Appendix D

Manifolds and Lie Groups

The purpose of this appendix is to collect the essential parts of manifold and Lie group theory in a convenient form for the body of the book. The philosophy of this appendix is to give the main definitions and to prove many of the basic theorems. Some of the more difficult results are stated with appropriate references that the careful reader who is unfamiliar with differential geometry can study.

D.1 C^∞ Manifolds

D.1.1 Basic Definitions

Let X be a Hausdorff topological space with a countable basis for its topology. Then an *n-chart* for X is a pair (U, Φ) of an open subset U of X and a continuous map Φ of U into $\mathbb{R}^n$ such that $\Phi(U)$ is open in $\mathbb{R}^n$ and Φ is a homeomorphism of U onto $\Phi(U)$. A *C^∞ n-atlas* for X is a collection $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in I}$ of n-charts for X such that

(1) the collection of sets $\{U_\alpha\}_{\alpha \in I}$ is an open covering of X,

(2) the maps $\Phi_\beta \circ \Phi_\alpha^{-1} : \Phi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \Phi_\beta(U_\alpha \cap U_\beta)$ are of class C^∞ for all $\alpha, \beta \in I$.

Examples

1. Let $X = \mathbb{R}^n$ and take $U = \mathbb{R}^n$ and Φ the identity map. Then (U, Φ) is an n-chart for X and $\{(U, \Phi)\}$ is a C^∞ atlas.

2. Let $X = S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$ with the topology as a closed subset of $\mathbb{R}^{n+1}$. Let $S^n_{i,+} = \{(x_1, \ldots, x_{n+1}) \in S^n : x_i > 0\}$ and $S_{i,-} = \{(x_1, \ldots, x_{n+1}) \in S^n : x_i < 0\}$ for $i = 1, \ldots, n+1$. Define

$$\Phi_{i,\pm}(x) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$$

for $x \in S^n_{i,\pm}$ (the projection of $S_{i,\pm}$ onto the hyperplane $\{x_i = 0\}$). Then $\Phi_{i,\pm}(S^n_{i,\pm}) = B_n = \{x \in \mathbb{R}^n \,:\, x_1^2 + \cdots + x_n^2 < 1\}$ and

$$\Phi^{-1}_{i,\pm}(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, \pm\sqrt{1 - x_1^2 - \cdots - x_n^2}, x_i, \ldots, x_n). \quad \text{(D.1)}$$

Thus each $(U, \Phi_{i,\pm})$ is a chart. The sets $S^n_{i,\pm}$ cover S^n. From (D.1) it is clear that

$$\{(S^n_{i,\epsilon}, \Phi_{n,\epsilon}) \,:\, 1 \le i \le n+1,\, \epsilon = \pm\}$$

is a C^∞ n-atlas for X.

3. Let $f_1, \ldots, f_k$ be C^∞ real-valued functions on $\mathbb{R}^n$ with $k \le n$. Let X be the set of points $x \in \mathbb{R}^n$ so that $f_i(x) = 0$ for all $i = 1, \ldots, k$ and some $k \times k$ minor of the $k \times n$ matrix

$$D(x) = \left[\frac{\partial f_i}{\partial x_j}(x) \right]$$

is nonzero (note that X might be empty). Give X the subspace topology in $\mathbb{R}^n$. For $1 \le i_1 < \cdots < i_k \le n$ let $D_{i_1,i_2,\ldots,i_k}(x)$ be the $k \times k$ matrix formed by rows $i_1, \ldots, i_k$ of $D(x)$. Define

$$U_{i_1,i_2,\ldots,i_k} = \{x \in X \,:\, \det D_{i_1,i_2,\ldots,i_k}(x) \neq 0\}\,.$$

Then these sets constitute an open covering of X.

We now construct an $(n-k)$-atlas for X as follows: Given $x \in X$, choose $1 \le i_1 < \cdots < i_k \le n$ so that $x \in U_{i_1,i_2,\ldots,i_k}$. Let $1 \le p_1 < \cdots < p_{n-k} \le n$ be the complementary set of indices:

$$\{i_1, \ldots, i_k\} \cup \{p_1, \ldots, p_{n-k}\} = \{1, \ldots, n\}\,.$$

For $y \in \mathbb{R}^n$ we define $u_q(y) = f_{i_q}(y)$ for $1 \le q \le k$ and $u_{k+q}(y) = y_{p_q}$ for $1 \le q \le n-k$. Then

$$\det \left[\frac{\partial u_i}{\partial x_j}(y) \right] \neq 0 \quad \text{for } y \in U_{i_1,i_2,\ldots,i_k}\,.$$

Set $\Psi(y) = (u_1(y), \ldots, u_n(y))$. The inverse function theorem (see Lang [1993]) implies that then there exists an open subset $V_x \subset \mathbb{R}^n$ containing x such that $W_x = \Psi(V_x)$ is open in $\mathbb{R}^n$ and Ψ is a bijection from V_x onto W_x with C^∞ inverse map. From this result we see that if we define

$$\Phi_x(y) = (y_{p_1}, \ldots, y_{p_{n-k}}) \quad \text{for } y \in U_{i_1,i_2,\ldots,i_k} \cap V_x\,,$$

then Φ_x is a homeomorphism onto its image, which is open in $\mathbb{R}^{n-k}$. Set $U_x = U_{i_1,i_2,\ldots,i_k} \cap V_x$. Then $\{(U_x, \Phi_x)\}_{x \in X}$ is a C^∞ $(n-k)$-atlas for X.

4. Let $X \subset \mathbb{C}^n$ be an irreducible affine variety of dimension m (Appendix A.1.5). Let X_0 be the set of smooth points of X (Appendix A.3.1, Example 3). Endow X_0 with the subspace topology as a subset of $\mathbb{C}^n$ (which we look upon as $\mathbb{R}^{2n}$). We

show how to find a $2m$-dimensional C^∞ structure on X_0. Let $f_1, \ldots, f_p$ generate the ideal of X. Each function f_i is a polynomial in the complex linear coordinates $z_1, \ldots, z_n$ on $\mathbb{C}^n$. We define the $p \times n$ matrix

$$F(x) = \left[\frac{\partial f_i}{\partial z_j}(x) \right] .$$

Fix $x \in X_0$. The tangent space $T_x(X)$ (in the sense of algebraic varieties) has dimension m over $\mathbb{C}$. This implies that there exists $1 \leq i_1 < \cdots < i_{n-m} \leq p$ and $j_1 < \cdots < j_{n-m}$ such that the $(n-m) \times (n-m)$ minor $N(x)$ formed using rows $i_1, \ldots, i_{n-m}$ and columns $j_1, \ldots, j_{n-m}$ of $F(x)$ is nonzero. Hence there is a Zariski-open subset U of $\mathbb{C}^n$ containing x such that $N(y) \neq 0$ for all $y \in U$. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by $z_j = x_j + \mathrm{i}\, x_{n+j}$, where $\mathrm{i} = \sqrt{-1}$ and $x_1, \ldots, x_{2n}$ are real coordinates on $\mathbb{R}^{2n}$. We write $f_i = g_i + \mathrm{i}\, g_{p+i}$ for $i = 1, \ldots, p$, with g_k a real-valued polynomial in $x_1, \ldots, x_{2n}$. Define the $2p \times 2n$ matrix

$$D(y) = \left[\frac{\partial g_i}{\partial x_j}(y) \right] .$$

Let $M(y)$ be the $2(n-m) \times 2(n-m)$ minor formed from rows $i_1, \ldots, i_{n-m}$, $p + i_1, \ldots, p + i_{m-n}$ and columns $j_1, \ldots, j_{n-m}, n + j_1, \ldots, n + j_{n-m}$ of $D(y)$. The Cauchy-Riemann equations imply that

$$M(y) = |N(y)|^2 \tag{D.2}$$

(see Helgason [1984, Chap. VIII, §2 (7)]). Hence $D(y)$ has rank $2(n-m)$ when $y \in U$.

Let

$$Y = \{ y \in \mathbb{C}^n \,:\, f_{i_k}(y) = 0 \text{ for } k = 1, \ldots, n-m \} .$$

Let V be a Zariski-open subset of $\mathbb{C}^n$ such that $x \in V$ and $V \cap Y$ is irreducible. Then $\dim V \cap Y = m$ and $V \cap X \subset V \cap Y$. Since $\dim V \cap X = m$, this implies that $V \cap X = V \cap Y$ (Theorem A.1.18). Now use $(D.2)$ to construct a C^∞ $2m$-atlas for $V \cap X$ by the method of Example 3. Given $x \in X_0$ choose an element (U, Φ) in this atlas with $x \in U$ and denote it as (U_x, Φ_x). Then $\{(U_x, \Phi_x) \,:\, x \in X_0\}$ defines a C^∞ atlas on X_0 that gives rise to a $2m$-dimensional C^∞ structure. ♦

Suppose X has a C^∞ n-atlas $\mathcal{A} = \{(U_\alpha, \Phi_\alpha)\}_{\alpha \in I}$. We say that an n-chart (U, Φ) is *compatible* with $\mathcal{A}$ if whenever $U \cap U_\alpha \neq \emptyset$ then the maps

$$\begin{aligned} \Phi \circ \Phi_\alpha^{-1} &: \Phi_\alpha(U \cap U_\alpha) \longrightarrow \Phi(U \cap U_\alpha) \quad \text{and} \\ \Phi_\alpha \circ \Phi^{-1} &: \Phi(U \cap U_\alpha) \longrightarrow \Phi_\alpha(U \cap U_\alpha) \end{aligned}$$

are of class C^∞.

If X is a Hausdorff space with a countable basis for its topology then an *n-dimensional C^∞ structure* on X is an n-atlas $\mathcal{A}$ for X so that every n-chart of X that is compatible with $\mathcal{A}$ is contained in $\mathcal{A}$.

Lemma D.1.1. *If $\mathcal{A}$ is a C^∞ n-atlas for X then $\mathcal{A}$ is contained in a unique n-dimensional C^∞ structure for X.*

Proof. If $\mathcal{B}$ is the collection of all n-charts compatible with $\mathcal{A}$, then the definition of atlas implies that $\mathcal{B} \supset \mathcal{A}$. The chain rule for differentiation implies that a chart compatible with $\mathcal{A}$ is compatible with $\mathcal{B}$. Thus $\mathcal{B}$ is a C^∞ structure on X. The chain rule also implies the uniqueness. ♦

Definition D.1.2. A pair $(X, \mathcal{A})$ of a Hausdorff topological space with a countable basis for its topology and an n-dimensional C^∞ structure on X will be called an n-dimensional C^∞ *manifold*.

If $M = (X, \mathcal{A})$ we will write $x \in M$ for $x \in X$, and we will say that a chart for X is a chart for M if it is in $\mathcal{A}$. Each example above is a C^∞ manifold with C^∞ structure corresponding to the atlas constructed there. We now give some more important examples.

5. (Products) Let $M = (X, \mathcal{A})$ and $N = (Y, \mathcal{B})$ be manifolds of dimensions m and n respectively. Given $(U, \Phi) \in \mathcal{A}$ and $(V, \Psi) \in \mathcal{B}$, we define $(\Phi \times \Psi)(x, y) = (\Phi(x), \Psi(y))$ for $(x, y) \in U \times V$. Then the set

$$\{(U \times V, \Phi \times \Psi) \,:\, (U, \Phi) \in \mathcal{A},\ (V, \Psi) \in \mathcal{B}\}$$

is a C^∞ $(m + n)$-atlas for $X \times Y$ (with the product topology). This defines the structure of a C^∞ $(m + n)$-dimensional manifold on $X \times Y$. We use the notation $M \times N$ for the corresponding C^∞ manifold and call it the *product manifold*.

6. (Submanifolds) Let $M = (X, \mathcal{A})$ be a C^∞ manifold and let U be an open subset of X. Then the set

$$\{(V \cap U, \Phi|_{U \cap V}) \,:\, (V, \Phi) \in \mathcal{A}\}$$

is a C^∞ atlas for U. Thus U is a C^∞ manifold called an *open submanifold* of M.

7. (Covering Manifolds) Let $M = (X, \mathcal{A})$ be a C^∞ manifold. Let $\pi : Y \longrightarrow X$ be a covering space that is a Hausdorff space with a countable basis for its topology. Recall that this means that

(a) π is a continuous surjective mapping;

(b) for all $x \in X$ there exists an open neighborhood U of x in X such that $\pi^{-1}(U) = \bigcup_\alpha V_\alpha$ is a disjoint union of open sets, and $\pi : V_\alpha \longrightarrow U$ is a homeomorphism.

We say that the neighborhood U in (b) is *evenly covered* by π. We will now show how to find a C^∞ atlas on Y. If $x \in M$ let $(W_x, \Psi_x) \in \mathcal{A}$ with $x \in W_x$. Let U_x be an evenly covered neighborhood of x. Set $V_x = U_x \cap W_x$ and $\Phi_x = \Psi_x|_{V_x}$. Then $(V_x, \Phi_x) \in \mathcal{A}$ and V_x is evenly covered by π. Let

$$\pi^{-1}(V_x) = \textstyle\bigcup_\alpha V_{x,\alpha}$$

with π a homeomorphism of $V_{x,\alpha}$ onto V_x. Set $\Phi_{x,\alpha}(y) = \Phi_x(\pi(y))$ for $y \in V_{x,\alpha}$. We note that the collection $\{V_{x,\alpha}\}$, with $x \in X$ and α running through all of the appropriate indices, is an open covering of Y. We leave it to the reader to check that $\{(V_{x,\alpha}, \Phi_{x,\alpha})\}$ is a C^∞ n-atlas for Y ($n = \dim M$) and that $\pi : V_{x,\alpha} \longrightarrow V_x$ is a diffeomorphism for all $x \in X$ and all α.

Definition D.1.3. Let $M = (X, \mathcal{A})$ and $N = (Y, \mathcal{B})$ be C^∞ manifolds. A C^∞ *map* $f : M \longrightarrow N$ is a continuous map $f : X \longrightarrow Y$ such that whenever $(V, \Psi) \in \mathcal{B}$ and $(U, \Phi) \in \mathcal{A}$ satisfy $U \subset f^{-1}(V)$, then the map $\Psi \circ f \circ \Phi^{-1} : \Phi(U) \longrightarrow \Psi(V)$ is of class C^∞.

We will look upon $\mathbb{C}$ as $\mathbb{R}^2$ with the C^∞ structure as in Example 1 above. We denote by $C^\infty(M; \mathbb{R}^n)$ the space of all C^∞ maps of M into $\mathbb{R}^n$ (with the C^∞ structure as in Example 1 above). We write $C^\infty(M) = C^\infty(M; \mathbb{R})$ and $C^\infty(M; \mathbb{C}) = C^\infty(M, \mathbb{R}^2)$. These latter examples are algebras under pointwise addition and multiplication of functions.

We return to the examples above. In Example 1 the usual advanced calculus notion of C^∞ coincides with our definition. We note that in Example 2 the map $\iota : S^n \longrightarrow \mathbb{R}^{n+1}$ with $\iota(x) = x$ is of class C^∞. The map $\iota : X \longrightarrow \mathbb{R}^n$ with $\iota(x) = x$ in Examples 3 and 4 is of class C^∞. In Example 5 the projections on each of the factors are of class C^∞.

Let M be a C^∞ manifold. If f is a real-valued function on M we denote by $\operatorname{supp}(f)$ (the *support* of f) the closure of the set $\{x \in M : f(x) \neq 0\}$. Let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of M as a topological space. Then a *partition of unity subordinate* to the covering is a countable set $\{\varphi_i : 1 \leq i < N\} \subset C^\infty(M)$ with $N \leq \infty$ (the set might be finite) such that the following holds for all $x \in M$:

(1) $0 \leq \varphi_i(x) \leq 1$ for all $1 \leq i < N$.

(2) There is an open neighborhood U of x such that $\operatorname{Card}\{i : \varphi_i \,|_U \neq 0\} < \infty$.

(3) $\sum_{i<N} \varphi_i(x) = 1$.

(4) If $1 \leq i < N$ then there exists $\alpha \in I$ such that $\operatorname{supp}(\varphi_i) \subset U_\alpha$.

Theorem D.1.4. *For each open covering $\mathcal{U}$ of M there exists a partition of unity subordinate to $\mathcal{U}$.*

For a proof see Warner [1983].

We now give an example of how partitions of unity are used. Let A and B be closed subsets of M such that $A \cap B = \emptyset$. Let $U = M \setminus A$ and $V = M \setminus B$. Then $U \cup V = M$. Let $\{\varphi_i\}$ be a partition of unity subordinate to the open covering $\{U, V\}$. Let $S = \{i : \operatorname{supp}(\varphi_i) \subset V\}$. Set $f(x) = \sum_{i \in S} \varphi_i(x)$. Note that condition (2) above implies that the sum is actually finite for each x and that f defines an element of $C^\infty(M)$. We note that if $x \in B$ then $f(x) = 0$. Now consider $x \in A$. If $i \notin S$ then $\operatorname{supp}(\varphi_i) \subset U$ by condition (4) above. Hence if $i \notin S$, $\varphi_i(x) = 0$. Thus

$$1 = \sum_i \varphi_i(x) = \sum_{i \in S} \varphi_i(x) = f(x).$$

We have thus shown that there exists $f \in C^\infty(M)$ such that f is identically 0 on B and identically 1 on A.

D.1.2 Tangent Space

Let M be an n-dimensional C^∞ manifold. If $x \in M$ then a *point derivation* of $C^\infty(M)$ at x is a linear functional L on $C^\infty(M)$ such that

$$L(fg) = f(x)Lg + g(x)Lf \quad \text{for all } \ f, g \in C^\infty(M)\,.$$

Note that if $f(x) = c$ for all $x \in M$ is a constant function, then $L(f^2) = 2cL(f)$. But $f^2 = cf$, so $cL(f) = 2cL(f)$. Hence $L(f) = 0$. If L_1 and L_2 are point derivations at x then so is $a_1L_1 + a_2L_2$ for $a_1, a_2 \in \mathbb{R}$. Thus the point derivations at x form a vector space over $\mathbb{R}$.

Definition D.1.5. A point derivation at x is called a *tangent vector*. The *tangent space* of M at x is the vector space $T(M)_x$ of all point derivations of M at x.

Lemma D.1.6. *Let $x \in M$ and $f \in C^\infty(M)$. If f vanishes in an open neighborhood U of x then $Lf = 0$ for all $L \in T(M)_x$.*

Proof. Let $B = \operatorname{supp}(f)$. Then $B \cap U = \emptyset$. Take $\varphi \in C^\infty(M)$ be such that $\varphi(x) = 0$ and φ is identically 1 on B; then $\varphi f = f$. If $L \in T(M)_x$ then

$$Lf = L(\varphi f) = \varphi(x)Lf + f(x)L\varphi = 0\,. \qquad \blacklozenge$$

Let M and N be C^∞ manifolds. If $f : M \longrightarrow N$ is a C^∞ map then we define $df_x : T(M)_x \longrightarrow T(N)_{f(x)}$ by

$$df_x(L)\varphi = L(\varphi \circ f) \quad \text{for } L \in T(M)_x \text{ and } \varphi \in C^\infty(N)\,.$$

The map df_x is called the *differential* of f at x.

Lemma D.1.7. *Let U be an open submanifold of the C^∞ manifold M, and define $\iota : U \longrightarrow M$ by $\iota(x) = x$. Then ι is a C^∞ map, and for each $x \in U$ the map $di_x : T(U)_x \longrightarrow T(M)_x$ is a linear bijection.*

Proof. Let $x \in U$. Let (V, Φ) be a chart for M such that $x \in V$. Then $\Phi(V)$ is open in $\mathbb{R}^n$ and by the definition of chart $\Phi(U \cap V)$ is also open in $\mathbb{R}^n$. Choose open sets W_1, W_2 in $\mathbb{R}^n$ such that

$$\Phi(x) \in W_1 \subset \overline{W_1} \subset W_2 \subset \Phi(U \cap V)\,.$$

Let $A = \Phi^{-1}(\overline{W}_1)$ and let $B = M \setminus \Phi^{-1}(W_2)$. Let φ be identically 1 on A and identically 0 on B. Given $f \in C^\infty(U)$, define

$$(\varphi f)(x) = \begin{cases} \varphi(x)f(x) & \text{for } x \in U\,, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\varphi f \in C^\infty(M)$. Suppose $L \in T(U)_x$ satisfies $d\iota_x(L) = 0$. Then for all $f \in C^\infty(U)$ we have $0 = d\iota_x(L)(\varphi f) = L((\varphi f)|_U)$. Since φf agrees with f in a neighborhood of x, Lemma D.1.6 implies that $Lf = 0$ for all $f \in C^\infty(U)$, and hence $L = 0$. This proves that $d\iota_x$ is injective. Let $L \in T(M)_x$ and for $f \in C^\infty(U)$ define $\widetilde{L}f = L(\varphi f)$. The first part of the argument implies that $d\iota_x(\widetilde{L}) = L$. So $d\iota_x$ is surjective. ♦

We now determine $T(\mathbb{R}^n)_p$ for each $p \in \mathbb{R}^n$. Given a point p, we define a linear functional $\Big(\dfrac{\partial}{\partial x_i}\Big)_p$ on $C^\infty(\mathbb{R}^n)$ by $\Big(\dfrac{\partial}{\partial x_i}\Big)_p(f) = \dfrac{\partial f}{\partial x_i}(p)$. The Leibniz rule implies that this gives a tangent vector at p.

Lemma D.1.8. *The set* $\Big\{\Big(\dfrac{\partial}{\partial x_1}\Big)_p, \ldots, \Big(\dfrac{\partial}{\partial x_n}\Big)_p\Big\}$ *is a basis for* $T_p(\mathbb{R}^n)$.

Proof. If $f \in C^\infty(\mathbb{R}^n)$, then by the fundamental theorem of calculus

$$f(x) = f(p) + \int_0^1 \frac{d}{dt} f(p + t(x-p))\, dt = f(p) + \sum_i (x_i - p_i) g_i(x)\,,$$

with $g_i(x) = \displaystyle\int_0^1 \frac{\partial f}{\partial x_i}(p + t(x-p))\, dt$. We note that $g_i(p) = \Big(\dfrac{\partial}{\partial x_i}\Big)_p(f)$. If $L \in T(\mathbb{R}^n)_p$ then

$$\begin{aligned} L(f) &= L(f(p)1) + \sum_i L(x_i - p_i)\Big(\frac{\partial}{\partial x_i}\Big)_p(f) + \sum_i (p_i - p_i) L(g_i) \\ &= \sum_i L(x_i - p_i)\Big(\frac{\partial}{\partial x_i}\Big)_p(f)\,. \end{aligned}$$

Since $(\partial/\partial x_i)_p(x_j) = \delta_{ij}$, the set $\{(\partial/\partial x_1)_p, \ldots, (\partial/\partial x_n)_p\}$ is linearly independent. The calculation above implies that this set spans the tangent space at p. ♦

Let M be an n-dimensional C^∞ manifold. Let U be an open subset of M. A *system of local coordinates* on U is a set $\{u_1, \ldots, u_n\}$ of C^∞ functions on U such that if $\Phi(x) = (u_1(x), \ldots, u_n(x))$, $x \in U$, then (U, Φ) is a chart for M. Obviously, if (U, Φ) is a chart for M and $\Phi(x) = (u_1(x), \ldots, u_n(x))$, then $\{u_1, \ldots, u_n\}$ is a system of local coordinates for M on U, and this is the way all systems are obtained.

Let $\{u_1, \ldots, u_n\}$ be a system of local coordinates on U and let (U, Φ) be the corresponding chart. If $x \in U$ then

$$d\Phi_x : T(U)_x \longrightarrow T(\Phi(U))_{\Phi(x)}.$$

We note that $\Phi^{-1} : \Phi(U) \longrightarrow U$ is also C^∞, and $d\Phi^{-1}_{\Phi(x)}$ is the inverse mapping. Thus $T(U)_x$ is isomorphic as a vector space with $T(\Phi(U))_{\Phi(x)}$, which is isomorphic with $T(\mathbb{R}^n)_{\Phi(x)}$ by the above lemmas. We write

$$\Big(\frac{\partial}{\partial u_i}\Big)_x = d\Phi^{-1}_{\Phi(x)}\Big(\frac{\partial}{\partial x_i}\Big)_{\Phi(x)}.$$

Then $\left\{\left(\frac{\partial}{\partial u_1}\right)_x, \ldots, \left(\frac{\partial}{\partial u_n}\right)_x\right\}$ is a basis of $T(U)_x$, and hence it is a basis for $T(M)_x$ by Lemma D.1.7.

It is convenient to look upon $T(\mathbb{R}^n)_x$ as $\mathbb{R}^n$ as follows: If $v \in \mathbb{R}^n$ define the *directional derivative* v_x at x by

$$v_x \cdot f = \frac{d}{dt} f(x+tv)\Big|_{t=0}.$$

Then Lemma D.1.8 implies that $T(\mathbb{R}^n)_x = \{v_x : v \in \mathbb{R}^n\}$. Thus the map $v \mapsto v_x$ gives a linear isomorphism between $\mathbb{R}^n$ and $T(\mathbb{R}^n)_x$.

Definition D.1.9. Let M and N be C^∞ manifolds with N a subset of M. Let $\iota(x) = x$ for $x \in N$. We call N a *submanifold* of M if ι is a C^∞ map and di_x is injective for each $x \in N$.

Examples 2 and 3 of Section D.1.1 are submanifolds of $\mathbb{R}^n$. A submanifold is not necessarily a topological subspace (*i.e.*, the topology on N as a manifold is not necessarily the relative topology coming from M).

Let M be a C^∞ manifold. Then a *vector field* on M is an assignment $p \mapsto X_p \in T(M)_p$ for $p \in M$ such that for all $f \in C^\infty(M)$, the function $p \mapsto X_p f$ is an element of $C^\infty(M)$. We write $(Xf)(x) = X_x(f)$. Thus a vector field defines an endomorphism of $C^\infty(M)$ as a vector space over $\mathbb{R}$.

Let M be a C^∞ manifold and let X and Y be vector fields on M. We note that if $x \in M$ then $[X,Y]_x f = X_x(Yf) - Y_x(Xf)$. A direct calculation shows then $[X,Y]_x$ is a point derivation at x (see Section 1.3.7). Hence the assignment $x \mapsto [X,Y]_x$ defines a vector field on M.

Example

When $M = \mathbb{R}^n$ and $v \in \mathbb{R}^n$, then $x \mapsto v_x$ is a vector field. More generally, if $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a C^∞ mapping then $x \mapsto F(x)_x$ is a vector field. Every vector field on $\mathbb{R}^n$ is given in this way (see Exercises D.1.4 #3).

D.1.3 Differential Forms and Integration

Let M be a C^∞ manifold. A *differential j-form* on M is an assignment $x \mapsto \omega_x$ with ω_x an alternating j-multilinear form on $T(M)_x$ (see Section B.2.4) such that if $X_1, \ldots, X_j$ are vector fields on M then $x \mapsto \omega_x((X_1)_x, \ldots, (X_j)_x)$ defines a C^∞ function on M. Let $\Omega^j(M)$ denote the space of all differential j-forms on M. If $f \in C^\infty(M)$ and $\omega \in \Omega^j(M)$ define $(f\omega)_x = f(x)\omega_x$. This makes $\Omega^j(M)$ into a $C^\infty(M)$-module.

If M and N are C^∞ manifolds and $f : M \longrightarrow N$ is a C^∞ map, then we define $f^* : \Omega^j(N) \longrightarrow \Omega^j(M)$ by

$$(f^*\omega)_x(v_1, \ldots, v_j) = \omega_{f(x)}(df_x(v_1), \ldots, df_x(v_j)).$$

If $U \subset M$ is an open submanifold (Example 5 of Section D.1.1) and if ι is the usual inclusion of U into M we write $\omega|_U = \iota^*\omega$ for $\omega \in \Omega^j(M)$.

Given $f \in C^\infty(M)$ we define $df_x(v) = v(f)$ for $v \in T(M)_x$. This notation is consistent with our earlier definition of differential if we identify $T(\mathbb{R})_x$ with $\mathbb{R}$. Let $\{u_1, \ldots, u_n\}$ be a system of local coordinates on an open subset U of M. Given indices $1 \le i_1 < i_2 < \cdots < i_j \le n$, a point $x \in U$, and $v_1, \ldots, v_j \in T(M)_x$, we set

$$(du_{i_1} \wedge du_{i_2} \wedge \cdots \wedge du_{i_j})_x(v_1, \ldots, v_j) = (du_{i_1})_x \wedge \cdots \wedge \left(du_{i_j}\right)_x (v_1, \ldots, v_j)\,.$$

It is easily seen that $du_{i_1} \wedge \cdots \wedge du_{i_j} \in \Omega^j(U)$. In fact,

$$du_{i_1} \wedge \cdots \wedge du_{i_j} = \Phi^*(dx_{i_1} \wedge \cdots \wedge dx_{i_j})$$

with $x_i(x) = x_i$ (the usual coordinates on $\mathbb{R}^n$). Since $\left\{ (du_1)_x, \ldots, (du_n)_x \right\}$ is the dual basis to the basis $\left\{(\partial/\partial u_1)_x, \ldots, (\partial/\partial_{u_n})x\right\}$ of $T(M)_x$ (see Section D.1.2), we see that if $\omega \in \Omega^j(U)$ then there exist unique functions $a_{i_1 \ldots i_j} \in C^\infty(U)$ such that

$$\omega = \sum_{1 \le i_1 < \cdots < i_j \le n} a_{i_1 \ldots i_j}\, du_{i_1} \wedge \cdots \wedge du_{i_j}\,. \tag{D.3}$$

Definition D.1.10. An assignment $x \mapsto \omega_x$ with ω_x a j-multilinear mapping of $T(M)_x$ to $\mathbb{R}$ is an element of $\Omega^j(M)$ if and only if for every system of local coordinates $\{u_1, \ldots, u_n\}$ on U there exist functions $a_{i_1 \ldots i_j} \in C^\infty(U)$ such that

$$\omega_x = \sum_{1 \le i_1 < \cdots < i_j \le n} a_{i_1 \ldots i_j}(x)\ (du_{i_1})_x \wedge \cdots \wedge \left(du_{i_j}\right)_x \quad \text{for all } x \in U\,.$$

We call ω a *differential j-form* on X.

We will be particularly interested in $\Omega^n(M)$ for $n = \dim M$ to define integration of functions on M in an intrinsic way.

Definition D.1.11. A manifold M is *orientable* if there exists an atlas $\mathcal{A}$ for M with the following property: If $(U, \Phi), (V, \Psi) \in \mathcal{A}$ and if $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are the corresponding systems of local coordinates on U and V, respectively, then

$$\frac{\partial(v_1, \ldots, v_n)}{\partial(u_1, \ldots, u_n)}(x) = \det\left[\left(\frac{\partial}{\partial u_i}\right)_x v_j\right] > 0 \quad \text{for all } x \in U \cap V\,. \tag{D.4}$$

By the chain rule condition (D.4) is equivalent to the condition

$$(dv_1 \wedge \cdots \wedge dv_n)_x = a(x)(du_1 \wedge \cdots \wedge du_n)_x \quad \text{for } x \in U \cap V$$

for some function $a \in C^\infty(U \cap V)$ with $a(x) > 0$ for all $x \in U \cap V$.

Assume M is orientable. An *orientation* of M is an atlas $\mathcal{A}$ for M such that condition (D.4) is satisfied for every pair of elements of $\mathcal{A}$ and $\mathcal{A}$ is maximal with respect to this property. If $\mathcal{A}$ is an orientation of M and (U, Φ) is a chart for M then (U, Φ) will be said to be *compatible with the orientation* if $(U, \Phi) \in \mathcal{A}$.

Theorem D.1.12. *An n-dimensional manifold M is orientable if and only if there exists $\omega \in \Omega^n(M)$ such that $\omega_x \neq 0$ for all $x \in M$. Assume this condition holds. The set of all charts (U, Φ) for M such that*

$$\omega|_U = a(du_1 \wedge \cdots \wedge du_n)\,,$$

with $u_1, \ldots, u_n$ the corresponding system of local coordinates on U and $a(x) > 0$ for all $x \in U$, forms an orientation of M.

Proof. Let $\mathcal{A}$ be an atlas defining an orientation of M. Take a partition of unity $\{\varphi_i\}$ subordinate to the covering $\{U : (U, \Phi) \in \mathcal{A}\}$ of M. Choose $(U_i, \Phi_i) \in \mathcal{A}$ such that $\operatorname{supp}(\varphi_i) \subset U_i$. Let $\omega_i = \Phi_i^*(dx_1 \wedge \cdots \wedge dx_n)$. Define

$$\omega_x = \sum_{\{i\,:\,x \in U_i\}} \varphi_i(x)\,(\omega_i)_x\ .$$

Then the positivity condition (D.4) implies that $\omega_x \neq 0$ for all $x \in M$. The formula for ω implies that $x \mapsto \omega_x$ defines an element of $\Omega^n(M)$. Proving the second part of the theorem is easier and is left to the reader. ♦

If $\omega \in \Omega^n(M)$ and $\omega_x \neq 0$ for all $x \in M$ then we will call ω a *volume form*. Our next task is to define integration with respect to volume forms.

Let ω be a volume form on M and let $\mathcal{A}$ be the orientation corresponding to ω. Let $C_c(M)$ denote the space of all continuous real-valued functions on M with compact support. Suppose that $f \in C_c(M)$ and $\operatorname{supp}(f) \subset U$ with $(U, \Phi) \in \mathcal{A}$. We can write $\omega|_U = a\Phi^*(dx_1 \wedge \cdots \wedge dx_n)$ with $a \in C^\infty(U)$ and $a(p) > 0$ for $p \in U$. We set

$$\int_M f\omega = \int_{\Phi(U)} a\big(\Phi^{-1}(x)\big) f\big(\Phi^{-1}(x)\big)\, dx_1 \cdots dx_n\,. \tag{D.5}$$

Suppose $(V, \Psi) \in \mathcal{A}$ is another chart with $\operatorname{supp}(f) \subset V$. We have

$$\omega|_V = b\Psi^*(dx_1 \wedge \cdots \wedge dx_n).$$

If $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ are the local coordinates corresponding to (U, Φ) and (V, Ψ), then

$$b\frac{\partial(v_1, \ldots, v_n)}{\partial(u_1, \ldots, u_n)} = a \quad \text{on } U \cap V\,.$$

The change of variables theorem of advanced calculus (Lang [1993]) implies that

$$\begin{aligned}
&\int_{\Phi(U)} a(\Phi^{-1}(x)) f(\Phi^{-1}(x))\, dx_1 \cdots dx_n \\
&\qquad = \int_{\Phi(U\cap V)} a(\Phi^{-1}(x)) f(\Phi^{-1}(x))\, dx_1 \cdots dx_n \\
&\qquad = \int_{\Psi(U\cap V)} b(\Psi^{-1}(x)) f(\Psi^{-1}(x))\, dx_1 \cdots dx_n \\
&\qquad = \int_{\Psi(V)} b(\Psi^{-1}(x)) f(\Psi^{-1}(x))\, dx_1 \cdots dx_n\,.
\end{aligned}$$

Thus the formula (D.5) is justified.

We will now define the integral of a general $f \in C_c(M)$ using a partition of unity. Let $\{\varphi_i\}$ and $\{\psi_j\}$ be partitions of unity subordinate to $\{U : (U, \Phi) \in \mathcal{A}\}$. Fix for each i (*resp.* j) $(U_i, \Phi_i) \in \mathcal{A}$ (*resp.* $(V_j, \Psi_j) \in \mathcal{A}$) such that $\operatorname{supp}(\varphi_i) \subset U_i$ (resp. $\operatorname{supp}(\psi_j) \subset V_j$). Then we assert that

$$\sum_i \int_M \varphi_i f \omega = \sum_j \int_M \psi_j f \, \omega \,. \tag{D.6}$$

Indeed, $\varphi_i f = \sum_j \psi_j \varphi_i f$ and $\psi_j f = \sum_i \varphi_i \psi_j f$ (note that since f has compact support each sum only has a finite number of nonzero terms). Thus we have

$$\sum_i \int_M \varphi_i f \, \omega = \sum_{j,i} \int_M \psi_j \varphi_i f \, \omega = \sum_j \int_M \psi_j f \, \omega \,.$$

This proves the assertion. We use formula (D.6) to define the *integral* of f with respect to ω and denote it by $\int_M f \, \omega$.

The basic properties of the advanced calculus notion of integral (such as additivity) carry over to our case. The following two results will be particularly important:

Lemma D.1.13. *Let $f \in C_c(M)$. If $f(x) \geq 0$ for all $x \in M$ and if $f(x) > 0$ for some $x \in M$ then $\int_M f \, \omega > 0$.*

This is clear from the definition.

Theorem D.1.14. *Let M and N be C^∞ manifolds. Let ω be a volume form on N and let $f \in C_c(N)$ and let Φ be a diffeomorphism of M onto N. Then*

$$\int_N f \, \omega = \int_M (f \circ \Phi) \, \Phi^* \omega \,.$$

When $M = N$ we can write this in a somewhat more suggestive form. In this case $\Phi^* \omega = \varphi \, \omega$ with $\varphi \in C^\infty(M)$. Then the formula in the theorem reads

$$\int_M f \omega = \int_M (f \circ \Phi) \, |\varphi| \, \omega \,. \tag{D.7}$$

This is proved by taking a partition of unity subordinate to the orientation determined by ω, pulling back via Φ, and then using the change of variables theorem one chart at a time.

If $f \in C_c(M, \mathbb{C})$ (the complex-valued continuous functions with compact support) we write $f = f_1 + \mathrm{i} f_2$, with $f_1, f_2 \in C_c(M)$ and $\mathrm{i} = \sqrt{-1}$, and we set

$$\int_M f \, \omega = \int_M f_1 \, \omega + \mathrm{i} \int_M f_2 \, \omega \,.$$

We end this section with one additional result that we will need in our analytic proof of the Weyl character formula. Let M be a C^∞ manifold with volume form

ω. Let $\pi : Y \longrightarrow M$ be a finite covering space (see Example 7, Section D.1.1). Here *finite* means that $\pi^{-1}(x)$ is finite for each $x \in M$. Endow Y with the manifold structure in Example 7, Section D.1.1. We will denote this manifold by N. We assume that M and N are connected; this implies that $|\pi^{-1}(x)|$ is independent of $x \in M$; we will denote this number by d (the *degree* of the covering). Also, since π is locally a diffeomorphism, $\pi^*\omega$ is a volume form for N.

Theorem D.1.15. *Let $f \in C_c(M)$. Then $\int_N (f \circ \pi)\pi^*\omega = d \int_M f\,\omega$.*

Proof. This is obvious if $\mathrm{supp}(f)$ is contained in an evenly covered chart. In general use a partition of unity subordinate to a covering by evenly covered charts. ♦

D.1.4 Exercises

1. Let $X = \mathbb{R}$. Set $\Psi(x) = x^3$. Show that $\{(X, \Psi)\}$ is a C^∞ atlas for X that is not contained in the C^∞ structure corresponding to Example 1 in Section D.1.1.

2. Let M be the C^∞ manifold corresponding to Example 3 in Section D.1.1. Let $\iota(x) = x$ for $x \in M$, $\iota : M \longrightarrow \mathbb{R}^n$. Show that $d\iota_x : T(M)_x \longrightarrow T(\mathbb{R}^n)_x$ is injective and that $di_x(T(M)_x) = \{v_x : v \in \mathbb{R}^n, (df_i)_x(v_x) = 0\}$. In particular, for the example of S^n conclude that

$$di_x(T(S^n)_x) = \{v_x : (x, v) = 0\},$$

where $(\cdot, \cdot)$ is the usual inner product on $\mathbb{R}^n$.

3. A C^∞ curve in a C^∞ manifold M is a C^∞ map $\sigma : (a, b) \longrightarrow M$ (here $(a, b) = \{t \in \mathbb{R} : a < t < b\}$). Define $\sigma'(t) = d\sigma_t(d/dt)_t$. Show that $T(M)_p$ is the set of all $\sigma'(t)$ with $\sigma(t) = p$.

4. Let $M = S^1 \times S^1$, where we take $S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\} \subset \mathbb{C}$ and $i = \sqrt{-1}$.

   (a) Show that the map $f : \mathbb{R}^2 \longrightarrow M$ given by $f(x, y) = (e^{ix}, e^{iy})$ is C^∞.

   (b) Let $Y = f(\{x, \sqrt{2}x) : x \in \mathbb{R}\})$ and define $\Phi(f(x, \sqrt{2}x)) = x$. Show that Φ is a bijection between Y and $\mathbb{R}$.

   (c) Endow Y with the topology that makes Φ a homeomorphism. Endow Y with the C^∞ structure containing the chart (Y, Φ). Let N denote this C^∞ manifold. Show that N is a submanifold of M.

   (d) Show that the relative topology of Y in M is not the same as the given topology.

5. Let M be an n-dimensional C^∞ manifold. Then a correspondence $p \mapsto X_p \in T(M)_p$ is a vector field if and only if for each system of local coordinates $\{u_1, \ldots, u_n\}$ on U there exist C^∞ functions $a_1, \ldots, a_n$ on U such that

$$X_p = \sum_i a_i(p) \left(\frac{\partial}{\partial u_i}\right)_p \quad \text{for } p \in U.$$

   Show that it is enough to check this for U that cover M.

6. Show that the vector fields on a C^∞ manifold form a Lie algebra under the vector field bracket.

7. Define $\omega \in \Omega^n(\mathbb{R}^{n+1})$ by

$$\omega = \sum_{i=1}^{n+1} (-x_i)^{i+1} dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n.$$

Let $\iota : S^n \longrightarrow \mathbb{R}^{n+1}$ be the usual injection ($\iota(x) = x$). Show that $\iota^*\omega$ defines a volume form on S^n.

8. Let $n = 2$ in the preceding exercise. Let $\Phi : \mathbb{R} \longrightarrow S^1$ be defined by $\Phi(t) = (\cos t, \sin t)$. Calculate $\Phi^*\omega$.

9. Let M be the submanifold of $\mathbb{R}^n$ corresponding to Example 3 in Section D.1.1. If $f \in C^\infty(\mathbb{R}^n)$ define $\nabla f(x) \in \mathbb{R}^n$ by $(\nabla f(x), v) = v_x(f)$ for $v \in \mathbb{R}^n$, where $(\cdot\,,\,\cdot)$ is the usual inner product on $\mathbb{R}^n$. Define $\omega \in \Omega^{n-k}(\mathbb{R}^n)$ by

$$\begin{aligned}&\omega_x\big((v_1)_x, \ldots, (v_{n-k})_x\big)\\ &\qquad = (dx_1 \wedge \cdots \wedge dx_n)_x \left(\nabla f_1(x), \ldots, \nabla f_k(x), (v_1)_x, \ldots, (v_{n-k})_x\right).\end{aligned}$$

Let $\iota : M \longrightarrow \mathbb{R}^n$ be the usual injection ($\iota(x) = x$). Show that $\iota^*\omega$ defines a volume form on M. Relate this exercise to exercise 9.

10. Verify formula (D.7).

D.2 Lie Groups

D.2.1 Basic Definitions

Let G be a C^∞ manifold such that the underlying set has the structure of a group. We write $m(x, y) = xy$ (the group multiplication) and $\eta(x) = x^{-1}$ (the group inverse). We say that G is a *Lie group* if $m : G \times G \longrightarrow G$ (see Example 5 in Section D.1.1) and $\eta : G \longrightarrow G$ are of class C^∞.

Examples

1. Let $G = \mathrm{GL}(n, \mathbb{R}) = \{X \in M_n(\mathbb{R}) : \det X \neq 0\}$. Then G is open in $M_n(\mathbb{R})$ (which we look upon as $\mathbb{R}^{n^2}$). The map m is clearly C^∞ and the map η is C^∞ by Cramer's rule.

2. Let $G \subset \mathrm{GL}(n, \mathbb{R})$ be any subgroup that is closed (relative to the topology of $\mathrm{GL}(n, \mathbb{R})$). In Section 1.3.5 we show that G has a Lie group structure that is compatible with its topology as a closed subspace of $\mathrm{GL}(n, \mathbb{R})$.

3. Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a linear algebraic group (see Section 1.4.1). Then we can view G as a closed subgroup of $\mathrm{GL}(2n, \mathbb{R})$, and hence give G a Lie group structure

compatible with the closed subgroup topology. In Section 1.4.4 we show that the Lie algebra of G is the Lie algebra in the sense of algebraic groups, but with scalars restricted to $\mathbb{R}$.

If G and H are Lie groups then a *Lie group homomorphism* of G to H is a group homomorphism $\varphi : G \longrightarrow H$ that is C^∞. We say that a Lie group homomorphism φ is a *Lie group isomorphism* if φ is a diffeomorphism.

If G and H are Lie groups with the underlying subset of H a subgroup of G then H is said to be a *Lie subgroup* of G if H is a submanifold of G. As in the case of submanifolds a Lie subgroup does not necessarily have the subspace topology (see Exercises D.1.4 #4).

If G and H are Lie groups then $G \times H$ is easily seen to be a Lie group with the product C^∞ structure and the product group structure.

D.2.2 Lie Algebra of a Lie Group

Let G be a Lie group. Let $L_y : G \longrightarrow G$ be the left translation map defined by $L_y x = yx$ for $y, x \in G$. Then L_y is of class C^∞ and $(L_y)^{-1} = L_{y^{-1}}$ by the associative rule. We look upon a vector field on G as a derivation of the algebra $C^\infty(G)$. That is, if X is a vector field and $f \in C^\infty(G)$ then $Xf \in C^\infty(G)$ is defined by $Xf(y) = X_y f$. The definition of tangent vector implies that

$$X(fg)(y) = (Xf)(y)g(y) + f(y)(Xg)(y) \quad \text{for } y \in G \text{ and } f, g \in C^\infty(G),$$

and hence X is a derivation of $C^\infty(G)$. We set $L_y^* f = f \circ L_y$. Then a vector field is said to be *left invariant* if for each $y \in G$ one has $L_y^* \circ X = X \circ L_y^*$. This property can be stated in terms of tangent vectors as

$$d(L_y)_x X_x = X_{yx} \quad \text{for all } x, y \in G. \tag{D.8}$$

We set $\mathrm{Lie}(G)$ equal to the vector space of all left invariant vector fields on G. Denote the identity element of G as 1.

Lemma D.2.1. *The map $X \mapsto X_1$ defines a linear bijection between $\mathrm{Lie}(G)$ and $T(G)_1$. If $X, Y \in \mathrm{Lie}(G)$ then $[X, Y] \in \mathrm{Lie}(G)$. Thus $\mathrm{Lie}(G)$ is a Lie algebra over $\mathbb{R}$ of dimension $n = \dim G$.*

Proof. Equation (D.8) implies that if $X_1 = 0$ then $X_g = 0$. Thus the map is injective. If $v \in T(G)_1$ then let $\sigma : (-\epsilon, \epsilon) \longrightarrow G$ be a C^∞ curve with $\sigma(0) = 1$ and $\sigma'(0) = v$ (see Exercises D.1.4 #3). Let $\Psi : G \times (-\epsilon, \epsilon) \longrightarrow G$ by $\Psi(g, t) = g\sigma(t)$. Since G is a Lie group, Ψ is a C^∞ map. Set

$$X_g f = \frac{d}{dt} f(g\sigma(t))\bigg|_{t=0}$$

for $f \in C^\infty(G)$. Then $g \mapsto X_g f$ defines a C^∞ function on G and so $g \mapsto X_g$

defines a vector field. We note that $X_1 = v$ and

$$\begin{aligned} d(L_g)_x X_x f &= X_x(f \circ L_g) = \frac{d}{dt}(f \circ L_g)(x\sigma(t))\Big|_{t=0} \\ &= \frac{d}{dt} f(gx\sigma(t))\Big|_{t=0} = X_{gx} f\,. \end{aligned}$$

Thus $X \in \mathrm{Lie}(G)$. So the map in the lemma is surjective.

Let $X, Y \in \mathrm{Lie}(G)$ and $g \in G$. Then for $f \in C^\infty(G)$ we have

$$\begin{aligned} L_g^*([X,Y]f) &= L_g^*(XYf - YXf) = L_g^*(XYf) - L_g^*(YXf) \\ &= X(L_g^*(Yf)) - Y(L_g^*(Xf)) = XYL_g^* f - YXL_g^* f \\ &= [X,Y]L_g^* f\,. \end{aligned}$$

Thus $[X,Y] \in \mathrm{Lie}(G)$. ♦

In light of the above lemma we call $\mathrm{Lie}(G)$ the *Lie algebra* of G.

Let G be a Lie group. The following basic theorem is proved in Section 1.3.6 when G is a closed subgroup of $\mathrm{GL}(n,\mathbb{R})$ (see Warner [1983] for a proof in general):

Theorem D.2.2. *There exists a unique C^∞ map $\exp : \mathrm{Lie}(G) \longrightarrow G$ with the properties:*

(1) $\exp(0) = 1$ *;*

(2) *for all $X \in \mathrm{Lie}(G)$, $f \in C^\infty(G)$, and $t \in \mathbb{R}$, one has*

$$\frac{d}{dt} f(\exp(tX)) = (Xf)(\exp(tX))\,.$$

Furthermore, if $X \in \mathrm{Lie}(G)$ and if $\sigma : \mathbb{R} \longrightarrow G$ is a C^∞ curve with $\sigma(0) = 1$ and $\sigma'(t) = X_{\sigma(t)}$ for all $t \in \mathbb{R}$, then $\sigma(t) = \exp(tX)$.

We note that if $X \in \mathrm{Lie}(G)$ then $\sigma_X(t) = \exp(tX)$ is a C^∞ curve with $\sigma_X(0) = 1$ and $\sigma'_X(0) = X_1$. But $\sigma'_X(0) = d\exp_0((X)_0)$, where $(X)_0$ is the tangent vector to the real vector space $\mathrm{Lie}(G)$ at 0 corresponding to X. Thus we see that $d\exp_0$ is a linear bijection between $T(\mathrm{Lie}(G))_0$ and $T(G)_1$. The inverse function theorem (using a chart containing 1) implies that there is an open neighborhood U_0 of 0 in $\mathrm{Lie}(G)$ such that $\exp(U_0) = U$ is open in G and $\exp : U_0 \longrightarrow U$ is a diffeomorphism. Let $\log : U \longrightarrow U_0$ be the inverse map to $\exp|_{U_0}$. Then $(U, \log)$ is a chart for G.

Corollary D.2.3. *Suppose G is a connected Lie group. Then G is generated (as a group) by $\exp(\mathrm{Lie}\, G)$.*

Proof. Since G is connected, it is generated by any neighborhood U of 1 (see Exercises D.2.5, #1). Take $U = \exp(U_0)$ with U_0 as above. ♦

Given Lie groups G and H and a Lie group homomorphism $\varphi : G \longrightarrow H$, we define $d\varphi : \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(H)$ by $d\varphi(X)_1 = d\varphi_1(X_1)$.

Lemma D.2.4. *For $X, Y \in \mathrm{Lie}(G)$ one has $d\varphi([X,Y]) = [d\varphi(X), d\varphi(Y)]$.*

Proof. If $f \in C^\infty(H)$ then $X(f \circ \varphi) = (d\varphi(X)f) \circ \varphi$ by the left-invariance of X. Hence $[X,Y](f \circ \varphi) = ([d\varphi(X), d\varphi(Y)]f) \circ \varphi$. This implies that $d\varphi([X,Y])_1 = ([d\varphi(X), d\varphi(Y)])_1$. ♦

Lemma D.2.5. *Let G and H be Lie groups. Suppose $\varphi : G \longrightarrow H$ is a Lie group homomorphism. Then $\varphi(\exp(X)) = \exp(d\varphi(X))$ for all $X \in \mathrm{Lie}(G)$.*

Proof. Let $\sigma(t) = \exp(tX)$ and $\mu(t) = \varphi(\exp(tX))$ for $t \in \mathbb{R}$. Then

$$\mu'(t) = d\varphi_{\sigma(t)}\sigma'(t) = d\varphi_{\sigma(t)}X_{\sigma(t)} = d\varphi(X)_{\mu(t)}\,.$$

Thus Theorem D.2.2 implies that $\mu(t) = \exp(td\varphi(X))$. ♦

Theorem D.2.6. *Let G be a Lie group and let H be a closed subgroup of G. Then H has a structure of a Lie group such that*

(1) *the inclusion map $\iota : H \longrightarrow G$ is a Lie group homomorphism;*

(2) $d\iota(\mathrm{Lie}(H)) = \{X \in \mathrm{Lie}(G) \,:\, \exp(tX) \in H \quad \textit{for all } \ t \in \mathbb{R}\}$.

Proof. In the case $G = \mathrm{GL}(n,\mathbb{R})$ this is Theorem 1.3.11. For a proof for nonlinear groups see Warner [1983]. ♦

For $g \in G$ we define $\mathrm{Inn}(g)(x) = gxg^{-1}$. Then $\mathrm{Inn}(g)$ defines a Lie group automorphism of G, called an *inner automorphism.* We define $\mathrm{Ad}(g) = d\ \mathrm{Inn}(g)$.

Lemma D.2.7. *The map $\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathrm{Lie}(G))$ is a Lie group homomorphism.*

This lemma is proved in Section 1.3.3 when G is a closed subgroups of $\mathrm{GL}(n,\mathbb{R})$. For a proof when G is not a linear group see Warner [1983].

Theorem D.2.8. *Let G be a topological group. Assume that*

(1) *there is an open neighborhood U of the identity element in G such that $u^{-1} \in U$ for all $u \in U$;*

(2) *there is a surjective homeomorphism $\Phi : U \longrightarrow B_r(0) \subset \mathbb{R}^m$ for some $r > 0$, and $\Phi(u^{-1}) = -\Phi(u)$ for all $u \in U$ (where $B_r(0) = \{x \in \mathbb{R}^m \,:\, ||x|| < r\}$);*

(3) *there is an s with $0 < s < r$ and a C^∞ map $F : B_s(0) \times B_s(0) \longrightarrow B_r(0)$ such that $uv \in U$ and $\Phi(uv) = F(\Phi(u), \Phi(v))$ for all $u, v \in \Phi^{-1}(B_s(0))$.*

Then there exists a Lie group structure on G compatible with the topological group structure.

Proof. Use the same argument as in the proof of Theorem 1.3.12, but with the logarithm map replaced by the map Φ. ♦

Theorem D.2.8 yields an easy proof of the fact that if $H \subset G$ is a closed normal subgroup of a Lie group G, then G/H has the structure of a Lie group.

Theorem D.2.9. *Suppose G is a connected Lie group and and $\pi : H \longrightarrow G$ is a covering space. Let e be the identity element of G and choose $e_0 \in \pi^{-1}(e)$. Then H has a structure of a Lie group with identity e_0 such that π is a Lie group homomorphism.*

Proof. Let $L_g : G \longrightarrow G$ be the left translation map $L_g(x) = gx$. Then for each $h \in H$ there exists a unique homeomorphism $\tilde{L}_h : H \longrightarrow H$ such that

$$\tilde{L}_h(e_0) = h \quad \text{and} \quad \pi(\tilde{L}_h(x)) = L_{\pi(h)}(\pi(x)) \quad \text{for all } x \in H\,.$$

We assert that the product $m(u, v) = \tilde{L}_u(v)$ makes H a group. The identity map has the same property assumed for $\tilde{L}_{e_0}$; thus $m(e_0, u) = u$. By definition $m(u, e_0) = u$. Hence e_0 is an identity element for the multiplication m. If $u \in H$ then there exists a unique $v \in H$ such that $\tilde{L}_u(v) = e_0$. To prove that H is a group it only remains to show that the associative rule is satisfied.

We note that $L_x \circ L_y = L_{xy}$ and $\pi(m(x, y)) = xy$. Since

$$m(x, y) = \tilde{L}_{m(x,y)}(e_0) = \tilde{L}_x \circ \tilde{L}_y(e_0)\,,$$

we have $\tilde{L}_{m(x,y)} = \tilde{L}_x \circ \tilde{L}_y$, which proves the associative rule.

We must now prove that $m : H \times H \longrightarrow H$ is continuous. We note that if we set $\mu(x, y) = xy$ for $x, y \in G$, then there is a unique liftn $\tilde{\mu} : H \times H \longrightarrow H$ of μ such that $\tilde{\mu}(e_0, e_0) = e_0$. Since m is another such lift we see that $m = \tilde{\mu}$. The existence of a Lie group structure on H such that π is a C^∞ map now follows from Theorem D.2.8. ♦

D.2.3 Homogeneous Spaces

Let G be a Lie group and let H be a closed subgroup of G. Then H is a Lie subgroup of G (Theorem D.2.6). Let $Y = G/H$ with the quotient topology. We now show how to put a C^∞ structure on Y yielding a C^∞ manifold M such that the map $G \times M \longrightarrow M$ given by $g, m \mapsto gm$ is of class C^∞.

Let $\mathfrak{g} = \mathrm{Lie}(G)$ and let $\mathfrak{h} = d\iota(\mathrm{Lie}(H))$ ($\iota(h) = h$, $h \in H$). Let V be a subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus V$. Define $\Phi : V \times \mathfrak{h} \longrightarrow G$ by

$$\Phi(v, X) = \exp(v)\exp(X)\,.$$

Then $d\Phi_{(0,0)}(v_0, X_0) = v_1 + X_1$. Thus there are open neighborhoods U_0 and W_0 of 0 in V and $\mathfrak{h}$, respectively, such that $\Phi(U_0 \times W_0) = U_1$ is open in G and Φ is a diffeomorphism from $U_0 \times W_0$ onto U_1. We now choose open neighborhoods of 0, $U_0' \subset U_0$ and $W_0' \subset W_0$, such that

(a) If $U_1' = \Phi(U_0' \times W_0')$ and $x, y \in U_1'$, then xy^{-1}, $y^{-1}x \in U_1$.

(b) $U_1' \cap H \subset \exp(W_0)$.

This can be done because the maps $G \times G \longrightarrow G$ given by $x, y \mapsto xy^{-1}$ and $x, y \mapsto y^{-1}x$ are C^∞ and because $\exp(W_0)$ contains a neighborhood of 1 in H.

We now observe that if $v_1, v_2 \in U_0'$ and if $h_1, h_2 \in H$ then

$$\exp(v_1)h_1 = \exp(v_2)h_2 \quad \text{implies } h_1 = h_2 \text{ and } v_1 = v_2\,. \tag{$\star$}$$

Indeed, we have $\exp(v_2)^{-1}\exp(v_1) = h_2h_1^{-1}$. Thus (a) implies that $h_2h_1^{-1} \in U_1 \cap H$ and (b) implies that $h_2h_1^{-1} = \exp(X)$ with $X \in W_0$. Thus

$$\Phi(v_1, 0) = \exp(v_1) = \exp(v_2)\exp(X) = \Phi(v_2, X).$$

We conclude that $v_1 = v_2$ and $X = 0$. Hence $h_2h_1^{-1} = 1$ and assertion ($\star$) is proved.

Let $\pi : G \longrightarrow G/H$ be defined by $\pi(g) = gH$. The map π is called the *natural projection.* Set $\bar{U} = \pi(U_1')$. Define $\Psi(\pi(\exp(v)) = v$ for $v \in U_0'$. Then $(\bar{U}, \Psi)$ defines a chart for G/H. Given $g \in G$ set $g(xH) = gxH$. This will be called the *natural action* of G on G/H. Define $\Psi_g(gx) = \Psi(x)$ for $x \in G/H$. Then an argument similar to the one above shows that $\{(g\bar{U}, \Psi_g)\}_{g\in G}$ is a C^∞ atlas for G/H. We have thus sketched the proof of the following result:

Theorem D.2.10. *Let G be a Lie group and let H be a closed subgroup of G. Then there exists a C^∞ structure of dimension $\dim G - \dim H$ on G/H such that the natural projection is of class C^∞ and the map $G \times G/H \longrightarrow G/H$ given by the natural action is of class C^∞.*

Examples

1. The standard action of the orthogonal group $\mathrm{O}(n+1, \mathbb{R})$ on $\mathbb{R}^{n+1}$ is transitive on the n-sphere S^n. This gives the homogeneous space $\mathrm{O}(n+1, \mathbb{R})/\mathrm{O}(n, \mathbb{R})$.

2. The Grassmann manifold $\mathrm{Grass}_k(\mathbb{R}^n)$ of k-planes in $\mathbb{R}^n$ is isomorphic to

$$\mathrm{O}(n, \mathbb{R})/\mathrm{O}(k, \mathbb{R}) \times \mathrm{O}(n-k, \mathbb{R})$$

as a homogeneous space for $\mathrm{O}(n, \mathbb{R})$.

D.2.4 Integration on Lie Groups and Homogeneous Spaces

Let G be a Lie group and set $\mathfrak{g} = \mathrm{Lie}(G)$. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$. From Lemma D.2.1 it follows that $(X_1)_g, \ldots, (X_n)_g$ is a basis of $T(G)_g$ for each $g \in G$. There is a unique element $\omega_g \in \bigwedge^n (T(G)_g)^*$ such that $\omega_g((X_1)_g, \ldots, (X_n)_g) = 1$ (here we identify $\bigwedge^n \mathfrak{g}^*$ with the n-multilinear alternating real functions on $\mathfrak{g}$; see Section B.2.4).

We claim that $g \mapsto \omega_g$ defines an element of $\Omega^n(G)$. To see this let $\{y_1, \ldots, y_n\}$ be a system of local coordinates on $U \subset G$. Then

$$(X_i)_g = \sum_j u_{ji}(g)\Big(\frac{\partial}{\partial y_j}\Big)_g \quad \text{with } u_{ij} \in C^\infty(U)\,.$$

Since $(X_1)_g, \ldots, (X_n)_g$ is a basis of $T(G)_g$, we have $\det[u_{ij}(g)] \neq 0$ for all $g \in U$. This implies that if X is a vector field on G then $X = \sum_i a_i X_i$ with $a_i \in C^\infty(G)$. Thus $g \mapsto \omega_g$ is indeed in $\Omega^n(G)$. Since $\omega_g \neq 0$ for all $g \in G$, ω is a volume form on G.

We also note that if $g \in G$ then

$$\begin{aligned}(L_g^*\omega)_x((X_1)_x, \ldots, (X_n)_x) &= \omega_{gx}(d(L_g)_x (X_1)_x, \ldots, d(L_g)_x (X_n)_x) \\ &= \omega_{gx}((X_1)_{gx}, \ldots, (X_n)_{gx}) \\ &= \omega_x((X_1)_x, \ldots, (X_n)_x).\end{aligned}$$

Thus $L_g^*\omega = \omega$. This calculation implies that if $f \in C_c(G)$ then

$$\int_G f\omega = \int_G (f \circ L_g)\, \omega. \tag{D.9}$$

Notice that ω is determined up to a scalar multiple. Having fixed ω, we write

$$\int_G f(g)\, dg = \int_G f\, \omega. \tag{D.10}$$

With this notation (D.9) becomes

$$\int_G f(xg)\, dg = \int_G f(g)\, dg \quad \text{for } x \in G\,. \tag{D.11}$$

For $g, x \in G$ we set $R_x g = gx$. Then R_x defines a diffeomorphism of G.

Lemma D.2.11. *If $f \in C_c(G)$ and $x \in G$ then*

$$\int_G f(gx)\, dg = |\det \mathrm{Ad}(x)| \int_G f(g)\, dg\,.$$

Proof. By the left invariance of ω we can write

$$\int_G f(gx)\, dg = \int_G f(x^{-1}gx)\, dg = \int_G f \circ \mathrm{Inn}(x^{-1})\, \omega\,.$$

Thus by Theorem D.1.14, with $\Phi = \mathrm{Inn}(x)$, we have

$$\int_G f(gx)\, dg = \int_G f\, \Phi^*\omega\,.$$

Now $d\Phi = \mathrm{Ad}(x)$, so from the definition of ω we obtain

$$\begin{aligned}(\Phi^*\omega)_g\big((X_1)_g, \ldots, (X_n)_g\big) &= \omega_1\big((\mathrm{Ad}(x)X_1)_1, \ldots, (\mathrm{Ad}(x)X_n)_1\big) \\ &= \det(\mathrm{Ad}(x))\omega_1\big((X_1)_1, \ldots, (X_n)_1\big) \\ &= \det(\mathrm{Ad}(x))\omega_g\big((X_1)_g, \ldots, (X_n)_g\big)\end{aligned}$$

for $g \in G$ and $X_1, \ldots, X_n \in \mathfrak{g}$. Hence $\mathrm{Inn}(x)^*\omega = \det(\mathrm{Ad}(x))\omega$. The lemma now follows from formula (D.7). ♦

We define the *modular function* δ of G to be $\delta(g) = |\det \mathrm{Ad}(g)|$. Since $\mathrm{Ad}(xy) = \mathrm{Ad}(x)\,\mathrm{Ad}(y)$, we have $\delta(xy) = \delta(x)\delta(y)$, so δ is a Lie homomorphism of G into $\mathbb{R}^\times = \mathrm{GL}(1, \mathbb{R})$. By Lemma D.2.11 we can write

$$\int_G f(gx)\delta(g)\,dg = \delta(x)\int_G f(g)\delta(gx^{-1})\,dg\,.$$

Since $\delta(gx^{-1}) = \delta(g)\delta(x)^{-1}$, we obtain the integral formula

$$\int_G f(gx)\delta(g)\,dg = \int_G f(g)\delta(g)\,dg \tag{D.12}$$

for $x \in G$ and $f \in C_c(G)$. This shows that $\delta(g)\,dg$ is a *right-invariant* measure on G.

We say that G is *unimodular* if $\delta(g) = 1$ for all $g \in G$. In this case the left-invariant measure dg on G is also right invariant (the converse also holds).

Lemma D.2.12. *A compact Lie group is unimodular.*

Proof. Since $\delta(g) > 0$ for all $g \in G$, $\delta(G)$ is a compact subgroup of the multiplicative group $\{x \in \mathbb{R}^\times : x > 0\}$. The only such subgroup is $\{1\}$, by the Archimedean property of the real numbers. ♦

If G is compact then $1 \in C_c(G)$ and $0 < c = \int_G \omega < \infty$. Replacing ω by $c^{-1}\omega$, we get

$$\int_G dg = 1\,. \tag{D.13}$$

We will call dg the *normalized invariant measure* on G if it satisfies this condition.

Assume that G is a Lie group and that H is a closed subgroup of G. Give G/H the C^∞ structure defined in Section D.2.3. We assume for simplicity that if $g \in G$ then $\det \mathrm{Ad}(g) = 1$ and that if $h \in H$ then $\det(\mathrm{Ad}(h)|_{\mathfrak{h}}) = 1$. We now show how to put a G-invariant volume form ω on G/H. For $g \in G$ let l_g be the transformation $l_g x = gx$ on G/H. Then the desired form ω must satisfy $l_g^*\omega = \omega$ for all $g \in G$.

We will use the notation of Section D.2.3. Let $m = \dim G - \dim H$. Let $\eta \in \Omega^m(G)$ be defined as follows: Write $\mathfrak{g} = V \oplus \mathfrak{h}$ for some linear subspace V. We look upon V^* as $\{\lambda \in \mathfrak{g}^* : \lambda(\mathfrak{h}) = 0\}$. Then we can consider $\bigwedge^m V^*$ to be a subspace of $\bigwedge^m \mathfrak{g}^*$. Let $0 \neq \nu \in \bigwedge^m V^*$. Define

$$\eta_g\big((X_1)_g, \ldots, (X_m)_g\big) = \nu(X_1, \ldots, X_m)\,.$$

Then, as in the case of the invariant volume form on G, we see that $\eta \in \Omega^m(G)$ and $L_g^*\eta = \eta$ for all $g \in G$. We note that if $Z_1, \ldots, Z_m \in \mathfrak{h}$ then

$$\nu(X_1 + Z_1, \ldots, X_m + Z_m) = \nu(X_1, \ldots, X_m)\,.$$

Let $\{X_1, \ldots, X_m\}$ be a basis of V and let $\{X_{m+1}, \ldots, X_n\}$ be a basis of $\mathfrak{h}$. If $h \in H$ then the linear transformation $\mathrm{Ad}(h)$ has a matrix of block form

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

relative to the basis $\{X_1, \ldots, X_n\}$. Here C is the matrix of $\mathrm{Ad}(h)|_{\mathfrak{h}}$ relative to the basis $\{X_{m+1}, \ldots, X_n\}$. Thus

$$\nu(\mathrm{Ad}(h)X_1, \ldots, \mathrm{Ad}(h)X_m) = \det(A)\nu(X_1, \ldots, X_m)\,.$$

Now $\det A \det C = \det(\mathrm{Ad}(h))$. Our assumption thus implies that $\det A = 1$. We therefore see that

$$L_g^*\eta = \eta, \quad R_h^*\eta = \eta \quad \text{for } g \in G,\, h \in H\,. \tag{D.14}$$

We are now ready to define the form ω. If $g \in G$ and $v_1, \ldots, v_m \in T(G/H)_{gH}$ let $X_1, \ldots, X_m \in \mathfrak{g}$ be such that $d\pi_g((X_i)_g) = v_i$. We assert that

$$\eta_g((X_1)_g, \ldots, (X_m)_g) \text{ depends only on } gH \text{ and } v_1, \ldots, v_m\,. \tag{$\star$}$$

Indeed, let $w_i = dl_{g^{-1}}v_i \in T(G/H)_H$. Since $l_g \circ \pi = \pi \circ L_g$, we have $w_i = d\pi_1\big((X_i)_1\big)$. If $X_i' \in \mathfrak{g}$ also satisfies $d\pi_1\big((X_i')_1\big) = w_i$ for $i = 1, \ldots, m$, then $X_i - X_i' \in \mathfrak{h}$. Thus

$$\eta_1\big((X_1)_1, \ldots, (X_n)_1\big) = \eta_1\big((X_1')_1, \ldots, (X_n')_1\big)\,.$$

If $\pi(g) = \pi(g')$ then $g' = gh$, $h \in H$. Hence (D.14) implies ($\star$).

We also note that if $\{X_1, \ldots, X_m\}$ is a basis for V then

$$\big\{\, d\pi_1\,(X_1)_1, \ldots, d\pi_1\,(X_m)_1 \,\big\}$$

is a basis of $T(G/H)_H$. Hence there is an open neighborhood U of 1 in G such that $\big\{\, d\pi_g\big((X_1)_g\big), \ldots, d\pi_g\big((X_m)_g\big) \,\big\}$ is a basis of $T(G/H)_{gH}$ for $g \in U$. Thus if we set

$$\omega_{gH}\big(d\pi_g\,(X_1)_g, \ldots, d\pi_g\,(X_m)_g\big) = \eta_g\big((X_1)_g, \ldots, (X_m)_g\big)\,,$$

then we have defined ω in $\Omega^m(\pi(U))$. Now $l_{g^{-1}} : l_g\pi(U) \longrightarrow \pi(U)$. Thus for each $g \in G$ we have a differential form $\omega^g \in \Omega^m(l_g\pi(U))$ given by $\omega^g = l_g^*\omega$. Property ($\star$) above implies that

$$\omega_x^{g_1} = \omega_x^{g_2} \quad \text{for } x \in l_{g_1}(\pi(U)) \cap l_{g_2}(\pi(U))\,.$$

We have thus defined ω on G/H. This proves the following result:

Theorem D.2.13. *Let G be a Lie group and let H be a closed subgroup of G. Assume that $\det \mathrm{Ad}(g) = 1$ and $\det \mathrm{Ad}(h)|_{\mathrm{Lie}(H)} = 1$ for all $g \in G$ and $h \in H$. Set $m = \dim G/H$. Then there exists a volume form $\omega \in \Omega^m(G/H)$ such that $l_g^*\,\omega = \omega$ for all $g \in G$.*

D.2.5 Exercises

1. We look upon $\mathbb{C}$ as $\mathbb{R}^2$ as usual. Then $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is a group under complex multiplication.

   (a) Use the C^∞ structure as in Example 2 of Section D.1.1 to show that S^1 is a Lie group.

   (b) Define $\mathbb{T}^1 = S^1$ and inductively define $\mathbb{T}^{n+1} = \mathbb{T}^n \times \mathbb{T}^1$ as product of Lie groups for $n \geq 1$. Show that the submanifold in Exercise 3 of Section D.1.2 is a Lie subgroup of $\mathbb{T}^2$.

2. For $1 \geq k < n$ let $P \subset \mathrm{GL}(n, \mathbb{R})$ be the group of block upper-triangular matrices $g = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ with $A \in \mathrm{GL}(k, \mathbb{R})$, $B \in M_{k,n-k}(\mathbb{R})$, and $D \in \mathrm{GL}(n-k, \mathbb{R})$. Given $g \in \mathrm{GL}(k, \mathbb{R})$, let $\mu(g) \subset \mathbb{R}^n$ be the subspace spanned by the first k columns of g. Show that μ sets up an isomorphism between $\mathrm{GL}(n, \mathbb{R})/P$ and the k-Grassmannian $\mathrm{Grass}_k(\mathbb{R})$ as differentiable manifolds (see Example 2 of Section D.2.3).

3. One has $G_1(\mathbb{R}^n) = \mathbb{P}^1(\mathbb{R}^n)$. Show that the natural map $S^{n-1} \longrightarrow \mathbb{P}^1(\mathbb{R}^n)$ is a C^∞ covering.

4. Show that $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{O}(n, \mathbb{R})$ are unimodular.

5. Let G be the subgroup of upper-triangular matrices in $\mathrm{GL}(n, \mathbb{R})$. Calculate the modular function of G.

6. Show that the volume form on S^{n-1} given as in Exercise D.1.4 #7 is a scalar multiple of the one given in Section D.2.4. (HINT: Use Example 1 in Section D.2.3.)