3 Hilbert-Mumford type theorems.

3.1 Basics on group actions.

3.1.1 Algebraic group actions.

Let X be an algebraic variety and let G be an algebraic group both over $\mathbb{C}$. Then an (algebraic group) *action* of G on X is a morphism $\Phi : G \times X \to X$ satisfying:

1. $\Phi(1, x) = x$ (1denoting the identity element of G) for all $x \in X$.

2, $\Phi(gh, x) = \Phi(g, \Phi(h, x))$ for all $g, h \in G$ and $x \in X$.

We will denote such an action by gx. The set Gx is called the *orbit* of x. Our main example is G a Z-closed subgroup of $GL(n, F)$, $X = F^n$ and gx is the matrix action of G on F^n. More generally, we define a *regular representation* of an algebraic group, G, to be a group morphism $\sigma : G \to GL(V)$ where V is a finite dimensional vector space over $\mathbb{C}$. We denote it (σ, V). One more bit of notation the isotropy group of $x \in X$ is the set $\{g \in G | gx = x\}$ and it will be denoted G_x.

We will confine our attention to the case when G is irreducible (that is connected in the Z-topology).

Lemma 1 *Let G be irreducible and act on an algebraic variety, X. Let $x \in X$ and let Y be the Z-closure of Gx. Then*

1. Y is irreducible.

2. Gx is Z-open in Y.

3. There is a Z-closed G orbit in Y.

4. Y is the S-closure of Gx.

Proof. We have seen in Theorem 26 of 1.3.4 that Gx has interior in Y. Since $y \to gy$ defines an automorphism of Y we see that Gx is a union of open subsets of Y. This proves 2. Since Gx is the image under a morphism of an irreducible variety it is irreducible. As it is dense in Y, Y is irreducible. Let $Z = Y - Gx$. Then Z is closed and G-invariant. If $Z = \emptyset$ then Y is closed. If not let $V = Gz$ be an orbit of minimal dimension in Z. If W is the closure of V then $W - V$ is closed in W and since W is irreducible $\dim(W - V) < \dim W$. Thus the dimension of any orbit in $W - V$ would be lower than the minimum possible. Thus we must have V is closed proving 3. We note that 4 is an immediate consequence of 2 and Theorem 20 of 1.1.2.7. ■

Proposition 2 *Let G be an affine algebraic group acting on an irreducible affine variety X. Then there exists an imbedding of $\phi : X \to \mathbb{C}^n$ as a Z-closed subset of $\mathbb{C}^n$ for some n and an algebraic group homomorphism, $\sigma : G \to GL(n, \mathbb{C})$ such that $\phi(gx) = \sigma(g)\phi(x)$ for all $x \in X$ and $g \in G$.*

Proof. We may assume that $X \subset \mathbb{C}^m$ is Z-closed, Let $f_i = x_{i_{|X}}$ for $i = 1, .., m$ (x_i the standard coordinates on X). Let the action of G on X be given by F.Then $F^*\mathcal{O}(X)$ is a subalgebra of $\mathcal{O}(G \times X) = \mathcal{O}(G) \otimes_{\mathbb{C}} \mathcal{O}(X)$. We also note that $F^*(f)(1, x) = f(x)$ also if we set $\tau(g)f(x) = f(g^{-1}x)$ then $F^*\tau(g)f(h, x) = F^*f(hg^{-1}, x)$. This implies that the linear span of $\{F^*\tau(g)f | g \in G\}$ is finite dimensional for each $f \in \mathcal{O}(X)$. Thus we see that the linear span, W, of $\{\tau(g)f_i | g \in G, i = 1, .., m\}$ is finite dimensional. Let $u_1, ..., u_d$ be a basis of W. Then the map $\phi(x) = (u_1(x), ..., u_d(x))$ defines an isomorphism of X into $\mathbb{C}^d$. We assert that $\phi(X)$ is S-closed in $\mathbb{C}^n$.Let $\{z_j\}$ be a sequence in X such that $\lim_{j\to\infty} \phi(z_j\} = v$ exists in $\mathbb{C}^d$. Then since the f_j are in the linear span of the u_i we see that $\lim_{j\to\infty}(x_1(z_j), ..., x_m(z_j)) = x$ exists. But this means $\lim_{j\to\infty} z_j = x$ in the S-topology. Thus $v = \phi(x)$. We now observe that since $\phi(X)$ contains interior in the Z-closure $\phi(X)$ we see that the Z-closure of $\phi(X)$ is equal to the S-closure which equals $\phi(X)$. We now note that if $g \in G$ then $\tau(g)u_i = \sum_j A_{ji}(g)u_j$. This implies that if $\sigma(g) = [A_{ij}(g^{-1})]$ then $\phi(gx) = \sigma(g)\phi(x)$. ■

We now assume that G is a symmetric subgroup of $GL(n, \mathbb{C})$ and $K = G \cap U(n)$. Let μ_K be the normalized invariant measure on K. If G acts on X an affine variety then we will use the notation $\mathcal{O}(X)^G = \{f \in \mathcal{O}(X) | f(gx) = f(x)$ for all $g \in G, x \in X\}$.

Theorem 3 *Assume that X is irreducible. The algebra $\mathcal{O}(X)^G$ is finitely generated over $\mathbb{C}$.*

Proof. Let $T(f)(x) = \mu_K(k \to f(kx))$. Then $T : \mathcal{O}(X) \to \mathcal{O}(X)^K$. Since K is Z-dense in G and $g \to f(gx) \in \mathcal{O}(G)$ we see that $\mathcal{O}(X)^K = \mathcal{O}(X)^G$. Also since $\mu_K(1) = 1$ we see that $T^2 = T$. We also note that $T(uf) = uT(f)$ if $u \in \mathcal{O}(X)^G$ and $f \in \mathcal{O}(X)$. We may assume that $X \subset \mathbb{C}^m$ is Z-closed and we have $\sigma : G \to GL(m, \mathbb{C})$ such that $gx = \sigma(g)x$ for $g \in G, x \in X$. We first prove the theorem with X replaced by $\mathbb{C}^m$. Let $\mathcal{O}(\mathbb{C}^m)^G_+ = \{f \in \mathcal{O}(\mathbb{C}^m)^G | f(0) = 0\}$. Let J be the ideal in $\mathcal{O}(\mathbb{C}^m)(= \mathbb{C}[x_1, ..., x_m])$ generated by $\mathcal{O}(\mathbb{C}^m)^G_+$. Let $f_1, ..., f_r \in \mathcal{O}(\mathbb{C}^m)^G_+$ be generators of the ideal. Then we assert that $f_1, ..., f_r$ generate $\mathcal{O}(\mathbb{C}^m)^G$ as an algebra over $\mathbb{C}$. We set $\mathcal{O}^k(\mathbb{C}^m)$

equal to the space of homogeneous polynomials of degree k in m variables. Then $\mathcal{O}(\mathbb{C}^m)^G = \oplus_{k\geq 0}\mathcal{O}(\mathbb{C}^m)^G \cap \mathcal{O}^k(\mathbb{C}^m)$. Furthermore, if $f \in \mathcal{O}^k(\mathbb{C}^m)$ then $T(f) \in \mathcal{O}^k(\mathbb{C}^m) \cap \mathcal{O}(\mathbb{C}^m)^G$. By replacing the f_j by the union of their homogeneous components we may assume that each f_i is homogeneous of positive degree d_i. Let R be the subalgebra of $\mathcal{O}(\mathbb{C}^m)^G$ generated by the f_j. Then $R = \oplus_{k\geq 0}R \cap \mathcal{O}^k(\mathbb{C}^m)$. We prove by induction on l that $\oplus_{k\leq l}R \cap \mathcal{O}^k(\mathbb{C}^m) = \oplus_{k\leq l}\mathcal{O}(\mathbb{C}^m)^G \cap \mathcal{O}^k(\mathbb{C}^m)$. This is clear if $l = 0$. Assume for l. If $f \in \mathcal{O}(\mathbb{C}^m)^G \cap \mathcal{O}^{l+1}(\mathbb{C}^m)$ then $f = \sum h_i f_i$ with $h_i \in \mathcal{O}^{l+1-d_i}(\mathbb{C}^m)$. Now $T(f) = f$ thus $f = \sum T(h_i)f_i$. Since $\deg T(h_i) < l+1$ and $T(h_i) \in \mathcal{O}(\mathbb{C}^m)^G$ the inductive hypothesis implies that $T(h_i) \in R$. This completes the proof for $X = \mathbb{C}^n$. We now prove the result for X Z-closed, irreducible and the action is given by $\sigma(g)x$ with σ an algebraic group homomorphism of G to $GL(m, \mathbb{C})$. If $f \in \mathcal{O}(X)$ then there exists $\phi \in \mathcal{O}(\mathbb{C}^m)$ such that $\phi_{|X} = f$. We note that $T(\phi)_{|X} = T(\phi_{|X})$ since X is G-invariant and hence K-invariant. This implies that $\mathcal{O}(\mathbb{C}^m)^G_{|X} = \mathcal{O}(X)^G$. Thus $\mathcal{O}(X)^G$ is generated by the $f_{i|X}$. ■

Let X be an irreducible affine variety with the symmetric group G acting on X. We define the *categorical quotient* of X by G to be $\text{spec}_{\max}\mathcal{O}(X)^G$ and we denote it by $X//G$. We see that since $\mathcal{O}(X)^G$ is an integral domain $X//G$ is irreducible. The main problems of geometric invariant theory involve giving a geometric interpretation of this variety. Here are a few examples.

Examples.

1. $G = SL(n, \mathbb{C}, n > 1$, acting on $\mathbb{C}^n$ by the matrix action. Then $Gx = \mathbb{C}^n - \{0\}$. Thus $\mathcal{O}(\mathbb{C}^m)^G = \mathbb{C}1$. The categorical quotient is thus a single point. But there are 2 orbits.

2. $G = \mathbb{C}^\times$ acting on $\mathbb{C}^2$ by $z(x.y) = (zx, z^{-1}y)$. Then $\mathcal{O}(\mathbb{C}^2)^G = \mathbb{C}[xy]$. Thus $\mathbb{C}^2//G$ is isomorphic with $\mathbb{C}$ as an affine variety. If $a \neq 0$ then the set $X_a = \{(x, y)|xy = a\}$ is a single orbit. If $a = 0$ then the set X_0 is the union of three orbits. Thus the orbit space is more complicated than the categorical quotient.

In both of these examples it seems clear that one should assign to the exceptional point the orbit $\{0\}$.We will see why later.

3.1.2 Lie group actions.

Let M be a C^∞ manifold and let G be a Lie group. Then *Lie group action* of G on M is a C^∞ map $\Phi : G \times M \to M$ satisfying the same conditions as in the previous section. That is

1. $\Phi(1, x) = x$ (1denoting the identity element of G) for all $x \in X$.

2, $\Phi(gh, x) = \Phi(g, \Phi(h, x))$ for all $g, h \in G$ and $x \in X$.

We will denote such an action by gx. The set Gx is called the *orbit* of x. As before the main example will be the case when $\sigma : G \to GL(n, \mathbb{R})$ is a Lie group homomorphism and $M = \mathbb{R}^n$ with the usual action. We also note that if G is an algebraic group over $\mathbb{C}$ and X is a variety on which G acts algebraically and if X^o is the set of smooth points of X then the action of G on X^o is a Lie group action. Although it is a theorem that every manifold can be imbedded in a high dimensional $\mathbb{R}^n$ the analogue of Proposition 2 in 3.1.1 is not true.

Let G be a Lie group and let H be a closed subgroup of G. We give the space of cosets of G relative to H the quotient topology. The results that we will be describing are more genera however the reader can assume that $G \subset GL(n, \mathbb{R})$ is a closed subgroup since this is the only case that we have defined an exponential map. Consider $\mathfrak{g} = Lie(G)$ and $\mathfrak{h} = Lie(H)$ and let $V \subset Lie(G)$ be a real subspace complementary to $Lie(H)$. We consider the map $F : V \times H \to G$ given by $F(v, h) = \exp(v)h$.

Proposition 4 *There exists an open neighborhood, U, of 0 in V such that*

1. *$F(U \times H)$ is open in G.*
2. *$F : U \times H \to G$ is a diffeomorphism.*

This result allows us to put a C^∞ structure on G/H by using the fact that $G/H = \cup_{g \in G} gF(U, H)/H$ and taking $F(U, H)/H$ to have the C^∞ structure coming from the map $U \to F(U, H)/H$ given by $u \longmapsto \exp(u)H$.

Proposition 5 *The C^∞ structure on G/H defined above is uniquely determined by the condition that the map $G \times G/H \to G/H$ given by $g, xH \longmapsto gxH$ is a Lie group action.*

If G acts on the manifold M and if $x \in M$ then we set $G_x = \{g \in G | gx = x\}$. Then G_x is closed. If use the previous notation with $H = G_x$ we see that the map $U \to M$ given by $v \longmapsto \exp(v)x$ induces a C^∞ map of G/H to M whose image is the orbit of x. We note that the subspace topology for the

orbit is not the same as the topology induced by the map of G/H onto the orbit.

Example. Let $M = T^2$ the two dimensional torus, $S^1 \times S^1$ which we look upon as $\mathbb{R}^2/\mathbb{Z}^2$. We look upon $\mathbb{R}$ as a lie group under addition and we set $x \cdot ((u,v) + \mathbb{Z}^2) = (u + x, v + xa) + \mathbb{Z}^2$. with a fixed and in $\mathbb{R}$. If a is rational then the stabilizer of $(0,0) + \mathbb{Z}^2$ is the subgroup of $\mathbb{R}$ consistent of the $z \in \mathbb{Z}$ such that $za \in \mathbb{Z}$. Thus the corresponding homogeneous space is homeomorphic to S^1 and the orbit is closed and homeomorphic with S^1 in the subspace topology. However, if a is irrational then the orbit is dense in the torus and the stabilizer of $(0,0) + \mathbb{Z}^2$ is $\{0\}$. If we endow the orbit with the subspace topology and pull this topology back to $\mathbb{R}$ we have a different topology on $\mathbb{R}$.

This indicates that the algebraic situation is tamer. we will for the most part be studying it but allowing ourself to use Lie theoretic methods when they are useful.

3.1.3 Algebraic homogeneous spaces.

In this section we will recall some standard results on algebraic homogeneous spaces. Let G be an affine algebraic group over $\mathbb{C}$ and let H be a closed subgroup. Then in the sense of Lie group theory there is a natural C^∞ structure on G/H We will now show that there is also a natural structure of a quasi-projective variety on G/H which is smooth and yields the C^∞ structure.

We start with the following

Proposition 6 *Let G be an affine algebraic group and let H be a Z-closed subgroup. Then there exists a regular representation (σ, V) and a vector $v \in V$ such that $H = \{g \in G | \sigma(g)v \in \mathbb{C}v\}$.*

We won't give a proof of this result here since it is adequately covered in the literature (and the usual proof is not terribly instructive).

Let $G, H, (\sigma, V)$ be as in the statement of the Proposition and barring some trivial examples we may assume that $\dim V > 0$. Let $\dim V = n$ and choose a basis of V establishing an isomorphism of V with $\mathbb{C}^{n+1}$. We let G act on $\mathbb{P}^n$ by $g[x] = [\sigma(g)x]$ for $x \in \mathbb{C}^{n+1} - \{0\}$. Then $H = \{g \in G | g[v] = [v]\}$. Thus the morphism $G \to \mathbb{P}^n$ given by $g \longmapsto g[v]$ yields a bijection ϕ between

G/H and the orbit $G[v]$. Now $G[v[$ is open in its closure $Y \subset \mathbb{P}^n$. Thus the orbit is a quasi-projective variety. This induces a structure a quasi-projective variety on the coset space. The basic result is that this structure is independent of the choices made to construct it. We will call this algebraic variety G/H.

The upshot is

Theorem 7 *Let X be a variety and let G be affine algebraic acting on X let $x \in X$ the the morphism G to Gx induces an isomorphism of G/G_x onto Gx (recall that $G_x = \{g \in G | gx = x\}$).*

3.2 Invariants and closed orbits.

3.2.1 Invariants.

Let G be a symmetric subgroup of $GL(m, \mathbb{C})$ and let (σ, V) be a regular representation of G. Let $\dim V = n$. Let $\mathcal{O}(V)^G$ denote that space of all regular functions on V that are invariant under $\sigma(G)$. That is,

$$\mathcal{O}(V)^G = \{f \in \mathcal{O}(V) | f(\sigma(g)v) = f(v), g \in G, v \in V\}.$$

Given an element $v \in V$ we will use the notation $X_v = \{w \in V | f(w) = f(v), f \in \mathcal{O}(V)^G\}$. Of particular interest is X_0 (0 the zero vector) which is usually called the *null cone.*

We note that we have an equivalence relation $v \sim w$ if $w \in X_v$. The equivalence classes relative to this relation form the point set of the categorical quotient $V//\sigma(G)$. The key result in this context is

Theorem 8 *Each set X_v contains a unique closed orbit.*

Before we prove this result we will study some consequences. If $v = 0$ then the theorem implies that $\{0\}$ is the unique closed orbit in X_0. This implies that if $w \in X_0$ then $0 \in \overline{\sigma(G)w}$. More generally if $\sigma(G)w_o$ is the closed orbit in X_v and if $w \in X_v$ then $\sigma(G)w_o \subset \overline{\sigma(G)w}$.

We now begin a proof of the theorem. We will need the following result which will be used later also.

Lemma 9 *Let X be an affine variety and let Y, Z be Z-closed disjoint subsets of X. Then there exists $\phi \in \mathcal{O}(X)$ such that $\phi_{|Y} = 0$ and $\phi_{|Z} = 1$.*

Proof. Let J denote the ideal of elements in $\mathcal{O}(X)$ vanishing on Y. Then $L = J_{|Z}$ is an ideal in $\mathcal{O}(Z)$. If there exists $z \in Z$ such that $f(z) = 0$ for all $f \in L$ then $z \in Y$. Since $Y \cap Z = \emptyset$ we see that L has no zeros. The nullstellensatz implies that $L = \mathcal{O}(Z)$ and thus $1 \in L$. ■

We will now prove the theorem. Suppose that there are two closed orbits in X_v, $A = Gu$ and $B = Gw$ then since orbits are disjoint if they are distinct there exists $f \in \mathcal{O}(V)$ such that $f_{|Gu} = 1$ and $f_{|Gw} = 0$. Let $K = G \cap U(m)$ and $\phi = \mu_K(k \to f \circ \sigma(k))$. Then since K is Z-dense in G, $\phi \in \mathcal{O}(V)^G$. Since the orbits are clearly G-invariant we have a contradiction since ϕ must be constant on X_v.

This result says that if X is affine and G acts on X then $X//G$ is a variety parametrizing the closed orbits. This explains one aspect of the categorical quotient. However, it leads to the question of how one can approach the closed orbit that is in the closure of a not necessarily closed orbit. This is particularly important for elements in the null cone. At this point our only characterization of these elements is that all invariants that vanish at 0 vanish. Form our earlier results there are only a finite number of conditions to satisfy. However in most cases it is not an easy task to find a finite number of invariants to test. It is the Hilbert-Mumford criterion that gives an effective method. In the next section we will emphasize the null cone.

We will also record a variant of the above lemma proved in the argument the proof of the Proof of Theorem 8.

Proposition 10 *Let X be an affine variety with an action of a symmetric subgroup G of $GL(n, \mathbb{C})$. Assume that Y and Z are closed and G-invariant subvarieties of X such that $Y \cap Z = \emptyset$. Then there exists $f \in \mathcal{O}(X)$ such that $f(gx) = f(x)$ for $g \in G$ and $x \in X$ such that $f_{|Y} = 0$ and $f_{|Z} = 1$.*

3.2.2 A Hilbert-Mumford theorem.

Let G be a symmetric subgroup of $GL(n, \mathbb{C})$ and let (σ, V) be a regular representation of G. The Hilbert Mumford theorem states

Theorem 11 *A necessary and sufficient condition for $v \in V$ to be in the null cone is that there exist an algebraic group homomorphism $\varphi : \mathbb{C}^\times \to G$ such that $\lim_{z \to 0} \sigma(\varphi(z))v = 0$.*

We will see in our proof that we can assume that $\varphi(\overline{z}) = \varphi(z)^*$.

We will first prove a result for real symmetric groups which will be a model for a generalization that replaces the closed orbit $\{0\}$with a general closed orbit.

Let G be a connected closed symmetric subgroup of $GL(n, \mathbb{R})$ and let $K = G \cap O(n)$. As usual we set $\mathfrak{g} = Lie(G) \subset M_n(\mathbb{R})$ and $\mathfrak{k} = Lie(K) \subset \mathfrak{g}$. Let $\theta(X) = -X^T$ and $\mathfrak{p} = \{X \in \mathfrak{g} | \theta X = -X\}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal subspace subject to the condition that $[X, Y] = 0$ if $X, Y \in \mathfrak{a}$. That is $\mathfrak{a}$ is a Cartan subspace. We set $A = \exp(\mathfrak{a})$. Then we have shown that

$$G = KAK.$$

Let (σ, V) be a representation of G on the finite dimensional real vector space V with an inner product $\langle ..., ... \rangle$such that K acts orthogonally and $\mathfrak{p}$ acts by self adjoint transformations. We wish to prove

Theorem 12 *Let $v \in V$, if $0 \in \overline{Gv}$ then there is an element $u \in Kv$ and $h \in \mathfrak{a}$ such that*

$$\lim_{t \to +\infty} \exp(th)u = 0.$$

Proof. We first note that since $\overline{Gv} = K\overline{(AKv)}$, the hypothesis implies that $0 \in \overline{(AKv)}$. Thus there exist sequences h_j with $h_j \in \mathfrak{a}$ and $k_j \in K$ such that

$$\lim_{j \to \infty} \exp(h_j) k_j v = 0.$$

Since K is compact we may assume $\lim_{j \to \infty} k_j = k \in K$. On $M_n(\mathbb{R})$ we use the Hilbert-Schmidt inner product $(\mathrm{tr}(XY^T) = \langle X, Y \rangle)$. Then we write $h_j = t_j u_j$ with $t_j > 0$ and $\|u_j\| = 1$. We may assume that $\lim_{j \to \infty} u_j = u \in \overline{\mathfrak{a}}$. If $\lambda \in \mathfrak{a}^*$ then define $V_\lambda = \{x \in V | hx = \lambda(h)v, h \in \mathfrak{a}\}$.Set $\Sigma = \{\lambda \in \mathfrak{a}^* | V_\lambda \neq 0\}$. If $x \in V$ then we write $x = \sum_{\lambda \in \Sigma} x_\lambda$ with $x_\lambda \in V_\lambda$. Then, since $\langle V_\lambda, V_\mu \rangle = 0$ if $\lambda \neq \mu$, we have

$$\lim_{j \to \infty} \sum_{\lambda} e^{t_j \lambda(u_j)} \|(k_j v)_\lambda\|^2 = 0.$$

Thus for every $\lambda \in \Sigma$ we have

$$\lim_{j \to \infty} e^{t_j \lambda(u_j)} \|(k_j v)_\lambda\|^2 = 0.$$

If $\lambda(u) > 0$ then $\lambda(u_j) > 0$ for $\lambda \geq N$ for some N. Thus $e^{t_j \lambda(u_j)} \|(k_j v)_\lambda\|^2 \geq \|(k_j v)_\lambda\|^2$. Hence we must have $(kv)_\lambda = 0$. Suppose that $\lambda(u) = 0$ and $(kv)_\lambda \neq 0$ then if for an infinite number of j we have $\lambda(u_j) \geq 0$ we will run into the same contradiction. We therefore see that we may assume that if $\lambda(u) = 0$ and $(kv)_\lambda \neq 0$ then $\lambda(u_j) < 0$ all j. Let N be so large that if $\lambda(u) < 0$ then $\lambda(u + u_N) < 0$. Then we have $\lambda(u + u_N) < 0$ for all λ such that $(kv)_\lambda \neq 0$ and $\lambda(u) \leq 0$, hence for all λ such that $(kv)_\lambda \neq 0$. Take $h = u + u_N$. Then $e^{th} kv \to 0$ as $t \to +\infty$. ■

We will now show how the above result implies Theorem 11. We return to the situation of a symmetric subgroup, G, of $GL(n, \mathbb{C})$. We take a maximal compact torus, T, in K and H to be the Z-closure of T in G. Then $H = T \exp(iLie(T))$. As usual we set $\mathfrak{a} = iLie(T)$ and $A = \exp(\mathfrak{a})$. We note that H is isomorphic with $(\mathbb{C}^\times)^m$ and any algebraic homomorphism of $\mathbb{C}^\times$ to $(\mathbb{C}^\times)^m$ is given by $z \to (z^{q_1}, z^{q_2}, ..., z^{q_m})$. Thus if we take as a basis of $Lie(H)$ the elements $e_1, ..., e_m$ with $\exp(\sum z_j e_j) = (e^{z_1}, e^{z_2}, ..., e^{z_m})$. Then an algebraic homomorphism $\phi : \mathbb{C}^\times \to H$ is given by $\phi(e^z) = \exp(z(\sum k_j e_j))$ with k_j in $\mathbb{Z}$. Now let $h = \sum h_i e_i$ be as in the Theorem above. Let $w_i \in \mathbb{Q}$ be such that $\sum |w_i - h_i| < \delta$ for some small δ (to be determined). If we take δ sufficiently small then of $\lambda \in \Sigma$ (see the notation in the proof of the Theorem above) is such that $v_\lambda \neq 0$ then $\lambda(\sum w_i e_i) < 0$. Now let p be a positive integer such that pw_i is an integer for each i. Set $X = -p \sum w_i e_i$. Then $\phi(e^z) = \exp(zX)$ defines an algebraic homomorphism from $\mathbb{C}^\times$ to H. Since $\lim_{z \to 0} \phi(z)u = 0$, Theorem 11 is proved.

3.2.3 A generalization of the Hilbert-Mumford theorem.

In this section we will given an exposition of some methods of Richardson that appeared in ??? that gives an analog of the Hilbert Mumford theorem with 0 replaced with any closed orbit. Such a theorem was conjectured by Mumford in great generality.

Let G and (σ, V) be as in Theorem 11 in the previous subsection and let A be as in the end of the subsection. In this subsection we will prove

Theorem 13 *Let $v \in V$ and let Gw be the closed orbit in $\overline{Gv}$. Then there exists an algebraic group homomorphism $\varphi : \mathbb{C}^\times \to G$ and an element such that $\lim_{z \to 0} \sigma(\varphi(z))v \in Gw$. Furthermore, φ can be chosen so that $\varphi(\overline{z}) = \varphi(z)^*$.*

The main ingredient of the proof is the following result based on Richardson's argument.

Theorem 14 *Let $v \in V$ and let Gw be the closed orbit in $\overline{Gv}$ then there is $k \in K$ such that $\overline{Akv} \cap Gw \neq \emptyset$.*

Proof. We will use the notation of the previous section. We may assume that $Gw \subset \overline{Gv} - Gv$. Let H be as before, the Z-closure of A. Then H is an algebraic torus in G. We note that it is enough to prove that there exists $k \in K$ such that $\overline{Hkv} \cap Gw \neq \emptyset$. Indeed, if we have shown this then if $T = H \cap U(n)$ then $H = TA$. thus $\overline{Hkv} = \overline{TAkv} = T\,\overline{Akv}$ since T is compact. Now Gw is T-invariant and hence $\overline{Akv} \cap Gw \neq \emptyset$. Set $Y = Gw$.Then by assumption it is Z-closed and disjoint from Gv. We will now prove the assertion for H by contradiction. Assume it is false. Then $\overline{Hkv} \cap Y = \emptyset$ for all $k \in K$. Since $\overline{Hkv}$ and Y are H invariant and H is symmetric Proposition ??? implies that for each $k \in K$ there exists $f_k \in \mathcal{O}(V)^H$ such that $f_{k|Y} = 0$ and $f_{k|\overline{Hkv}} = 1$. Let $U_k = \{x \in V | f_k(x) \neq 0\}$ Then U_k is open in V and contains kv. This implies since Kv is compact that there exist $k_1, ..., k_m \in K$ such that $Kv \subset U_{k_1} \cup \cdots \cup U_{k_m}$. Let

$$f(x) = \sum_{i=1}^{m} |f_{k_i}(x)|$$

for $x \in V$. Then f is continuous, H-invariant and $f(x) > 0$ if $x \in Kv$. This implies that f attains a minimum $\xi > 0$ on the compact set Kv. Thus the H-invariance implies that $f(x) \geq \xi$ for $x \in \overline{HKv}$. Since $f(y) = 0$ this implies that $\overline{HKv} \cap Y = \emptyset$. Thus $\cup_{k \in K} k\overline{HKv} \cap kY = \emptyset$. Since $kY = Y$ this implies that $K\overline{HKv} \cap Y = \emptyset$. Since $K\overline{HKv} = \overline{KHKv} = \overline{Gv}$ we have our desired contradiction. ■

We will be using the rest of this section to prove Theorem 13 using Theorem 14. For this we will need two lemmas the latter taken from ??? the former a very special case of a general separation result for convex sets.

Lemma 15 *Let C be a closed convex subset of $\mathbb{R}^n$ not containing 0. Then there exist $f \in (\mathbb{R}^n)^*$ such that $f(x) > 0$ for all $x \in C$.*

Proof. We may assume that $C \neq \emptyset$. Let $m = \inf_{x \in C} \|x\|$. Then, since C is closed there exists $x \in C$ such that $\|x\| = m$. We assert that $f(y) = \langle x, y \rangle$

has the desired property. If $y \in C$ and $f(y) = 0$ then we consider $z(t) = (1-t)x + ty$ for $0 \leq t \leq 1$. Then we leave it to the reader to check that

$$\|z(t)\|^2 < \|x\|^2 \text{ for } 0 < t < \frac{2\|x\|^2}{\|x\|^2 + \|y\|^2}.$$

This contradicts the definition of m. Thus $f(y) \neq 0$ for $y \in C$. The assertion about f follows since $f(x) > 0$ and C is connected ■

Lemma 16 *Let $v_1, ..., v_m \in \mathbb{Q}^n$ be such that there exists $1 \leq k \leq m$ such that*

1. If $r_1 v_1 + ... + r_m v_m = 0$ with $r_i \in \mathbb{Q}$ and $r_i \geq 0$ then $r_i = 0$ for $1 \leq i \leq k$;

2. There exist $s_{k+1}, ..., s_m \in \mathbb{Q}$ such that $s_i > 0$ for $k+1 \leq i \leq m$ such that $\sum_{i=k+1}^m s_i v_i = 0$.

Then there exists $f \in (\mathbb{Q}^n)^*$ such that

$$\begin{cases} f(v_i) > 0, 1 \leq i \leq k \\ f(v_i) = 0, k+1 \leq i \leq m \end{cases}.$$

Proof. Let $W = span_{\mathbb{R}}\{v_{k+1}, ..., v_m\}$. Let $Z = \mathbb{R}^n/W$ and let $z_i = v_i + W$ for $i = 1, ..., k$. Consider the convex hull, C, of $\{z_1, ..., z_k\}$and assume that $0 \in C$. Thus $0 = a_1 z_1 + ... + a_k z_k$ with $a_i \geq 0$ and $\sum a_i = 1$. The a_i are thus either 0 or $a_i > 0$. Let

$$S = \{c = (c_1, ..., c_k) \in \mathbb{R}^k | c_i = 0 if a_i = 0 and \sum c_i z_i = 0\}.$$

The space $\mathbb{Q}^k \cap S$ is dense in S. Thus one would have a relation $b_1 z_1 + ... + b_k z_k = 0$ with $b_i \in \mathbb{Q}$, $b_i = 0$ if $a_i = 0$ and $b_i > 0$ if $a_i > 0$. This implies that there exist u_j with $u_j \in \mathbb{Q}$, $j = k+1, ..., m$ such that $b_1 v_1 + ... + b_k v_k + u_{k+1} v_{k+1} + ... + u_m v_m = 0$. Now using 2. we see that we can assume that $u_i > 0$. This contradicts 1. in the statement. We therefore see that $0 \notin C$. The lemma above now implies that there exists $g \in Z^*$ such that $g_{|C} > 0$. We may as above assume that $g(z_i) \in \mathbb{Q}$. Now take $f(x) = g(x + W)$. ■

We are now ready to prove Theorem 13. We first replace v by kv (k as in Theorem 14). We show that it is enough to prove the result for H. Indeed, Theorem 14 implies $\overline{Hv} \cap Gw \neq \emptyset$. Let Hu be the unique closed H-orbit in $\overline{Hv}$. Then Hu must be contained in $\overline{Hv} \cap Gw$ (since both sets are closed and the unique closed orbit is contained in every H-orbit closure contained

in $\overline{Hv}$). We can also replace H by its image in $GL(V)$. We note that we can choose a basis of V such that H consists of diagonal matrices that is we look upon V as $\mathbb{C}^m$. We may also assume that $v = (v_1, ..., v_m)$ with $v_j \neq 0$ for all $j = 1, ..., m$ since we can project $\overline{Hv}$ into off of the zero coordinates of v. Let H_m be the group of diagonal, invertible, $m \times m$ matrices and let $D_m(F)$ be the vector space of $m \times m$ diagonal matrices over F. Under this hypothesis the map $T : D_m(\mathbb{C}) \to \mathbb{C}^m$ given by $T(z) = zv$ is an isomorphism of affine spaces. We can obviously assume that Hv is not closed. $Lie(H)$ is a subspace of $D_m(\mathbb{C})$ and $\mathfrak{a} = Lie(A)$ is a subspace of $D_m(\mathbb{R})$. We define $\delta_i(diag(z)) = z_i$. We also set $u_i = \delta_{i|\mathfrak{a}}$. Define

$$X = \{a \in \mathbb{Q}^m | \sum a_i u_i = 0\}, X^+ = \{a \in X | a_i > 0, i = 1, ..., m\}.$$

We also set $J = \{j | \text{there exists } a \in X^+, \text{ such that } a_j > 0\}$. We may assume that $J = \{k+1, ..., m\}$ with $k \geq 0$. We assert that $k > 0$. Indeed if $k = 0$ then $J = \{1, ..., m\}$. However, assume j is in J then there exists $a \in X^+$ with $a_j > 0$. But then if $z \in A$ then

$$\prod z_i^{a_i} = 1.$$

This implies that if z is in the closure of A in $D_n(\mathbb{R})$ then the above product is also 1. Thus we must have that if $a \in \overline{A}$ then $a_j \neq 0$. This implies since $H = TA$ if $z \in \overline{H}$ then $z_j \neq 0$. We therefore see that if $k = 0$ then $\overline{H} \subset H_m$. Since H is closed in H_m this would imply $\overline{H} = H$. We therefore see that $k > 0$.

We now observe that if we use the basis $e_1, ..., e_n$ of $Lie(H)$ in the proof of Theorem 11 (at the end of the last section) then $u_i(e_j) \in \mathbb{Q}$. We $\mathfrak{a}^*_{\mathbb{Q}}$ with the space spanned by the dual basis $e_1^*, ..., e_n^*$ of $e_1, ..., e_n$ over $\mathbb{Q}$. We now observe that $u_1, ..., u_m$ and k satisfy Lemma 16. condition 1. is by our definition of k and condition 2. is satisfied by taking the sum of $\beta_j \in X^+$ each giving one choice of $\beta_j \in X^+$ that has jth coefficient positive. We conclude that there exists $f \in (\mathfrak{a}^*_{\mathbb{Q}})^*$ such that $f(u_i) > 0$ for $1 \leq i \leq k$ and $f(u_i) = 0$ for $k+1 \leq i \leq m$. Multiplying f by a positive integer we may assume that $f(e_i^*) \in \mathbb{Z}$ for all i. Thus $h = \sum f(e_i^*) e_i \in \mathfrak{a}$ and there exists a algebraic group homomorphism $\varphi : \mathbb{C}^\times \to H$ such that $\phi(e^z) = e^{zh}$. By the definition of f we see that

$$\lim_{z \to 0} \varphi(z) v = (0, ..., 0, v_{k+1}, ..., v_m) = w.$$

To complete the proof we must show that Hw is closed. If it were note closed that we could use exactly the same argument to show that there exists $m \geq p > k$ such that $(0, ..., 0, v_{p+1}, ..., v_m) \in \overline{Hw} \subset \overline{Hv}$ but this contradicts the fact that the coordinates of elements in $\overline{Hv}$ are non-zero for $j \geq k+1$.

We have finally finished the proof.

3.2.4 The Kempf-Ness Theorem.

In this section we will derive the Kempf-Ness theorem from the theorem in the preceding section.

Let G be a connected symmetric subgroup of $GL(n, \mathbb{C})$ and let, as usual, $K = U(n) \cap G$. Let $\langle ..., ... \rangle$ be the usual $U(n)$ invariant inner product on $\mathbb{C}^n$. We say that $v \in \mathbb{C}^n$ is *critical* if $\langle Xv, v \rangle = 0$ for all $X \in Lie(G)$. We note that the set of critical points is invariant under the action of K. Here is the theorem.

Theorem 17 *Let G, K be as above. Let $v \in \mathbb{C}^n$.*

1. *If v is critical if and only if $\|gv\| \geq \|v\|$ for all $g \in G$.*
2. *If v is critical and $w \in Gv$ is such that $\|v\| = \|w\|$ then $w \in Kv$.*
3. *If v is critical then Gv is closed.*
4. *If Gv is closed then there exists a critical element in Gv.*

As we wiill see part 3. of the theorem is the hardest part. Before we give a proof we record a corollary

Corollary 18 *If v, w are critical then $w \in Gv$ implies $w \in Kv$.*

Proof. If $w = gv$ with $g \in G$ then $\|w\| \geq \|v\|$ by 1. But also $v = hw$ with $h \in G$ so $\|v\| \geq \|w\|$. The result now follows from 2. ■

We will devote the rest of this section to a proof of the theorem.

We note that $G = K \exp(\mathfrak{p})$ with $\mathfrak{p} = iLie(K)$. If $X \in \mathfrak{p}$ then

$$\frac{d}{dt} \|\exp(tX)v\|^2 = \frac{d}{dt} \langle \exp(tX)v, \exp(tX)v \rangle =$$

$$\begin{aligned} & \langle \exp(tX)Xv, \exp(tX)v \rangle + \langle \exp(tX)v, \exp(tX)Xv \rangle \\ = \; & 2 \langle \exp(tX)Xv, \exp(tX)v \rangle . \end{aligned}$$

since $X^* = X$. Arguing in the same way we have

$$\frac{d^2}{dt^2} \|\exp(tX)v\|^2 = 4 \langle \exp(tX)Xv, \exp(tX)Xv \rangle .$$

We now begin the proof. Suppose v is critical then if $X \in \mathfrak{p}$ and $k \in K$ we have

$$\|k \exp(tX)v\|^2 = \|\exp(tX)v\|^2 = \alpha(t).$$

By the above $\alpha'(0) = 2 \langle Xv, v \rangle = 0$ and $\alpha''(t) = 4 \langle \exp(tX)Xv, \exp(tX)Xv \rangle \geq 0$. This implies that $t = 0$ is a minimum for α. In particular, if $g = k \exp X$ as above then

$$\|gv\|^2 = \|k \exp(X)v\|^2 = \|\exp(X)v\|^2 \geq \|v\|^2 .$$

If $v \in \mathbb{C}^n$ and $\|gv\| \geq \|v\|$ for all $g \in G$ and if $X \in \mathfrak{p}$ then

$$0 = \frac{d}{dt} \|\exp(tX)v\|^2_{|t=0} = 2 \langle Xv, v \rangle .$$

Since $Lie(G) = \mathfrak{p} + i\mathfrak{p}$, we see that $\langle Yv, v \rangle$ for all $Y \in Lie(G)$. This proves 1.

We will now prove 2. Suppose $w = k \exp(X)v$ and assume that $\|v\| = \|w\|$. Then defining $\alpha(t) = \|\exp(tX)v\|^2$ as above we see that if $Xv \neq 0$ then $\alpha''(t) > 0$ for all t. Since $\alpha'(0) = 0$ this implies that $\alpha(t) > \alpha(0)$ for $t \neq 0$. Since $\alpha(1)$ is assumed to be equal to $\alpha(0)$ we conclude that $\exp(X)v = v$ so $gv = kv$.

We now prove 4. Suppose that Gv is closed (we may assume that this is in either the S or the Z topology). Let $m = \inf_{g \in G} \|gv\|$. Then since Gv is closed in the S-topology there exists $w \in Gv$ with $\|w\| = m$. 4. now follows from 1.

We are left with 3. Let v be critical and assume that Gv is not closed. Let Y be the closed orbit contained in $\overline{Gv}$ We will derive a contradiction. By Theorem 13 there exists $\varphi : \mathbb{C}^\times \to G$ algebraic homomorphism such that $\varphi(e^t) = \exp(tX)$ with $X \in \mathfrak{p}$ such that $\lim_{t \to -\infty} e^{tX}v = y \in Y$. Since X is self adjoint $\mathbb{C}^n$ has an orthonormal basis $u_1, ..., u_n$ such that $Xu_i = \lambda_i u_i$ for $i = 1, ..., n$. Thus if $v = \sum v_i u_i$ we have $\exp(tX)v = \sum e^{t\lambda_i} v_i u_i$. Now $\langle Xv, v \rangle = 0$ so

$$\sum \lambda_i |v_i|^2 = 0.$$

We conclude that if some $\lambda_i \neq 0$ with $v_i \neq 0$ then there must be $\lambda_j \neq 0$ with $v_j \neq 0$ such that $\lambda_i \lambda_j < 0$. However, we have

$$\|y\|^2 = \lim_{t \to -\infty} \sum e^{2\lambda_l t} |v_l|^2.$$

We conclude that we must have all the $\lambda_i = 0$ and this provides the desired contradiction.

3.2.5 The Matsushima criterion.

In this section we will (following ideas of Ivan Losev) show how one can give a proof of the Matsushima criterion in the following form. This version also gives a proof that a linearly reductive group over $\mathbb{C}$ is symmetric in our sense without invoking the theory of semisimple Lie algebras.

Theorem 19 *Let G be a symmetric subgroup of $GL(n, \mathbb{C})$ and let H be a Z-closed subgroup such that G/H is affine. Then H is conjugate in G to a symmetric subgroup of $Gl(n, \mathbb{C})$.*

Proof. We shown that there exists an embedding of G/H into $\mathbb{C}^m$ with Z-closed image X and a regular representation σ of G on $\mathbb{C}^m$ such that X is $\sigma(G)$-invariant. Thus $X = \sigma(G)v$ with $v \in \mathbb{C}^n$ a closed orbit. The Kempf-Ness theorem implies that there exists a critical vector $w \in X$. Let $w = \sigma(u)v$. Then replacing H by uHu^{-1} we may assume v is critical. Let $K = G \cap U(n)$ and let $\langle ..., ... \rangle$denote a $\sigma(K)$-invariant inner product on $\mathbb{C}^m$. Let $\omega(x, y) = \operatorname{Im} \langle x, y \rangle$for $x, y \in \mathbb{C}^m$. Then if we look upon $\mathbb{C}^m$ as $\mathbb{R}^{2m}$ in the usual way, ω defines a symplectic structure on $\mathbb{C}^m$. If $W \subset \mathbb{C}^m$ is a complex subspace then the restriction of ω to the space is non-degenerate. We will leave off the σ in writing the action of G and $Lie(G) = \mathfrak{g}$ on $\mathbb{C}$.We will also use the notation $\mathfrak{k} = Lie(K)$. Let $X, Y \in \mathfrak{k}$. We calculate $\omega(Xv, Yv) = \operatorname{Im} \langle Xv, Yv \rangle = -\operatorname{Im} \langle v, XYv \rangle$. Similarly $\omega(Xv, Yv) = -\omega(Yv, Xv) = -\operatorname{Im} \langle Yv, Xv \rangle = \operatorname{Im} \langle v, YXv \rangle$ .Hence we have

$$\omega(Xv, Yv) = \frac{1}{2} \operatorname{Im} \langle [X, Y]v, v \rangle = 0$$

since v is critical. This implies since ω is non-degenerate on $\mathfrak{g}v$ that

$$\dim_{\mathbb{R}} \mathfrak{k}v \leq \frac{1}{2} \dim_{\mathbb{R}} \mathfrak{g}v = \dim_{\mathbb{C}} \mathfrak{g}v.$$

On the other hand if K_v (resp. G_v) is the stabilizer in K (resp. G)of v then we have $G_v \cap K = K_v$ and

$$\dim_{\mathbb{R}} \mathfrak{k} v = \dim_{\mathbb{R}} \mathfrak{k} - \dim Lie(K_v) = \dim_{\mathbb{C}} \mathfrak{g} - \dim(K_v.).$$

Applying the preceding inequality this implies that $\dim_{\mathbb{C}} G - \dim K_v \leq \dim_{\mathbb{C}} G - \dim_{\mathbb{C}} G_v$. Thus $\dim K_v \geq \dim_C G_v$. But G_v contains the Z-closure of K_v in G hence G_v must be the Z-closure of K_v so it is symmetric. ■

Corollary 20 *If $G \subset GL(n, \mathbb{C})$ is linearly reductive then G is conjugate in $GL(n, \mathbb{C})$ to a symmetric subgroup.*

Proof. Since G is linearly reductive $\mathcal{O}(GL(n, \mathbb{C}))^G$ (relative to the action of right multiplication) is finitely generated (Theorem 3) and Theorem 8 is true. This implies that $GL(n, \mathbb{C})/G$ is the categorical quotient and hence affine. To complete the proof we will prove the two assertions. ■

We assume that $G \subset GL(n, \mathbb{C})$ is linearly reductive. Then if G acts on an affine space, X, then we set $\tau(g)f(x) = f(x^{-1}x)$ for $g \in G$ and$x \in X$. If $f \in \mathcal{O}(X)$ then dimspan$_{\mathbb{C}}\tau(g)f < \infty$ and the restriction of $\tau(g)$ to span$_{\mathbb{C}}\tau(g)f$ defines a regular representation. This implies that if $U \subset \mathcal{O}(X)$ is finite and if

$$W_U = \text{span}_{\mathbb{C}}\{\tau(g)f | f \in U\}$$

then $W_U = W_U^G \oplus Z_U$ a direct sum of G invariant spaces. Furthermore, by complete reducibility Z_U is a direct sum of non-trivial irreducible representations. Let P_U be the natural projection of W_U onto W_U^G. We assert that if $U \subset \mathcal{O}(X)$is finite and if $f \in W_U$ then $P_{\{f\}}f = P_U f$. Indeed, We write $f = P_U f + u$ with $u \in Z_U$. Thus $W_{\{f\}} \subset \mathbb{C}P_U f \oplus Z_U$. Thus $W_{\{f\}} = \mathbb{C}P_U f \oplus Z_U \cap Z_{\{f\}}$. Thus P_U leaves $W_{\{f\}}$ invariant and so $P_{U|W_{\{f\}}} = P_{\{f\}}$. We therefore have defined a linear projection of $\mathcal{O}(X)$ onto $\mathcal{O}(X)^G$. We will denote it by R. Similarly, if (σ, V) is a finite dimensional regular representation of G then $V = V^G \oplus Z_V$ with Z_V a direct sum of irreducible non-trivial representations of G and we denote the projection onto V^G by R. We have

Lemma 21 *If (ρ, V) and (σ, W) are regular representations of G and $T \in Hom_G(V, W)$ then $TRv = RTv$.*

Proof. Let $V = V^G \oplus Z_V$ then we must have $TV^G \subset W^G$ and $TZ_V \subset Z_W$. Thus if $v \in V$ we have $v = Rv + (I - R)v$ so $Tv = TRv + T(I - R)V$ now $T(I - R)V \subset Z_W$ thus $RTv = RTRv$. But R is the identity on W^G. ■

Corollary 22 *Let G act on the affine space X then if Y is a G-invariant Z-closed subset and if $f \in \mathcal{O}(X)$ then $(Rf)_Y = R(f_Y)$.*

Corollary 23 *Theorem 8 is true in the context of linearly reductive groups.*

Lemma 24 *If X is an affine variety with G acting algebraically then if $f \in \mathcal{O}(X)^G$ and $\phi \in \mathcal{O}(X)$ then $R(f\phi) = fR\phi$.*

Proof. $W_{\{\phi,f\phi\}} = W_{\{\phi\}} + fW_{\{\phi\}}$. Now $W_{\{\phi\}} = W^G_{\{\phi\}} \oplus Z_{\{\phi\}}$ and so $fW_{\{\phi\}} = fW^G_{\{\phi\}} + fZ_{\{\phi\}}$. Now $RfZ_{\{\phi\}} = 0$ so $Rf\phi = RfR\phi + Rf(I-R)f = RfR\phi = fR\phi$. ■

Now the proof of the finite generation is proved in exactly the same way as Theorem 3.