$. As before we say that $p \in X$ is smooth if $\dim T_p^*(X) = \dim X$. If $X = \cup_{j=1}^m X_j$is its decomposition into irreducible components and if $X' = \cup_{p \in X_j} X_j$ then $\mathcal{O}_{X,p} = \mathcal{O}_{X',p}$and the cotangent space depends only on this closed subvariety.

1.2.6 The formal algebra at a smooth point.

Let (X, R) be an affine variety over F and algebraically closed field. We assume that $p \in X$ is a smooth point. Let $u_1, ..., u_d \in \mathfrak{m}_{X,p}$ be such that if $\overline{u_i} = u_i + \mathfrak{m}_{X,p}^2$ form a basis of $\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2$. If $\alpha = (a_1, ..., \alpha_d)$ is a multiindex then we set (as usual) $u^\alpha = u_1^{\alpha_1} \cdots u_d^{\alpha_d}$. Let $p_k : R \to R/\mathfrak{m}_{X,p}^k$.

Lemma 15 *$u_1, .., u_d$ are algebraically independent over F.*

Lemma 16 *In the notation above the set $\{p_l u^\alpha | 0 \leq |\alpha| < l\}$is a basis of $\mathcal{O}_{X,p}/\mathfrak{m}_{X,p}^l$ over F.*

Corollary 17 *In the notation above*

$$\varprojlim O_{X,p}/\mathfrak{m}_{X,p}^k \cong F[[x_1, ..., x_d]]$$

the formal power series in the indeterminates $x_1, ..., x_d$.

Theorem 18 *Let (X, R) be an affine variety over F of dimension d and let p be a smooth point of X. Let $X = X_1 \cup \cdots \cup X_m$ be the decomposition of X into irreducible components and let $X' = \cup_{p \in X_j} X_j$ (the union of the irreducible components containing X). Then X' is irreducible.*

Proof. We may assume that $X = X'$. Thus we need only show that if p is a smooth point of X and if every irreducible component of X contains p then X is irreducible. We first note that the map $O_{X,p} \to \varprojlim O_{X,p}/\mathfrak{m}_{X,p}^k$ given by $a \longmapsto \{a + \mathfrak{m}_{X,p}^k\}$ is injective. Indeed, if $a \mapsto 0$ then $a \in \mathfrak{m}_{X,p}^k$ for all k and thus Corollary 29 in the appendix (section 1.4) implies that $a = 0$. This implies that $\mathcal{O}_{X,p}$ is an integral domain. We assert that this implies that R is an integral domain. Indeed, we assert that the map $a \longmapsto a/1$ injects R into $\mathcal{O}_{X,p}$. To see this we note that $a/1 = 0$ implies that there exists $\phi \in R$ such that $\phi(p) \neq 0$ and $\phi a = 0$. This implies that if Y is an irreducible component of X then $\phi a_{|Y} = 0$. Since $\phi(p) \neq 0$ the set $\{y \in Y | \phi(y) \neq 0\}$ is Z-dense in Y and thus $a_{|Y} = 0$. We have assumed that every irreducible component of X contains p sot $a = 0$.This implies that Rinjects into an integral domain so $X$is irreducible. ■

1.2.7 Relations between the standard and Zariski topologies.

We now look at the case when $F = \mathbb{C}$.

We look upon $\mathbb{C}$ with the standard topology as $\mathbb{R}^2$ and $\mathbb{C}^n$ as $\mathbb{R}^{2n}$. If f function from a set S to $\mathbb{C}$ we will write $f(s) = f_R(s) + if_I(s)$ with f_R and f_I real valued functions on S. If $\delta > 0$ we set $B^n_\delta = \{x \in \mathbb{C}^n | \, \|x\| < \delta\}$thought of as the δ ball in $\mathbb{R}^{2n}$. We will now prove

Theorem 19 *Let X be Z-closed in $\mathbb{C}^n$ and irreducible be of dimension d. Let $p \in X$ be a smooth point then there exists $\delta > 0$ and a C^∞ map $\Phi : B^d_\delta \to \mathbb{C}^n$ such that $\Phi(0) = p$, $\Phi(B^d_\delta) \subset X$ is an open neighborhood of p in the standard topology and $\Phi : B^d_\delta \to \Phi(B^d_\delta)$ is a homeomorphism in the standard topology.*

Proof. Let $f_1, ..., f_m$ generate $\mathcal{I}_X$. After possible relabeling we may assume that $(df_1)_p, ..., (df_{n-d})_p$ are linearly independent elements of $(\mathbb{C}^n)^*$. Let $u_i = (f_i)_R$ and $u_{n-d+i} = (f_i)_I$ then we note that $(du_i)_p = ((df_i)_p)_R$ and $(du_{n-d+i})_p = ((df_i)_p)_I$ where du_j is the standard calculus differential on $\mathbb{R}^{2n}$. This implies that $(du_1)_p, ..., (du_{2n-2d})_p$are linearly independent at p. Now after relabeling we may assume that the restrictions of $x_1 - p_1, ..., x_d - p_d$ to X form a basis of $\mathcal{M}_p$ modulo $\mathcal{M}^2_p$. Set $u_{2n-2d+i} = (x_i - p_i)_R$ and $u_{2n-d+i} = (x_i - p_i)_I$. Then $(du_1)_p, ..., (du_{2n})_p$ form a basis of $(\mathbb{R}^{2n})^*$. The real analytic inverse function theorem implies that there exists an open standard neighborhood U of p in $\mathbb{R}^{2n}$ such that if $\Psi = (u_1, ..., u_{2n})$ then $\Psi(U)$ is open and $\Psi : U \to \Psi(U)$ is a (real analytic) diffeomorphism. Let $\delta > 0$ be so small that $B^{n-d}_\delta \times B^d_\delta \subset \Psi(U)$. Then

$$\Psi^{-1}(0 \times B^d_\delta) = \mathbb{C}^n(f_1, ..., f_{n-d}) \cap \Psi^{-1}(B^{n-d}_\delta \times B^d_\delta).$$

We set $\Phi(z) = \Psi^{-1}(0, z)$ for $z \in B^d_\delta$.

Let $Z = \mathbb{C}^n(f_1, ..., f_{n-d})$ and let $Z = \cup^r_{i=1} Z_i$ be the decomposition of Z into irreducible components. Since X is irreducible we may assume that $X \subset Z_1$. On the other hand since $f_1, ..., f_{n-d} \in \mathcal{I}_{Z_1}$we see that $\dim T_p(Z_1) \leq d = \dim X$. Thus $Z_1 = X$.

Let W be the Z-closure of $\Phi(B^d_\delta)$. Then $W \subset \mathbb{C}(f_1, ..., f_{n-d})$. We also note that $x_i \circ \Phi$ defines an analytic function on B^d_δ for $i = 1, ..., n$. This implies that Φ^* defines an injective homomorphism of $\mathcal{O}(W)$ into the analytic functions on B^d_δ. Since this algebra is an integral domain this implies that $\mathcal{O}(W)$ is an integral domain and hence that W is irreducible. Furthermore the functions $x_i - p_i$, $i = 1, .., d$ are algebraically independent on $\Phi(B^d_\delta)$and hence on W.

thus the dimension of W is at least d. Since $p \in W$ we see that $\dim W = d$ (the tangent space at p is at most of dimension d). Thus W is an irreducible component of Y containing p. Since p is smooth we must have $Z = X$ the unique irreducible component of Y containing p. ■

In our application of these results we will use (the case $n = 1$) of the following theorem.

In general the standard metric topology (which we will denote as the S-topology) is much finer than the Z-topology on an affine variety (they are equal if and only if the variety is finite). However, we will show that the S-closure of a Z-open set in an affine variety is the Z-closure.

Theorem 20 *Let X be Z-closed in $\mathbb{C}^n$ and irreducible and let $U \neq \emptyset$ be Z-open in X. Then the S-closure of U is X.*

Proof. We prove this result by induction on $\dim X$. If $\dim X = 1$ then $X - U$ is finite. Let $p \in X - U$. Let $f : Y \to X$ be the normalization of X then f is surjective and continuous in the S-topology. Theorem 13 implies that Y is smooth. Let $q \in f^{-1}(p)$ as in Theorem 12 choose $\delta > 0$ and $\Phi : B_\delta^1 \to \Phi(B_\delta^1)$ be a homomorphism in the S-topology onto an S-open subset of Y with $\Phi(0) = q$. By taking δ possibly smaller we may assume that $\Phi(B_\delta^1) \cap f^{-1}(X - U) = \{q\}$. Choose a sequence $z_j \in B_\delta^1 - \{0\}$ such that $\lim_{j\to 0} z_j = 0$. Then $x_j = f(\Phi(z_j)) \in U$ and in the S-topology $\lim_{j\to\infty} x_j = p$.

Now assume that the result has been proved for all X of dimension $n \geq 1$. Assume $\dim X = n + 1$. Let $p \in X - U$. Let $Y = X - U = Y_1 \cup \cdots \cup Y_r$ be its decomposition into irreducible components and assume that $Y_1, ..., Y_s$ are exactly the ones that are not equal to $\{p\}$ (usually $s = r$). Let $p_i \in Y_i$ be such that $p_i \neq p$. Let $\phi \subset \mathcal{O}(X)$ be such that $\phi(p) = 0$ and $\phi(p_j) \neq 0$ for $j = 1, ..., s$. Set $Z = \{x \in X | \phi(x) = 0\}$. Let $Z = Z_1 \cup \cdots \cup Z_t$ be the decomposition of Z into irreducible components. Assume that some $Z_j \subset X - U$. Then since X is irreducible and U is non-empty we see that the Y_i are all of dimension at most n. Since Z_j is irreducible it must be contained in one of the Y_j. On the other hand $\dim Z_k = n$ for all k. Thus we must have $Z_j = Y_i$ for some i.But $p_i \notin Z$. This contradiction implies $Z_i \not\subseteq X - U$. Let Z_j be such that $p \in Z_j$. Then Z_j has dimension n and $U \cap Z_j \neq \emptyset$. Thus the inductive hypothesis implies that p is in the S-closure of $U \cap Z_j$ hence of U. ■

Exercises.

1. Why didn't we start the induction from dimension 0(where the result is trivial)?

2. Prove that if X is an affine variety and U is a Z-open subset of X then the Z-closure of U is the same as the S-closure.

1.3 Projective and quasiprojective varieties.

1.3.1 Projective algebraic sets.

We define *projective n-space* to be the set of all one dimensional subspaces of $F^{n+1}$and denote it $\mathbb{P}^n(F)$ or $\mathbb{P}^n$ if F is understood. If $L \in \mathbb{P}^n$ then $L = Fv$ with $v \in L - \{0\}$. Thus we can look upon $\mathbb{P}^n$ as the set $F^{n+1} - \{0\}$ modulo the equivalence relation $v \equiv w$ if there exists $c \in F^\times = F - \{0\}$ such that $cw = v$. If $(x_0, x_1, ..., x_n) \in F^{n+1} - \{0\}$ then we denote the corresponding line by $[x_0, x_1, ..., x_n]$ and $x_0, x_1, ..., x_n$ are called the *homogeneous coordinates* of the point in $\mathbb{P}^n$.

Let $f \in F[x_0, x_1, ..., x_n]$ then f is said to be *homogeneous of degree* m if $f(cx) = c^m f(x)$ for $x \in F^{n+1}$and $c \in F$. Assuming that F is infinite this condition is equivalent to saying that f is an F-linear combination of monomials $x^\alpha = x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $|\alpha| = \alpha_0 + ... + \alpha_n = m$. We denote by $F^m[x_0, ..., x_n]$ the space of polynomials that are homogeneous of degree m. We note that if $x \in F^{n+1} - \{0\}$ and if $f \in F^m[x_0, ..., x_n]$ then whether or not $f(x) = 0$ depends only on $[x]$. Define the closed sets in the Zariski topology on $\mathbb{P}^n$ to be the sets of the form

$$\mathbb{P}^n(S) = \{[x] \in \mathbb{P}^n | f(x) = 0, f \in S\}$$

and S is a set of homogeneous polynomials. One checks that with these sets satisfy the conditions necessary to be the closed sets of a topology.

Of particular interest are the Z-open sets $\mathbb{P}^n_j = \mathbb{P}^n - \mathbb{P}^n(x_j)$.Let $[x] \in \mathbb{P}^n_j$ (that is $x_j \neq 0$) then

$$[x] = [\frac{1}{x_j} x] = [\frac{x_0}{x_j}, ..., \frac{x_{j-1}}{x_j}, 1, \frac{x_{j+1}}{x_j}, ..., \frac{x_n}{x_j}].$$

We therefore see that we have a bijective mapping from F^n to $\mathbb{P}^n_j$, Ψ_j, given by

$$\Psi_j : (a_1, ..., a_n) \mapsto [a_1, ..., a_j, 1, a_{j+1}, ..., a_n].$$

Suppose that $f \in F[x_1, ..., x_n]$ is of degree m. Then the j-th *homogenization* of f is the polynomial obtained as follows:

Let $f(x_1,...,x_n) = \sum_{|\alpha|\leq m} c_\alpha x^\alpha$. Write

$$f_j(x_o,...,x_n) = \sum_{|\alpha|\leq m} c_\alpha x_j^{m-|\alpha|} x_o^{\alpha_1} \cdots x_{j-1}^{\alpha_j} x_{j+1}^{\alpha_j+1} \cdots x_n^{\alpha_n}.$$

Then f_j is homogeneous of degree m in the variables $x_0,...,x_n$ and $\Psi_j^* f_j = f$. Thus if we endow $\mathbb{P}_j^n$ with the subspace Zariski topology then $\Psi_j : F^n \to \mathbb{P}_j^n$ defines a Z-homeomorphism.

If $f \in F^m[x_0,...,x_n]$ then define $R_i f(x_1,...,x_n) = f(x_1,...,x_j,1,x_{j+1},...,x_n)$. Then R_i is a linear bijection between $F^m[x_0,...,x_n]$ and $\oplus_{j=0}^m F^m[x_1,...,x_n] = F_m[x_1,...,x_n]$.

Let X be a Z-closed subset of $\mathbb{P}^n$. Then we have an open covering $X = \cup_{i=0}^n \mathbb{P}_i^n \cap X$. Let Ψ_i be as above. If $X = \mathbb{P}^n(S)$ then $\Psi_i^{-1}(\mathbb{P}_i^n \cap X) = F^n(R_i S)$.

1.3.2 Sheaves of functions.

Let X be a topological space then a *sheaf of functions* on X is an assignment $\mathcal{F} : U \to \mathcal{F}(U)$ with $\mathcal{F}(U)$ a subalgebra of the algebra of all functions from U to F satisfying the following properties:

1. If $V \subset U$ then $\mathcal{F}(U)_{|V} \subset \mathcal{F}(V)$.

2. If $\{V_\alpha\}_{\alpha\in I}$ is an open covering of U then if $\gamma_\alpha \in \mathcal{F}(V_\alpha)$ is given such that if $x \in V_\alpha \cap V_\beta$ then $\gamma_\alpha(x) = \gamma_\beta(x)$ then there exists $\gamma \in \mathcal{F}(U)$ such that $\gamma_{|V_\alpha} = \gamma_\alpha$ for all α.

Examples.

1. It is easily seen that if (X,R) is an affine variety then $\mathcal{O}_X$ is a sheaf of functions on X. We call $\mathcal{O}_X$ the *structure sheaf* of X.

2. If $\mathcal{F}$ is a sheaf of functions on X and if U is an open subset of X then we define $\mathcal{F}_{|U}$ to be the assignment $V \to \mathcal{F}(V)$ for V open in U(hence in X).

3. If X is a topological space and if $F = \mathbb{R}$ or $\mathbb{C}$ then the assignment of the assignment $U \to C(U,F)$ (the continuous functions from U to F) is a sheaf or functions.

4. If $F = \mathbb{R}$ then the assignment U open in $\mathbb{R}^n$ to $C^\infty(U)$ (the functions from U to $\mathbb{R}$ such that all partial derivatives of all orders are continuous) is a sheaf of functions denoted $C^\infty_{\mathbb{R}^n}$. Similarly,$U \to C^\omega(U)$ the assignment to U the real analytic functions from U to $\mathbb{R}$.

We consider the category of all pairs $(X,\mathcal{F})$ with X a topological space and $\mathcal{F}$ a sheaf of functions on X. We will call this the category of spaces

with structure. With morphisms between $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ continuos maps $f : X \to Y$ such that $f^*\mathcal{G}(U) \subset \mathcal{F}(f^{-1}U)$.

Example.4. The category of smooth manifolds is the full subcategory of spaces with structures $(X, \mathcal{F})$ where X is a Hausdorff space such that for each $p \in X$ there is an open neighborhood of p, U, such that $(U, \mathcal{F}_{|U})$ is isomorphic with $(V, C^\infty_{\mathbb{R}^n|V})$ for V an open set of $\mathbb{R}^n$ for some fixed n. We will denote the sheaf $\mathcal{F}$ by C^∞_X and call n the dimension of X.

We are now ready to give a definition of algebraic variety.

1.3.3 Algebraic varieties.

The example in the previous subsection indicates how one should define an algebraic variety. We define a *pre-variety* to be a topological space with a structure $(X, \mathcal{O}_X)$ such that for every $p \in X$ there exists an open neighborhood, U, of p in X and an affine variety $(Y, \mathcal{O}_Y)$ such that $(U, \mathcal{O}_{X_{|U}})$ is isomorphic with $(Y, \mathcal{O}_Y)$.

Examples.

1.We have seen that the structure that we attached to an affine variety (X, R) which we denoted as $(X, \mathcal{O}_X)$ is a pre-variety.

2. Let X be a Z-closed subset of $\mathbb{P}^n$. Set $X_i = X \cap \mathbb{P}^n_i$ and let Ψ_i be as in 1.3.1. Then in 1.3.1 we observed that $\Psi_i^{-1}(X_i)$ is Z-closed in F^n. We pull back the structure as in example on $Y_i = \Psi_i^{-1}(X_i)$. Then getting sheaves of functions $\mathcal{O}_{X_i}$. If $U \subset X_i \cap X_j$ then $\mathcal{O}_{X_i|U} = \mathcal{O}_{X_j|U}$. We define for $U \subset X$, Z-open $\mathcal{O}_X(U)$ to be the functions γ on U such that $\gamma_{|U\cap X_i} \in \mathcal{O}_{X_i}(U \cap X_i)$. Then $(X, \mathcal{O}_X)$ is a pre-variety.

3. If U is open in X a Z-closed subset in $\mathbb{P}^n$ then we take $\mathcal{O}_U$ to be the restriction of $\mathcal{O}_X$ to U and $(U, \mathcal{O}_U)$ is a pre-variety.

We now come to the notion of algebraic variety. For this we need to define products of pre-varieties. If $X \subset F^n$ and $Y \subset F^m$ are Z-closed sets then we note that $X \times Y$ is Z-closed in $F^n \times F^m = F^{m+n}$ (as a topological space–recall that this is not the product topology). To see this we take F^n to be $F^n \times 0$ and F^m to be $0 \times F^m$ then if $X = F^n(S)$ and $Y = F^m(T)$ then we look at $f \in S$ as a function of $x_1, ..., x_n$ and $g \in T$ as a function of $x_{n+1}, ..., x_{n+m}$ then $X \times Y = F^{n+m}(S \cup T)$.

This defines a product of affine varieties. One can do this intrinsically by defining $(X, R) \times (Y, S)$ to be $(X \times Y, R \otimes_F S)$. Here $(\gamma \otimes \delta)(x, y) = \gamma(x)\delta(y)$.

If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are pre-varieties then we can define the pre-variety $(X \times Y, \mathcal{O}_{X\times Y})$ as follows: If $U \subset X$ and $V \subset Y$ are open and $(U, \mathcal{O}_{X_{|U}})$ and $(V, \mathcal{O}_{Y|V})$ are isomorphic to affine varieties then take $\mathcal{O}_{X\times Y|U\times V}$ to be $\mathcal{O}_{U\times V}$ as in 1.3.2 Example 2. This defines a product (here the topology on $U \times V$ is the topology of the product of affine varieties defined as above). One can prove that this is the categorical product in the category of pre-varieties.

We can finally define an algebraic variety. Let $(X, \mathcal{O}_X)$ be a pre-variety then it is an *algebraic variety over* F if $\Delta(X) = \{(x, x)|x \in X\}$ is a closed set in X. This condition is usually called *separable*. We have (c.f. Lemma A.4.2 [GW])

Theorem 21 *Let X be Z-open in the Z-closed subset, Y, in $\mathbb{P}^n$. Then the corresponding pre-variety (as in example 3 above) is an algebraic variety.*

A variety as in the above theorem will be called *quasi-projective*. An algebraic variety isomorphic with Z-closed subset of $\mathbb{P}^n$will be called a *projective variety*.

Theorem 22 *If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are quasi-projective then so is $(X \times Y, \mathcal{O}_{X\times Y})$.*

For this it is enough to show that $\mathbb{P}^n \times \mathbb{P}^m$ is isomorphic with a Z-closed subset of $\mathbb{P}^N$ for some N. To prove this we consider the map $(F^{n+1} - \{0\}) \times (F^{m+1} - \{0\}) \to F^{n+1} \otimes F^{m+1} - \{0\}$ given by

$$(x, y) \longmapsto x \otimes y.$$

If we take the standard bases e_j of F^{n+1} and F^{m+1} then we can identify $F^{n+1} \otimes F^{m+1}$ with $F^{(m+1)(n+1)}$using the basis $e_i \otimes e_j$ for $0 \leq i \leq n$ and $0 \leq j \leq m$. One checks that this defines a morphism of $(F^{n+1} - \{0\}) \times (F^{m+1} - \{0\}) \to F^{n+1} \otimes F^{m+1} - \{0\}$.Also, $[x \otimes y] = [x' \otimes y']$ with $(x, y) \in (F^{n+1} - \{0\}) \times (F^{m+1} - \{0\})$ if and only if $[x] = [x']$ and $[y] = [y']$. Further, one sees that if z_{ij} is the coordinate of $z \in F^{n+1} \otimes F^{m+1}$ then the condition that $[z]$ is in the image of this map is that $z_{ij}z_{kl} = z_{kj}z_{il} = z_{il}z_{kj}$. These are homogeneous equations of degree 2. Thus the image is closed. We have thus embedded $\mathbb{P}^n \times \mathbb{P}^m$ as a closed subset of $\mathbb{P}^{nm+n+m}$. This defines the topology on the corresponding product of algebraic varieties.

Exercise. Prove the assertions in the argument above (hit if $z = x \otimes y$ then $z_{ij} = x_i y_j$). Also show that the images of the $\mathbb{P}^n_i \times \mathbb{P}^m_j$ are open in the image and isomorphic to F^{n+m}.

The following theorem is Theorem A.4.8 in [GW].

Theorem 23 *If $(X, \mathcal{O}_X)$ is an irreducible projective variety then $\mathcal{O}_X(X)$ consists of the constants.*

1.3.4 Local properties of algebraic varieties.

It is almost counter-intuitive that the assertion that for X a topological space the assertion that $Y \subset X$ is closed is a local property. That is if for each $p \in X$ there exists $U \subset X$ open with $U \cap Y$ closed in U then Y is closed in X. To see this we note that this local condition implies that there is an open covering of X, $\{U_\alpha\}_{\alpha \in I}$ such that $U_\alpha \cap Y$ is closed in U_α. This implies that $U_\alpha - U_\alpha \cap Y$ is open in U_α hence in X. But $\cup_{\alpha \in I}(U_\alpha - U_\alpha \cap Y\ ) = X - Y$.

One notes that the topology that we have put on an algebraic variety is Noetherian so algebraic varieties have a unique (up to order) decomposition into irreducible algebraic varieties.

Since any two non-empty open sets in an irreducible topological space must intersect we see that we can define the dimension of an irreducible algebraic set to be the dimension of any open non-empty affine subvariety. We define the dimension of an algebraic variety to be the maximum of the dimensions of the irreducible components.

We will say that a point in an irreducible algebraic variety is smooth if it is smooth in an affine open set containing it. Thus the set of all smooth points is open in the variety.

We note that if $F = \mathbb{C}$ then a variety has in addition a standard metric topology. To distinguish the two topologies we will use the terms Z-topology and S-topology.

Theorem 19 now implies

Theorem 24 *If $(X, \mathcal{O}_X)$ is an irreducible d-dimensional algebraic variety over $\mathbb{C}$ then the set of smooth points endowed with the S-topology is a C^∞ manifold of dimension $2d$.*

In addition using the fact that closedness is a local property we have

Theorem 25 *Let $(X, \mathcal{O}_X)$ be an algebraic variety over $\mathbb{C}$. If U is Z-open in X then the S-closure of U is the same as the Z-closure.*

In our study of algebraic groups we will be using the following theorem the local version is e.g. Theorem A.2.7 in [GW].

Theorem 26 *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be non-empty irreducible algebraic varieties. Then if $f : X \to Y$ is a morphism such that the image of X is dense in Y (here we are using the Z-topology if $F = \mathbb{C}$) and if U is open and non-empty in X then $f(U)$ contains an open non-empty subset of Y.*

Corollary 27 *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be non-empty irreducible algebraic varieties over $\mathbb{C}$. Then if $f : X \to Y$ is a morphism such that the image of X is Z-dense in Y then it is S-dense.*

1.4 Appendix. Some local algebra.

1.4.1 The Artin Rees Lemma.

Let R be a Noetherian algebra over a field F. In this context we recall the Artin Rees lemma.

Lemma 28 *Let I be an ideal in R. Let M be a finitely generated module over R and let N be an R-submodule of M. Then there exists $l \in \mathbb{N}$ such that $I^{l+j}M \cap N = I^j(I^l M \cap N)$ for all $j \in \mathbb{N}$.*

Proof. Let t be an indeterminate and consider the subalgebra

$$R^* = R + tI + t^2I^2 + ...$$

of $R[t]$,this algebra is sometimes called the Rees algebra of R with respect to I. If I is generated by $r_1, ..., r_m$ (i.e. $T = Rr_1 + ... + Ri_m$) then R^* is the algebra over R generated by $tr_1, ..., tr_m$. Thus R^* is Noetherian. We consider the R^*-module

$$M^* = M + tIM + t^2I^2M + ...$$

If M is generated as a module by $m_1, ..., m_r$ as an R-module then so is M^* as an R^* module. Let

$$N_1 = N + tIM \cap N + t^2I(IM \cap N) + ... + t^{r+1}I^r(IM \cap N) + ...$$

$$N_2 = N + tIM \cap N + t^2(I^2M \cap N) + ... + t^{r+2}I^r(I^2M \cap N) + ...$$

and in general

$$N_m = N + tIM \cap N + t^2(I^2M \cap N) + ... + t^m(I^mM \cap N) +$$
$$t^{m+1}I(I^mM \cap N) + ... + t^{m+j}I^j(I^mM \cap N) + ...$$

Then $N_1 \subset N_2 \subset ...$ and all are R^*-submodules. Thus there is an index l such that $N_{l+j} = N_l$ for all j. Comparing coefficients of t yields the lemma. ■

Corollary 29 *Let $\mathcal{M}$ be a maximal proper ideal in R. If M is an R-module then $\cap_{k\geq 1}\mathcal{M}^kM = 0$.*

Proof. Let $N = \cap_{k\geq 1}\mathcal{M}^kM$. Let l be as in the Artin Rees lemma. Then

$$N = \mathcal{M}^{l+j}M \cap N = \mathcal{M}^j(\mathcal{M}^lM \cap N) = \mathcal{M}^jN.$$

Thus $N = \mathcal{M}N$. Now Nakayama's lemma (below) implies that $N = 0$. ■

We now recall Nakayama's Lemma.

Lemma 30 *Let R be a commutative ring with identity and let I be an ideal in R such that $1 + a$ is invertible for all $a \in I$ (e.g. I is maximal). Then if M is a finitely generated R-module such that $IM = M$ then $M = 0$.*

Proof. Let $M = Rm_1 + ... + Rm_r$. Then there exist $a_{ij} \in I$ such that $\sum_{j=1}^r a_{ij}m_j = m_i$. Thus we have

$$\sum_{j=1}^{r}(\delta_{ij} - a_{ij})m_j = 0.$$

If we apply Cramer's rule we have $\det(I - [a_{ij}])m_k = 0$ for all k. But $\det(I - [a_{ij}]) = 1 + a$ with $a \in I$. Thus $m_k = 0$ all k. ■

1.4.2 Localization.

Let R be a finitely generated algebra over F. Let $S \subset R$ be closed under multiplication containing 1 and doesn't contain 0we will call S a *multiplicative subset*. Then we define $R_{(S)}$ to be the set $R \times S$ modulo the equivalence relation (r, s) is equivalent with (t, u) if there exists $v \in S$ such that $v(ru - ts) = 0$. We observe that if (r, s) is equivalent with (r', s') and (t, u) is equivalent with (t', u') then (rt, su) is equivalent with $((r't', s'u')$ and $(ru + ts, su)$ is equivalent with $(r'u' + t's', s'u')$ we can thus define an algebra structure on $R_{(S)}$. We write the equivalence class of (r, s) as r/s. We have an algebra homomorphism of R to $R_{(S)}$ given by $r \to r/1$.

Lemma 31 *If R and S are as above then $R_{(S)}$ is Noetherian.*

Let $\mathcal{M}$ be an ideal in R such that $S = R - \mathcal{M}$ is multiplicative. This condition is equivalent to the primality of $\mathcal{M}$. Then we set $R_{\mathcal{M}} = R_{(S)}$. Then $R_{\mathcal{M}}$ has the unique maximal ideal $\mathfrak{m} = \mathcal{M}R_{(S)}$. This is the definition of a *local ring*.

Let R be a local ring with maximal ideal $\mathcal{M}$. Let $j_{kl} : R/\mathcal{M}^l \to R/\mathcal{M}^l$.for $l \geq k$ be the natural projection. We define

$$\varprojlim R/\mathcal{M}^k$$

to be the set of sequences $\{a_k\}$ with $a_k \in R/\mathcal{M}^k$ and $j_{kl}a_l = a_k$ for $l \geq k$. Then this ring with addition and multiplication given by component wise operations is called the completion of R and denoted $\overline{R}$.