2 Lie groups and algebraic groups.

2.1 Basic Definitions.

In this subsection we will introduce the class of groups to be studied. We first recall that a *Lie group* is a group that is also a differentiable manifold and multiplication $(x, y \longmapsto xy)$ and inverse $(x \mapsto x^{-1})$ are C^∞ maps. An algebraic group is a group that is also an algebraic variety such that multiplication and inverse are morphisms.

Before we can introduce our main characters we first consider $GL(n, \mathbb{C})$ as an affine algebraic group. Here $M_n(\mathbb{C})$ denotes the space of $n \times n$ matrices and $GL(n, \mathbb{C}) = \{g \in M_n(\mathbb{C}) | \det(g) \neq)\}$. Now $M_n(\mathbb{C})$ is given the structure of affine space $\mathbb{C}^{n^2}$ with the coordinates x_{ij} for $X = [x_{ij}]$. This implies that $GL(n, \mathbb{C})$ is Z-open and as a variety is isomorphic with the affine variety $M_n(\mathbb{C})_{\{\det\}}$. This implies that $\mathcal{O}(GL(n, \mathbb{C})) = \mathbb{C}[x_{ij}, \det^{-1}]$.

Lemma 1 *If G is an algebraic group over an algebraically closed field, F, then every point in G is smooth.*

Proof. Let $L_g : G \to G$ be given by $L_g x = gx$. Then L_g is an isomorphism of G as an algebraic variety $(L_g^{-1} = L_{g^{-1}})$. Since isomorphisms preserve the set of smooth points we see that if $x \in G$ is smooth so is every element of $Gx = G$. ■

Proposition 2 *If G is an algebraic group over an algebraically closed field F then the Z-connected components*

Proof. Theorem 18 in section 1.2.6 implies that every element of G is contained in a unique irreducible component. ■

Theorem 3 *A closed subgroup of $GL(n, \mathbb{C})$ is a Lie group.*

This theorem is a special case of the fact that a closed subgroup of a Lie group is a Lie group. We should also explain what "is" means in these contexts. The result needed is

Theorem 4 *Let G and H be Lie groups then a continuous homomorphism $f : G \to H$ is C^∞.*

This implies that there is only one Lie group structure associated with the structure of G as a topological group.

If G is a closed subgroup of $GL(n,\mathbb{C})$ then we define the Lie algebra of G to be

$$Lie(G) = \{X \in M_n(\mathbb{C}) | e^{tX} \in G \text{ for all } t \in \mathbb{R}\}.$$

Some explanations are in order. First we define $\langle X, Y\rangle = \mathrm{tr} XY^*$ and $\|X\| = \sqrt{\langle X, X\rangle}$ We use this to define the metric topology on $M_n(\mathbb{C})$. We note that $\|XY\| \leq \|X\| \, \|Y\|$ so if we set

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}$$

then this series converges absolutely and uniformly in compacta. In particular the implies that

$$X \to e^X$$

defines a C^∞ map of $M_n(\mathbb{C})$ to $GL(n,\mathbb{C})$ in fact real (even complex) analytic. We also note that if $\|X\| < 1$ then the series

$$\log(I - X) = \sum_{m=1}^{\infty} \frac{X^m}{m}$$

converges absolutely and uniformly on compacta. This says that if $\delta > 0$ is so small that if $\|X\| < \delta$ then $\left\|I - e^X\right\| < 1$ we have $\log(e^X) = \log(I - (I - e^X)) = X$.

Proposition 5 *Let G be a closed subgroup of $GL(n,\mathbb{C})$ then $Lie(G)$ is an $\mathbb{R}$-subspace of $M_n(\mathbb{C})$such that if $X, Y \in Lie(G)$, $XY - YX = [X, Y] \in Lie(G)$.*

Proof. We have

$$e^{sX} e^{tY} = e^{sX + tY + \frac{st}{2}[X,Y] + O((|t|+|s|)^3)}.$$

To to sketch a proof this we take s and t so small that $\left\|I - e^{sX} e^{tY}\right\| < 1$. Expand the series $\log(I - (I - e^{sX} e^{tY}))$ ignoring terms that are $O((|t|+|s|)^3)$. (c.f. [GW], ???). For details.

We now prove the Lemma. We note that for fixed t

$$(e^{\frac{t}{m}X} e^{\frac{t}{m}Y})^m = e^{t(X+Y)+O(\frac{1}{m})}.$$

Taking the limit as $m \to \infty$ shows that $Lie(G)$ is a subspace. Also for fixed t

$$(e^{\frac{t}{m}X}e^{\frac{t}{m}Y}e^{-\frac{t}{m}X}e^{-\frac{t}{m}Y})^{m^2} = e^{t^2[X,Y]+O(\frac{1}{m})}.$$

Thus $e^{t^2[X,Y]} \in G$. Take inverses to get negative multiples of $[X,Y]$. ■

Theorem 6 *The Lie algebra of a linear algebraic group,$G \subset GL(n,\mathbb{C})$, is a complex subspace of $M_n(\mathbb{C})$. Furthermore $Lie(G) = T_I(G)$in the sense of algebraic geometry.*

G acts on $Lie(G)$ via $Ad(g)X = gXg^{-1}$, We note that $ge^Xg^{-1} = e^{Ad(g)X}$so $gLie(G)g^{-1} = Lie(G)$ for $g \in G$. We also note that if $X, Y \in M_n(\mathbb{C})$ then

$$\frac{d}{dt}e^{tX}Ye^{-tX} = Ad(e^{tX})[X,Y].$$

Then is

$$\frac{d}{dt}Ad(e^{tX}) = Ad(e^{tX})ad(X)$$

with $ad(X)Y = [X,Y]$. This implies that as an endomorphism of $M_n(\mathbb{C})$ $Ad(e^{tX}) = e^{tad(X)}$. If $X \in \mathfrak{g}$ the $ad(X)\mathfrak{g} \subset \mathfrak{g}$ so these formulas are true for any closed subgroup of $GL(n,\mathbb{C})$.

2.1.1 Some remarks about compact groups.

Let $K \subset GL(n,\mathbb{C})$ be a compact subgroup. Then K is closed and hence a Lie group. We will denote by μ_K the left invariant, positive normalized measure on K(Haar measure). We recall what this means. We define a (complex) measure on K to be a continuous linear map, μ, of $C(K)$ (continuous functions from K to $\mathbb{C}$) to $\mathbb{C}$ where $C(K)$ is endowed with the uniform topology. That is, we set $\|f\| = \max_{x\in K}|f(x)|$ and a linear map $\mu : C(K) \to \mathbb{C}$ is a measure if there exists a constant such that

$$|\mu(f)| \leq C\|f\|, f \in C(K).$$

We say that μ is positive if $\mu(f) \geq 0$ if $f(K) \subset [0,\infty)$. We say that it is normalized if it is positive and $\mu(1) = 1$ (1 the constant function taking the value 1). Finally, if $k \in K$ we define $L_kf(x) = f(k^{-1}x)$ for $k, x \in K$ and $f \in C(K)$. Then $L_k : C(K) \to C)(K)$. Left invariant means that $\mu \circ L_k = \mu$. One proves that a normalized invariant measure on K is unique and that it is given by integration of a differential form. For our purposes we will only need the following property (and its proof) that follows from the fact that it is given by the integration against a differential form.

Theorem 7 *Let V be a finite dimensional vector space over $\mathbb{C}$. Let $\sigma : K \to GL(V)$ be a continuous homomorphism. Then there exists a Hermitian inner product, $(...,...)$, on V such that $(\sigma(k)v, \sigma(k)w) = (v,w)$ for $v, w \in V$ and $k \in K$.*

Proof. Let $\langle ...,... \rangle$ be an inner product on V (e.g. choose a basis and use the standard inner product). Define $(z,w) = \mu_K(k \to \langle \sigma(k)^{-1}z, \sigma(k)^{-1}w \rangle)$. Then one checks that this form is Hermitian, and positive definite since the measure μ_K is positive. We also note that if $u \in K$ then

$$(\sigma(u)z, \sigma(u)w) = \mu_K(k \to \left\langle \sigma(k)^{-1}\sigma(u)z, \sigma(k)^{-1}\sigma(u)w \right\rangle) =$$

$$\mu_k(k \to \left\langle \sigma(u^{-1}k)^{-1}z, \sigma(u^{-1}k)^{-1}w \right\rangle) = \mu_K(k \to \left\langle \sigma(k)^{-1}z, \sigma(k)^{-1}w \right\rangle).$$

Let W be the orthogonal complement of V with respect to $(...,...)$ and let P be the projection onto V corresponding to the decomposition $\mathbb{C}^m = V \oplus W$. ■

Theorem 8 *Let K be a compact subgroup of $GL(n,\mathbb{C})$ and let G be the Zariski closure of K in $GL(n,\mathbb{C})$ then K is a maximal compact subgroup of G.*

Proof. Let U be a compact subgroup of G containing K. Suppose that $K \neq U$ we show that this leads to a contradiction. Then if $u \in U - K$ then $Ku \cap K = \emptyset$. Since both sets are compact there exists a continuous function, f, on U such that $f_{|K} = 1$ and $f_{|Ku} = 0$. The Stone-Weierstrass theorem implies that there is $\phi \in \mathbb{C}[x_{ij}]$ such that $|f(x) - \phi(x)| < \frac{1}{4}$ for $x \in U$. Thus $|\phi(ku)| < \frac{1}{4}$ and $|\phi(k)| > \frac{3}{4}$ for all $k \in K$. Let $\gamma(X) = \mu_K(k \to \phi(kX))$. Then γ is a polynomial on $M_n(\mathbb{C})$ (expand in monomials and note that we are just integrating coefficients). We note that $|\gamma(ku)| \leq \frac{1}{4}$ and $|\gamma(k)| \geq \frac{3}{4}$ for all $k \in K$. On the other hand, if $Y \in M_n(\mathbb{C})$ then then the function $X \to \gamma(XY)$ is a polynomial on $M_n(\mathbb{C})$ thus in $\mathcal{O}(GL(n,\mathbb{C}))$ and it takes the constant value $\gamma(Y)$ on K. Thus since G is the Z-closure of K we see that $\gamma(gY) = \gamma(Y)$ for all $g \in G$ hence for all $g \in U$. But then $\gamma(x) = \gamma(I)$ for all $u \in U$. This is a contradiction since $|\gamma(I)| \geq \frac{3}{4}$ and $|\gamma(u)| = 0$. ■

2.2 Symmetric Subgroups.

2.2.1 Definition.

We will take as the main examples over $\mathbb{C}$ the Z-closed subgroups, G, of $GL(n,\mathbb{C})$ that have the additional property: if $g \in G$ then $g^* \in G$ (here, as

usual, $[g_{ij}]^* = [\overline{g_{ji}}]$). We will call such a group *symmetric*. Let $U(n)$ denote the group of all $g \in GL(n,\mathbb{C})$ such that $gg^* = I$. Then $U(n)$ is a compact Lie group (in fact every row of $g \in G$ is an element of the $2n-1$ sphere so the group is topologically a closed subset of a product of n spheres).

Examples.

1. Obviously. $GL(n,\mathbb{C})$ is a symmetric subgroup of itself.

2. $SL(n,\mathbb{C}) = \{g \in GL(n,\mathbb{C}) | \det g = 1\}$. This group is a hypersurface in $M_n(\mathbb{C})$.

3. $O(n,\mathbb{C}) = \{g \in GL(n,\mathbb{C}) | gg^T = 1\}$ (here $[g_{ij}]^T = [g_{ji}]$). Thus the equations defining $O(n,\mathbb{C})$ are

$$\sum_k x_{ik}x_{kj} = \delta_{ij}.$$

4. $SO(n,\mathbb{C}) = \{g \in O(n,\mathbb{C}) | \det g = 1\}$.

5. $Sp(n,\mathbb{C}) = Sp_{2m}(\mathbb{C}) = \{g \in GL(2n,\mathbb{C}) | gJg^T = J\}$. Here

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

with I the $n \times n$ identity matrix.

6. (Most general example) Let $K \subset U(n)$ be a closed (hence compact subgroup). Let G be the Z-closure of K in $GL(n,\mathbb{C})$. If $f \in \mathcal{O}(GL(n,\mathbb{C}))$vanishes on K then $\overline{f(X^*)}$ (the overbar denotes complex conjugation) is also in $\mathcal{O}(GL(n,\mathbb{C}))$ and vanishes on K. Thus if $g \in G$ then g^* is in G.

Exercise. Prove that $Sp(n,\mathbb{C})$ is symmetric. Also, show that $\det g = 1$ if $g \in Sp(n,\mathbb{C})$.

2.2.2 Some decompositions of symmetric groups.

We define the exponential map to be $\exp : M_n(\mathbb{C}) \to GL(n,\mathbb{C})$ given by $\exp(X) = e^X$.

Theorem 9 *If G is a symmetric subgroup of $GL(n,\mathbb{C})$ and if $K = G \cap U(n)$ then*

$$Lie(G) = Lie(K) + iLie(K).$$

Furthermore, the map

$$K \times iLie(K) \to G$$

given by

$$k, Z \to ke^Z$$

defines a homeomorphism.

We note that the map is actually a diffeomorphism but we will not need this slightly harder fact. This theorem is a special case of a more general result that we will explain after deriving a few consequences.

Corollary 10 *If G is a symmetric subgroup of $GL(n,\mathbb{C})$ then G is the Z-closure of $K = G \cap U(n)$.*

Proof. Let $\phi \in \mathcal{O}(GL(n,\mathbb{C}))$ be such that $\phi_{|K} = 0$. Then $\phi(ke^{tX}) = 0$ for $k \in K, X \in Lie(K)$ and $t \in \mathbb{R}$. But then by looking at this function in t as the restriction of a holomorphic function on $\mathbb{C}$ we see that the above equation is also true for $t \in \mathbb{C}$. Thus $\phi(ke^Z) = 0$ for $k \in K$ and $Z \in iLie(K)$.Hence G is contained in the Z-closure of K. Since G is Z-closed G is the Z-closure. ■

Corollary 11 *If G is a symmetric subgroup of $GL(n,\mathbb{C})$ then G is irreducible in the Z-topology if and only if G is connected in the S-topology.*

We note that this is a special case of fact that an irreducible algebraic variety over $\mathbb{C}$ is connected in the S-toplogy.

We will now describe the generalization. Let G be a subgroup of $GL(n,\mathbb{R})$ that is given as the locus of zeros of a set of polynomials on $M_n(\mathbb{R})$ (thought of as $\mathbb{R}^{n^2}$). We will say that G is a symmetric real group if whenever $g \in G, g^T \in G$. Let $O(n)$ be the compact Lie group $O(n,\mathbb{C}) \cap U(n)$ and set $K = G \cap O(n)$.

Assume that we have such a group. Let $\mathfrak{g} = Lie(G)$ and let $\mathfrak{k} = Lie(K)$. Define $\langle X, Y\rangle = \mathrm{tr}XY^T$ is real valued on, positive definite and invariant under the action of $Ad(K)$. Set $\mathfrak{p} = \mathfrak{k}^\perp \cap \mathfrak{g}$. Then $Ad(k)\mathfrak{p} \subset \mathfrak{p}$ for $k \in K$.

We note that if $X \in \mathfrak{p}$ then

$$\langle ad(X)Y, Z\rangle = \mathrm{tr}[X,Y]Z^T = \mathrm{tr}(XYZ^T - YXZ^T) =$$

$$\mathrm{tr}(YZ^TX - YXZ^T) = \mathrm{tr}(Y[X,Z]^T) = \langle Y, ad(X)Z\rangle\,.$$

This implies that if $X \in \mathfrak{p}$ then adX is self adjoint relative to $\langle ..., ...\rangle$ .It is therefore diagonalizable as an endomorphism of $\mathfrak{g}$. Let $\mathfrak{a}$ be a subspace of $\mathfrak{p}$ that is maximal subject to the condition that $[X,Y] = 0$ for $X, Y \in \mathfrak{a}$. Such a subspace is called a Cartan subspace.

The following are basic theorems of E. Cartan.

Theorem 12 *Let $G, K, \mathfrak{p}$ be as above then the map $K \times \mathfrak{p} \to G$ given by $k, X \longmapsto ke^X$ is a homeomorphism. In particular, G is connected if and only if K is connected.*

We will use the following Lemma of Chevalley.

Lemma 13 *Let ϕ be a polynomial on $M_n(\mathbb{C})$ and let $X \in M_n(\mathbb{C})$ be such that $X^* = X$. Then if $\phi(e^{mX}) = 0$ for all $m \in \mathbb{N} = \{0, 1, 2, ...\}$ then $\phi(e^{tX}) = 0$ for all $t \in \mathbb{R}$.*

Proof. There exists $u \in U(n)$ such that uXu^{-1} is diagonal with real diagonal entries $a_i, i = 1, ..., n$. Replacing ϕ by $\phi \circ Ad(u)^{-1}$ we may assume that X is diagonal with the indicated diagonal entries. Now observing that the only monomials in the x_{ij} that are non-zero on e^{tX} are of the form $\gamma_m = x_{11}^{m_1} x_{22}^{m_2} \cdots x_{nn}^{m_n}$ and $\gamma_m(e^{tX}) = e^{tA}$ with $A = \sum_i m_i a_i$. Thus if $\phi(e^{tX})$ is not identically 0 then it must be of the form

$$\phi(e^{tX}) = \sum_{j=1}^{r} c_j e^{tA_j}$$

for some $c_j \in \mathbb{C}$ and the A_j are of the form $\sum m_{ij} a_i$ with $m_{ij} \in \mathbb{N}$. We group the terms so that $A_1 > ... > A_r$ and we may assume $c_1 \neq 0$. We show that this leads to a contradiction. In fact,

$$e^{-tA_1}\phi(e^{tX}) = c_1 + \sum_{j=2}^{r} c_j e^{-t(A_1 - A_j)}.$$

But $\phi(e^{mX}) = 0$ for $m \in \mathbb{N}$. Taking the limit of $e^{-mA_1}\phi(e^{mX})$ as $m \to \infty$ in $\mathbb{N}$ yields the contradiction $c_1 = 0$. ■

We will now prove the theorem.

We note that the polar decomposition of an element of $GL(n, \mathbb{R})$ as kp with $k \in O(n)$ and $p \in M_n(\mathbb{R})$ with $p^T = p$ and all eigenvalues real and strictly positive let P_n be the set of such matrices. This decomposition is unique. We also note that if $\mathfrak{p}_n = \{X \in M_n(\mathbb{R}) | X^T = X\}$ then P_n is an open subset and $\exp : \mathfrak{p}_n \to P_n$ is a homeomorphism.

Let $\phi_1, ..., \phi_m$ be polynomials defining $G \subset GL(n, \mathbb{R})$. Then if $g \in G$ we have $g = ke^X$ with $k \in O(n)$ and $X \in \mathfrak{p}_n$. Now $g^T g = e^X k^T k e^X = e^{2X}$. Thus since G is invariant under transposition we have $e^{2X} \in G$. Thus $e^{2mX} \in G$

for all $m \in \mathbb{Z}$. Thus implies that $\phi_i(e^{2mX}) = 0$ for all i and $m \in \mathbb{Z}$. The Lemma above implies that $\phi_i(e^{2tX}) = 0$ For all $t \in \mathbb{R}$. Thus $X \in Lie(G)$. Thus $X \in \mathfrak{p}$. This implies that $k \in K$. But then $G = K\exp(\mathfrak{p})$. That the map in the statement is a homeomorphism follows from the assertion for the polar decomposition.

Theorem 14 *Let $G, K, \mathfrak{p}$ be as above. Then if $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are Cartan subspaces of $\mathfrak{p}$ then there exists $k \in K$ such that $Ad(k)\mathfrak{a}_1 = \mathfrak{a}_2$.*

For the proof we need the following

Lemma 15 *If $\mathfrak{a}$ is a Cartan subspace of $\mathfrak{p}$ then there exists $H \in \mathfrak{a}$ such that $\mathfrak{a} = \{X \in \mathfrak{p} | [H, X] = 0\}$.*

Proof. If $\lambda \in \mathfrak{a}^*$define $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} | [h, X] = \lambda(h)X, h \in \mathfrak{a}\}$. Since $\mathfrak{a}$ consists of elements that commute and so $[ad(h_i), ad(h_2)] = 0$ for $h_1, h_2 \in \mathfrak{a}$ we see that the elements of $ad(\mathfrak{a})$ simultaneously diagonalize and we therefore see that $\mathfrak{g} = \mathfrak{g}_0 \oplus \oplus_{\lambda \neq 0}\mathfrak{g}_\lambda$. Let $\Lambda(\mathfrak{g}, \mathfrak{a})$ denote the set of all $\lambda \in \mathfrak{a}^*$ with $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$. Then $\Lambda(\mathfrak{g}, \mathfrak{a})$ is finite so there exists $H \in \mathfrak{a}$ such that $\lambda(H) \neq 0$ if $\lambda \in \Lambda(\mathfrak{g}, \mathfrak{a})$. Fix such an H. If $X \in \mathfrak{g}$ and $[H, X] = 0$ then $X \in \mathfrak{g}_0$. By definition of Cartan subspace $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$. This proves the Lemma. ■

We will now prove the theorem. The argument we will give is due to Hunt. Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be Cartan subspaces of $\mathfrak{p}$. Let $H_i \in \mathfrak{a}_i$ be as in the Lemma. Since K is compact the function $f(k) = \langle Ad(k)H_1, H_2\rangle$ achieves a minimum k_o. Thus if $X \in \mathfrak{k}$ then

$$\frac{d}{dt_{t=0}} \left\langle Ad(e^{tX})Ad(k_o)H_1, H_2 \right\rangle = 0.$$

Using ??? this implies that

$$0 = \langle [X, Ad(k_o)H_1], H_2\rangle = \langle X, [Ad(k_o)H_1, H_2]\rangle .$$

If $x, y \in \mathfrak{p}$ then $[x, y] \in \mathfrak{k}$. This implies that since X is an arbitrary element of $\mathfrak{k}$ we must have $[Ad(k_o)H_1, H_2] = 0$. This implies that $H_2 \in Ad(k_o)\mathfrak{a}_1$. Since $\mathfrak{a}_1$ is abelian this implies that $\mathfrak{a}_2 \subset Ad(k_o)\mathfrak{a}_1$. Maximality implies equality.

Corollary 16 *Let G, K be as above and let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{p}$. Set $A = \exp(\mathfrak{a})$. Then $G = KAK$.*

Proof. If $X \in \mathfrak{p}$ then since $[X, X] = 0$ it is contained in a Cartan subspace. The preceding theorem now implies that

$$\mathfrak{p} = Ad(K)\mathfrak{a}.$$

Thus $\exp(\mathfrak{p}) = \cup_{k\in K} kAk^{-1}$. Thus $G = K\exp(\mathfrak{p}) \subset KAK$. ■

2.2.3 Compact and Algebraic Torii.

We note that $GL(1, F) = \{[x]|x \in F^\times\}$ We will therefore think of $F^\times$as the algebraic group $GL(1, F)$. An algebraic group isomorphic with $(F^\times)^n$ will be called an n-dimensional *algebraic torus*. Thus the group of diagonal matrices in $GL(n, F)$ is an n dimensional algebraic torus.

Examples.

1. Consider the subgroup, H, of diagonal matrices in $SL(n, F)$. We can write such a matrix as

$$z = \begin{bmatrix} z_1 & 0 & \cdots & 0 & 0 \\ 0 & z_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & z_n & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{z_1 z_2 \cdots z_n} \end{bmatrix}.$$

Thus the map $(F^\times)^n \to H$ with $(z_1, ..., z_n) \longmapsto z$ (as displayed) defines an isomorphism. So H is an n-dimensional algebraic torus.

2. Consider the group, B, of all matrices of the form

$$\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} | a^2 + b^2 = 1 \right\}.$$

Then if i is a choice of $\sqrt{-1}$ then the map

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \to a + ib$$

is a group isomorphism of B onto $\mathbb{C}^\times$, the one dimensional algebraic torus.

We note that $S^1 = \{z \in \mathbb{C}||z| = 1\}$ is a compact subgroup of $\mathbb{C}^\times$. A *compact n-torus* is a Lie group isomorphic with $(S^1)^n$ (the n-fold product

group). We will write T^n for the group $(S^1)^n$. We can realize T^n as the group of diagonal elements of $U(n)$. It is easily seen that the Zariski closure of this realization in $GL(n,\mathbb{C})$ is an algerbraic n-torus. Conversely, if H is an algebraic n-torus taken to be a Z-closed subgroup of $GL(m,\mathbb{C})$ for some m then the subgroup, T, of H corresponding to T^n under the isomormphism with $(\mathbb{C}^\times)^n$ is a compact subgroup of $GL(m,\mathbb{C})$ we leave it to the reader to check that H is the Z-closure of T. One can also prove that a compact connected commutative Lie group is a compact torus.

Now let G be a symmetric subgroup of $GL(n,\mathbb{C})$. Let $K = G \cap U(n)$. Let $T \subset K$ be a maximal compact torus in K. Then $iLie(T) \subset iLie(K) = \mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ containing $iLie(T)$. Then $i\mathfrak{a}$ is an abelian subalgebra of $Lie(K)$. Let T_1 be the S-closure of $\exp(i\mathfrak{a})$ then T_1 is the closure of a connected set so T_1 is a connected compact abelian group. Since T is a maximal such group we see that $T_1 = T$. This implies $\mathfrak{a} = iLie(T)$.

We now put all of these observations together.

Theorem 17 *Let G be a symmetric subgroup of $GL(n,\mathbb{C})$ and let $K = G \cap U(n)$. Then*

1. A maximal compact torus of K is of the form $\exp(i\mathfrak{a})$ with $\mathfrak{a}$ a Cartan subspace of $\mathfrak{p}$.

2. All maximal compact tori are conjugate in K.

3. If T is a maximal compact torus in K and $\mathfrak{a} = iLie(T)$ then $T\exp(\mathfrak{a})$ is a maximal algebraic torus in G.

4. If $\mathfrak{a}$ is a Cartan subspace of $\mathfrak{p}$ then $\exp(i\mathfrak{a})$ is a maximal compact torus in K.

We also have

Theorem 18 *Let G and K be as in the previous theorem. Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{p} = iLie(K)$. Let H be the unique maximal algebraic torus containing $\exp(i\mathfrak{a})$ then $H = \exp(\mathfrak{a} + i\mathfrak{a}) = T\exp(\mathfrak{a})$ with $T = \exp(i\mathfrak{a})$ a maximal compact torus in K, Furthermore, $G = KHK$.*

Proof. The first assertions are a direct consequence of the previous result. The last follows since $HK = AK$ in the notation of Corollary 16. ■

2.2.4 General reductive algebraic groups over $\mathbb{C}$.

We will say that an affine algebraic group, G, is linearly reductive if whenever $\sigma : G \to GL(V)$ is an algebraic group homomorphism with V a finite dimensional vector space over $\mathbb{C}$ whenever W is a subspace invariant under $\sigma(G)$ there exists V_1 a subspace invariant under $\mathbb{C}$ such that $V = W \oplus V_1$. There is a more general notion of reductive over any algebraically closed field which is equivalent to linearly reductive in characteristic 0. The following theorems are true. We will not need them in these lectures.

Theorem 19 *Let G be an affine algebraic, reductive group then G is isomorphic with a symmetric subgroup of $GL(n, \mathbb{C})$ for some n.*

Theorem 20 *Let G be a linearly reductive affine algebraic group that all maximal compact subgroups (in the S-topology) are conjugate.*

Theorem 21 *If K is a compact Lie group then there exists n and a closed subgroup H in $U(n)$ such that K is Lie group isomorphic with H.*