Appendix A

Algebraic Geometry

We develop the aspects of algebraic geometry needed for the study of algebraic groups over $\mathbb{C}$ in this book. Although we give self-contained proofs of almost all of the results stated, we do not attempt to give an introduction to the field of algebraic geometry or to give motivating examples. We refer the interested reader to Cox, Little, and O'Shea [1992], Harris [1992], Shafarevich [1994], and Zariski and Samuel [1958] for more details.

A.1 Affine Algebraic Sets

A.1.1 Basic Properties

Let V be a finite-dimensional vector space over $\mathbb{C}$. A complex-valued function f on V is a polynomial of degree $\leq k$ if for some basis $\{e_1, \ldots, e_n\}$ of V one has

$$f\big(\textstyle\sum_{i=1}^n x_i e_i\big) = \sum_{|I|\leq k} a_I\, x^I$$

Here for a multi-index $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$ we write $x^I = x_1^{i_1} \cdots x_n^{i_n}$. This definition is obviously independent of the choice of basis for V. If there exists a multi-index I with $|I| = k$ and $a_I \neq 0$, then we say that f has degree k. Let $\mathcal{P}(V)$ be the set of all polynomials on V, $\mathcal{P}_k(V)$ the polynomials of degree $\leq k$, and $\mathcal{P}^k(V)$ the polynomials of degree k. Then $\mathcal{P}(V)$ is a commutative algebra, relative to pointwise multiplication of functions. It is freely generated as an algebra by the linear coordinate functions $x_1, \ldots, x_n$. A choice of a basis for V thus gives rise to an algebra isomorphism $\mathcal{P}(V) \cong \mathbb{C}[x_1, \ldots, x_n]$, the polynomial ring in n variables.

Definition A.1.1. A subset $X \subset V$ is an *affine algebraic set* if there exist functions $f_j \in \mathcal{P}(V)$ such that $X = \{v \in V \,:\, f_j(v) = 0 \text{ for } j = 1, \ldots, m\}$.

When X is an affine algebraic set, we define the *affine ring* $\mathcal{O}[X]$ of X to be the functions on X that are restrictions of polynomials on V:

$$\mathcal{O}[X] = \{f|_X \,:\, f \in \mathcal{P}(V)\}\,.$$

We call these functions the *regular functions* on X. Define

$$\mathcal{I}_X = \{f \in \mathcal{P}(V) \,:\, f|_X = 0\}\,.$$

Then $\mathcal{I}_X$ is an ideal in $\mathcal{P}(V)$, and $\mathcal{O}[X] \cong \mathcal{P}(V)/\mathcal{I}_X$ as a commutative algebra.

Theorem A.1.2 (Hilbert basis theorem)**.** *Let $\mathcal{I} \subset \mathcal{P}(V)$ be an ideal. Then there is a finite set of polynomials $\{f_1, \ldots, f_d\} \subset \mathcal{I}$ so that every $g \in \mathcal{I}$ can be written as $g = g_1 f_1 + \cdots + g_d f_d$ with $g_i \in \mathcal{P}(V)$.*

Proof. Let $\dim V = n$. Then $\mathcal{P}(V) \cong \mathbb{C}[x_1, \ldots, x_n]$. Since a polynomial in $x_1, \ldots, x_n$ with coefficients in $\mathbb{C}$ can be written uniquely as a polynomial in x_n with coefficients that are polynomials in $x_1, \ldots, x_{n-1}$, there is a ring isomorphism

$$\mathcal{P}(V) \cong \mathcal{R}[x_n], \qquad \text{with } \mathcal{R} = \mathbb{C}[x_1, \ldots, x_{n-1}]\,. \tag{A.1}$$

We call an arbitrary commutative ring $\mathcal{R}$ *Noetherian* if every ideal in $\mathcal{R}$ is finitely generated. For example, the field $\mathbb{C}$ is Noetherian since its only ideals are $\{0\}$ and $\mathbb{C}$. To prove the theorem, we see from (A.1) that it suffices to prove the following:

(N) If a ring $\mathcal{R}$ is Noetherian, then the polynomial ring $\mathcal{R}[x]$ is Noetherian

We first show that the Noetherian property for a ring $\mathcal{R}$ is equivalent to the *ascending chain condition* for ideals in $\mathcal{R}$:

(ACC) If $\mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots \subset \mathcal{R}$ is an ascending chain of ideals in $\mathcal{R}$, then there exists an index p so that $\mathcal{I}_j = \mathcal{I}_p$ for all $j \geq p$.

Indeed, suppose $\mathcal{R}$ satisfies (ACC). Given an ideal $\mathcal{I} \subset \mathcal{R}$, we take $f_1 \in \mathcal{I}$ and set $I_1 = \mathcal{R}f_1$. If $\mathcal{I}_1 \neq \mathcal{I}$, we take $f_2 \in \mathcal{I}$ with $f_2 \notin \mathcal{I}_1$ and set $I_2 = \mathcal{R}f_1 + \mathcal{R}f_2$. Then $\mathcal{I}_1 \subset \mathcal{I}_2 \subset \mathcal{I}$. Continuing in this way, we obtain an ascending chain $\mathcal{I}_1, \mathcal{I}_2, \ldots$ of finitely generated ideals in $\mathcal{I}$. By (ACC) there is a finite n with $\mathcal{I}_n = \mathcal{I}$, and hence $\mathcal{I}$ is finitely generated. Conversely, suppose $\mathcal{R}$ is Noetherian. Given an ascending chain $\mathcal{I}_1 \subset \mathcal{I}_2 \subset \ldots$ of ideals in $\mathcal{R}$, set $\mathcal{I} = \bigcup_i \mathcal{I}_i$. Then $\mathcal{I}$ is an ideal in $\mathcal{R}$ and hence has a finite set of generators $f_1, \ldots, f_d$ by the Noetherian property. But there exists an index p so that $f_j \in \mathcal{I}_p$ for all j. Thus $\mathcal{I} = \mathcal{I}_p$, so $\mathcal{I}_j = \mathcal{I}_p$ for all $j \geq p$. Hence $\mathcal{R}$ satisfies (ACC).

We now prove (N). Assume that $\mathcal{R}$ is a Noetherian ring and let $\mathcal{I} \subset \mathcal{R}[x]$ be an ideal. Choose a nonzero polynomial $f_1(x) \in \mathcal{I}$ of minimum degree and form the ideal $\mathcal{I}_1 = \mathcal{R}[x]f_1(x) \subset \mathcal{I}$. If $\mathcal{I}_1 = \mathcal{I}$, we are done. If $\mathcal{I}_1 \neq \mathcal{I}$, take $f_2(x)$ as a nonzero polynomial of minimum degree among all elements of $\mathcal{I} \setminus \mathcal{I}_1$ and set $\mathcal{I}_2 = \mathcal{R}[x]f_1(x) + \mathcal{R}[x]f_2(x) \subset \mathcal{I}$. If $\mathcal{I} = \mathcal{I}_2$ we are done. Otherwise, we continue this process of choosing $f_j(x)$ and forming the ideals $\mathcal{I}_j = \mathcal{R}[x]f_1(x) + \cdots + \mathcal{R}[x]f_j(x)$. As long as $\mathcal{I}_j \neq \mathcal{I}$, we can choose $f_{j+1}(x)$ as a nonzero polynomial of minimum degree among all elements of $\mathcal{I} \setminus \mathcal{I}_j$. If $\mathcal{I}_d = \mathcal{I}$ for some d, then $f_1(x), \ldots, f_d(x)$ is a finite set of generators for $\mathcal{I}$ and we are done. We assume, for the sake of obtaining a contradiction, that $\mathcal{I}_j \neq \mathcal{I}$ for all j.

If $f(x) = c_m x^m + \cdots + c_1 x + c_0 \in \mathcal{R}[x]$ and $c_m \neq 0$, we call $c_m x^m$ the *initial term* and c_m the *initial coefficient* of $f(x)$. Let $a_j \in \mathcal{R}$ be the initial coefficient of $f_j(x)$ and let $\mathcal{J} \subset \mathcal{R}$ be the ideal generated by the set $\{a_1, a_2, \ldots\}$ of all initial coefficients. Since $\mathcal{R}$ is Noetherian, there is an integer m so that $\mathcal{J}$ is generated by $a_1, \ldots, a_m$. In particular, there are elements $u_j \in \mathcal{R}$ so that $a_{m+1} = \sum_{j=1}^m u_j a_j$. Let d_j be the degree of $f_j(x)$. By the choice of $f_j(x)$ we have $d_j \leq d_k$ for all $j < k$, since $f_k(x) \in \mathcal{I} \setminus \mathcal{I}_{j-1}$ and $f_j(x)$ has minimum degree among all nonzero elements in $\mathcal{I} \setminus \mathcal{I}_{j-1}$. Consider

$$g(x) = \textstyle\sum_{j=1}^m u_j f_j(x) x^{d_{m+1}-d_j} .$$

The initial coefficient of $g(x)$ is a_{m+1} and $g(x) \in \mathcal{I}_m$. Hence the polynomial $h(x) = f_{m+1}(x) - g(x)$ has strictly smaller degree than $f_{m+1}(x)$. Also, $h(x) \neq 0$ since $f_{m+1}(x) \notin \mathcal{I}_m$. But such a polynomial $h(x)$ cannot exist, by the choice of $f_{m+1}(x)$. This contradiction proves (N). ⧫

Suppose $\mathcal{A} \subset \mathbb{C}[x_1, \ldots, x_n]$ is any collection of polynomials. Let

$$X = \{x \in \mathbb{C}^n \,:\, f(x) = 0 \quad \text{for all } f \in \mathcal{A}\} ,$$

and let $\mathcal{I}_X \subset \mathbb{C}[x_1, \ldots, x_n]$ be the set of *all* polynomials that vanish on X. Then $\mathcal{A} \subset \mathcal{I}_X$ and $\mathcal{I}_X$ is an ideal. By the Hilbert basis theorem there are polynomials $f_1, \ldots, f_d$ that generate $\mathcal{I}_X$. Hence

$$X = \{x \in \mathbb{C}^n \,:\, f_j(x) = 0 \quad \text{for } j = 1, \ldots, d\} .$$

Thus it is no loss of generality to require that algebraic sets be defined by a finite number of polynomial equations.

Let $a \in X$. Then

$$\mathfrak{m}_a = \{f \in \mathcal{O}[X] \,:\, f(a) = 0\}$$

is a maximal ideal in $\mathcal{O}[X]$, since $f - f(a) \in \mathfrak{m}_a$ for all $f \in \mathcal{O}[X]$ and hence $\dim \mathcal{O}[X]/\mathfrak{m}_a = 1$. We will show that every maximal ideal is of this form. This basic result links algebraic properties of the ring $\mathcal{O}[X]$ to geometric properties of the algebraic set X.

Theorem A.1.3 (Hilbert Nullstellensatz)**.** *Let X be an affine algebraic set. If $\mathfrak{m}$ is a maximal ideal in $\mathcal{O}[X]$, then there is a unique point $a \in X$ such that $\mathfrak{m} = \mathfrak{m}_a$.*

Proof. Consider first the case $X = \mathbb{C}^n$. Define a representation ρ of the algebra $\mathcal{A} = \mathbb{C}[z_1, \ldots, z_n]$ on the vector space $V = \mathbb{C}[z_1, \ldots, z_n]/\mathfrak{m}$ by multiplication:

$$\rho(f)(g + \mathfrak{m}) = fg + \mathfrak{m} \quad \text{for } f, g \in \mathcal{A} .$$

The representation (ρ, V) is irreducible since $\mathfrak{m}$ is a maximal ideal. Since $\mathcal{A}$ is commutative, we have $\mathcal{A} \subset \mathrm{End}_{\mathcal{A}}(V)$. Hence Schur's lemma (Lemma 4.1.4) implies that $\rho(f)$ is a scalar multiple of the identity. In particular, there exist $a_i \in \mathbb{C}$ such that $z_i - a_i \in \mathfrak{m}$ for $i = 1, \ldots, n$. Set $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$. Then it follows that $f - f(a) \in \mathfrak{m}$ for all $f \in \mathbb{C}[z_1, \ldots, z_n]$. Hence $\mathfrak{m} = \mathfrak{m}_a$.

Now consider an arbitrary algebraic set $X \subset \mathbb{C}^n$ and maximal ideal $\mathfrak{m} \subset \mathcal{O}[X]$. Let $\mathfrak{m}'$ be the inverse image of $\mathfrak{m}$ under the canonical restriction map $f \mapsto f|_X$ from $\mathbb{C}[z_1, \ldots, z_n]$ onto $\mathcal{O}[X]$. Then the quotient rings $\mathcal{O}[X]/\mathfrak{m}$ and $\mathbb{C}[z_1, \ldots, z_n]/\mathfrak{m}'$ are naturally isomorphic. Since $\mathfrak{m}$ is a maximal ideal, these rings are fields, and hence $\mathfrak{m}'$ is a maximal ideal in $\mathbb{C}[z_1, \ldots, z_n]$. By the result just proved there exists $a \in \mathbb{C}^n$ so that $f - f(a) \in \mathfrak{m}'$ for all $f \in \mathbb{C}[z_1, \ldots, z_n]$. Hence $f|_X - f(a) \in \mathfrak{m}$ for all f. If $f \in \mathcal{I}_X$, then $f|_X = 0$, and so we have $f(a) \in \mathfrak{m}$. But $\mathfrak{m}$ is a proper ideal; thus we conclude that $f(a) = 0$ for all $f \in \mathcal{I}_X$. Therefore $a \in X$ and $\mathfrak{m} = \mathfrak{m}_a$. ♦

If A is an algebra with 1 over $\mathbb{C}$, then $\mathrm{Hom}(A, \mathbb{C})$ is the set of all linear maps $\varphi : A \longrightarrow \mathbb{C}$ such that $\varphi(1) = 1$ and $\varphi(a'a'') = \varphi(a')\varphi(a'')$ for all $a', a'' \in A$ (the *multiplicative linear functionals* on A). When X is an affine algebraic set and $A = \mathcal{O}[X]$, then every $x \in X$ defines a homomorphism φ_x by evaluation:

$$\varphi_x(f) = f(x) \quad \text{for } f \in \mathcal{O}[X]\,.$$

The coordinate functions from an ambient affine space separate the points of X, so the map $x \mapsto \varphi_x$ is injective.

Corollary A.1.4. *Let X be an affine algebraic set, and let $A = \mathcal{O}[X]$. The map $x \mapsto \varphi_x$ is a bijection between X and* $\mathrm{Hom}(A, \mathbb{C})$.

Proof. Let $\varphi \in \mathrm{Hom}(A, \mathbb{C})$. Then $\mathrm{Ker}(\varphi)$ is a maximal ideal of A, so by Theorem A.1.3 there exists $x \in X$ with $\mathrm{Ker}(\varphi) = \mathrm{Ker}(\varphi_x)$. Hence $0 = \varphi(f - f(x)) = \varphi(f) - f(x)$ for all $f \in A$, since $\varphi(1) = 1$. Thus $\varphi = \varphi_x$. ♦

Corollary A.1.5. *Let $f_1(x), \ldots, f_s(x)$ and $f(x)$ be polynomials on $\mathbb{C}^n$. Assume that $f(x) = 0$ whenever $f_i(x) = 0$ for $i = 1, \ldots, s$. Then there exists an integer r and polynomials $g_i(x)$ such that*

$$f(x)^r = \sum_{i=1}^{s} g_i(x) f_i(x)\,. \tag{A.2}$$

Proof. Let $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$. Introduce a new variable x_0 and the polynomial $f_0(x_0, x) = 1 - x_0 f(x)$. Take (x_0, x) as coordinates on $\mathbb{C}^{n+1}$, and view $f_1, \ldots, f_s$ as polynomials on $\mathbb{C}^{n+1}$ depending only on x. The assumption on $\{f_i\}$ and f implies that $\{f_0, f_1, \ldots, f_s\}$ have no common zeros on $\mathbb{C}^{n+1}$. Hence by Theorem A.1.3 the ideal generated by these functions contains 1. Thus there are polynomials $h_i(x_0, x)$ on $\mathbb{C}^{n+1}$ such that

$$1 = h_0(x_0, x)(1 - x_0 f(x)) + \sum_{i=1}^{s} h_i(x_0, x) f_i(x)\,. \tag{A.3}$$

Let r be the maximum degree of x_0 in the polynomials h_i. Substituting $x_0 = 1/f(x)$ in Equation (A.3) and multiplying by $f(x)^r$, we obtain Equation (A.2), with $g_i(x) = f(x)^r h_i(1/f(x), x)$. ♦

A.1.2 Zariski Topology

Let V be a finite-dimensional vector space over $\mathbb{C}$ and $X \subset V$ an algebraic subset.

Definition A.1.6. A subset $Y \subset X$ is *Zariski-closed* in X if Y is an algebraic subset of V. If $0 \neq f \in \mathcal{O}[X]$, then the *principal open subset* of X defined by f is $X^f = \{x \in X : f(x) \neq 0\}$.

Lemma A.1.7. *The Zariski-closed sets of X give X the structure of a topological space. The finite unions of principal open sets X^f, for $0 \neq f \in \mathcal{O}[X]$, are the nonempty open sets in this topology (the* Zariski topology*).*

Proof. We must check that finite unions and arbitrary intersections of algebraic sets are still algebraic. Suppose Y_1 is the zero set of polynomials $f_1, \ldots, f_r$ and Y_2 is the zero set of polynomials $g_1, \ldots, g_s$. Then $Y_1 \cup Y_2$ is the zero set of the functions $\{f_i g_j : 1 \leq i \leq r, 1 \leq j \leq s\}$ and is thus algebraic.

Given an arbitrary family $\{Y_\alpha : \alpha \in I\}$ of algebraic sets, their intersection is the zero set of a (possibly infinite) collection of polynomials $\{f_1, f_2, \ldots\}$. This intersection is still an algebraic set, however, by the Hilbert basis theorem. By definition, the complement of a proper algebraic subset of X is the union of finitely many sets of the form X^f. ⧫

Unless otherwise stated, we will use the term *closed set* in this appendix to refer to a *Zariski-closed set*. Let $x_1, \ldots, x_n$ be linear coordinate functions on V determined by a basis for V. Notice that a point $a = (a_1, \ldots, a_n) \in V$ is a closed set, since it is the zero set of the translated coordinate functions $\{x_i - a_i : 1 \leq i \leq n\}$. If $X \subset V$ is any set then $\overline{X}$ will denote the *closure* of X in the Zariski topology (the smallest closed set containing X).

Let V and W be finite-dimensional complex vector spaces. Suppose $X \subset V$ and $Y \subset W$ are algebraic sets and $f : X \longrightarrow Y$. If g is a complex-valued function on Y define $f^*(g)$ to be the function $f^*(g)(x) = g(f(x))$ for $x \in X$. We say that f is a *regular map* if $f^*(g)$ is in $\mathcal{O}[X]$ for all $g \in \mathcal{O}[Y]$. In terms of linear coordinates $x_1, \ldots, x_m$ for V and $y_1, \ldots, y_n$ for W, a regular map is given by the restriction to X of n polynomial functions $y_i = f_i(x_1, \ldots, x_m)$, for $i = 1, \ldots, n$. In particular, when $Y = \mathbb{C}$ this notion of regular map is consistent with our previous definition. It is clear from the definition that the composition of regular maps is regular.

Lemma A.1.8. *A regular map f between algebraic sets is continuous in the Zariski topology.*

Proof. Let $Z \subset Y$ be a closed set, defined by a set of polynomials $\{g_j\}$, say. Then $f^{-1}(Z)$ is the zero set of the polynomials $\{f^* g_j\}$, and hence is closed. ⧫

A.1.3 Products of Affine Sets

Let V, W be vector spaces and let $X \subset V$, $Y \subset W$ be affine algebraic sets. Then $X \times Y$ is an affine algebraic set in the vector space $V \oplus W$. To see this, let

$f_1, \ldots, f_m \in \mathcal{P}(V)$ be defining functions for X, and let $g_1, \ldots, g_n \in \mathcal{P}(W)$ be defining functions for Y. Extend f_i and g_i to polynomials on $V \times W$ by setting $f_i(v, w) = f_i(v)$ and $g_i(v, w) = g_i(w)$. Then $\{f_1, \ldots, f_m, g_1, \ldots, g_n\}$ is a set of defining functions for $X \times Y$.

By the universal property of tensor products relative to bilinear maps, there is a unique linear map $\mu : \mathcal{P}(V) \otimes \mathcal{P}(W) \longrightarrow \mathcal{P}(V \oplus W)$ such that $\mu(f' \otimes f'')(v, w) = f'(v)f''(w)$. This map clearly preserves multiplication of functions.

Lemma A.1.9. *The map μ induces an isomorphism of commutative algebras*

$$\nu : \mathcal{O}[X] \otimes \mathcal{O}[Y] \longrightarrow \mathcal{O}[X \times Y] .$$

Proof. Since μ is a vector-space isomorphism (see Proposition C.1.4) and the functions in $\mathcal{O}[X \times Y]$ are the restrictions of polynomials on $V \oplus W$, it is clear that ν is surjective and preserves multiplication. We will show that ν is injective. Suppose

$$0 \neq f = \textstyle\sum_i f'_i \otimes f''_i \in \mathcal{O}[X] \otimes \mathcal{O}[Y] .$$

Here we may assume that $\{f''_i\}$ is linearly independent and $f'_1 \neq 0$. Choose $g'_i \in \mathcal{P}(V)$ and $g''_i \in \mathcal{P}(W)$ with $f'_i = g'_i|_X$ and $f''_i = g''_i|_Y$ and set

$$g(v, w) = \textstyle\sum_i g'_i(v) g''_i(w) .$$

Then $\nu(f) = g|_{X \times Y}$. Choose $x_0 \in X$ such that $f'_1(x_0) \neq 0$. Then by the linear independence of the functions $\{f''_i\}$, we have

$$\textstyle\sum_i f'_i(x_0)\, f''_i \neq 0 .$$

Hence the function $y \mapsto g(x_0, y)$ on Y is nonzero, proving that $\nu(f) \neq 0$. ♦

A.1.4 Principal Open Sets

Assume X is a Zariski-closed subset of a vector space V. Let $f \in \mathcal{O}[X]$ with $f \neq 0$. Define $\psi : X^f \longrightarrow V \times \mathbb{C}$ by $\psi(x) = (x,\ 1/f(x))$. This map is injective, and we use it to define the structure of an affine algebraic set on the principal open set X^f as follows: Assume X is defined by $f_1, \ldots, f_n \in \mathcal{P}(V)$. Choose $\tilde{f} \in \mathcal{P}(V)$ so that $\tilde{f}|_X = f$. Then

$$\psi(X^f) = \{(v, t) \in V \times \mathbb{C} \,:\, f_i(v) = 0 \text{ for all } i \text{ and } \tilde{f}(v)t - 1 = 0\} .$$

Thus $\psi(X^f)$ is an algebraic set. We define the ring of regular functions on X^f by pulling back the regular functions on $\psi(X^f)$:

$$\mathcal{O}[X^f] = \{\psi^*(g) \,:\, g \in \mathcal{P}(V \times \mathbb{C})\} .$$

On $\psi(X^f)$ the coordinate t has the same restriction as $1/\tilde{f}$. Hence we see that the regular functions on X^f are all of the form $g(x_1, \ldots, x_n,\ 1/\tilde{f})$ with $g(x_1, \ldots, x_n, t)$ a polynomial in $n + 1$ variables. In particular, $1/f$ is a regular function on X^f. Note that $1/f$ is not the restriction to X^f of a polynomial on V unless f is constant.

A.1.5 Irreducible Components

Let $X \subset V$ be a nonempty closed set. We say that X is *reducible* if there are nonempty closed subsets $X_i \neq X$, $i = 1, 2$ such that $X = X_1 \cup X_2$. We say that X is *irreducible* if it is not reducible.

Lemma A.1.10. *An algebraic set X is irreducible if and only if $\mathcal{I}_X$ is a prime ideal ($\mathcal{O}[X]$ has no zero divisors).*

Proof. Suppose X is reducible. There are polynomials $f_i \in \mathcal{I}_{X_i}$ such that f_1 does not vanish on X_2 and f_2 does not vanish on X_1. Hence $f_i \notin \mathcal{I}_X$ but $f_1 f_2 \in \mathcal{I}_X$. Thus $\mathcal{I}_X$ is not a prime ideal.

Conversely, if $\mathcal{I}_X$ is not prime, then there exist f_1 and f_2 in $\mathcal{O}[X]$ with $f_1 f_2$ vanishing on X but f_i not vanishing on X. Set $X_i = \{x \in X : f_i(x) = 0\}$. Then $X = X_1 \cup X_2$ and $X \neq X_i$. Hence X is reducible. ⧫

We shall have frequent use for the following density property of irreducible algebraic sets:

Lemma A.1.11. *Let X be an irreducible algebraic set. Every nonempty open subset Y of X is dense in X. Furthermore, if $Y \subset X$ and $Z \subset X$ are nonempty open subsets, then $Y \cap Z$ is nonempty.*

Proof. By assumption, $X \setminus Y$ is a proper closed subset of X. Since $X = (X \setminus Y) \cup \overline{Y}$, the irreducibility of X implies that $\overline{Y} = X$, where $\overline{Y}$ is the closure of Y (in X) in the Zariski topology.

Let Y and Z be nonempty open subsets. If $Y \cap Z = \emptyset$, then $X = Y^c \cup Z^c$ and $Y^c \neq X$, $Z^c \neq X$. This contradicts the irreducibility of X. ⧫

Lemma A.1.12. *If X is any algebraic set, then there exists a finite collection of irreducible closed sets X_i such that*

$$X = X_1 \cup \cdots \cup X_r \text{ and } X_i \not\subset X_j \text{ for } i \neq j\,. \tag{A.4}$$

Furthermore, such a decomposition (A.4) *is unique up to a permutation of the indices and is called an* incontractible decomposition *of X. The sets X_i are called the* irreducible components *of X.*

Proof. Suppose the lemma is false for some X. Then there must be a decomposition $X = X_1 \cup X_1'$ into closed sets with $X \supset X_1$ properly and the lemma false for X_1. Continuing, we get an infinite strictly decreasing chain $X \supset X_1 \supset X_2 \supset \cdots$ of closed subsets. But this gives an infinite strictly increasing chain $\mathcal{I}_X \subset \mathcal{I}_{X_1} \subset \cdots$ of ideals, which contradicts the ascending chain condition for ideals in $\mathcal{P}(V)$.

By a deletion process, any decomposition of X as a finite union of closed subsets can be written in the form (A.4) with no proper containments among the X_i. Suppose $X = X_1' \cup \cdots \cup X_s'$ is another incontractible decomposition. Then for each index i

$$X_i' = X \cap X_i' = \textstyle\bigcup_{j=1}^{r} X_j \cap X_i'\,.$$

Since X_i' is irreducible, this decomposition has only one nonempty intersection. Thus there exists an index j such that $X_i' \subset X_j$. Similarly, for each index j there exists an index k such that $X_j \subset X_k'$. Hence $i = k$ and $X_i' = X_j$. So $r = s$ and there is a permutation σ such that $X_i' = X_{\sigma(i)}$. ⧫

Lemma A.1.13. *If X is an irreducible algebraic set then so is X^f for any nonzero function $f \in \mathcal{O}[X]$.*

Proof. Let X be a Zariski-closed subset of $\mathbb{C}^n$. Suppose $u, v \in \mathcal{O}[X^f]$ with $u \neq 0$ and $uv = 0$. There are polynomials g, h in $n+1$ variables such that $u(x) = g(x, 1/f(x))$ and $v(x) = h(x, 1/f(x))$, for $x = (x_1, \ldots, x_n) \in X^f$. Hence there is an integer k sufficiently large so that $f^k u$ and $f^k v$ are the restrictions to X^f of polynomials $\tilde{u}$ and $\tilde{v}$, respectively. But

$$f(x)\tilde{u}(x)\tilde{v}(x) = f(x)^{2k+1}u(x)v(x) = 0 \quad \text{for } x \in X^f,$$

and obviously $f(x)\tilde{u}(x)\tilde{v}(x) = 0$ if $f(x) = 0$. Since $\mathcal{O}[X]$ is an integral domain and $\tilde{u} \neq 0$, we conclude that $\tilde{v} = 0$. Thus if $x \in X^f$, then $v(x) = \tilde{v}(x)/f(x)^k = 0$. This proves that $\mathcal{O}[X^f]$ is an integral domain. ⧫

Lemma A.1.14. *Let V and W be finite-dimensional vector spaces. Suppose $X \subset V$ and $Y \subset W$ are irreducible algebraic sets. Then $X \times Y$ is an irreducible algebraic set in $V \oplus W$.*

Proof. We already verified that $X \times Y$ is algebraic. Suppose there are closed subsets Z_1, Z_2 in $V \oplus W$ such that $X \times Y = Z_1 \cup Z_2$. For each $x \in X$ the set $\{x\} \times Y$ is irreducible, since any decomposition of it into proper closed sets would give a decomposition of Y into proper closed sets. Hence for each $x \in X$ either $\{x\} \times Y \subset Z_1$ or else $\{x\} \times Y \subset Z_2$. This induces a decomposition $X = X_1 \cup X_2$, where

$$X_i = \{x \in X \,:\, \{x\} \times Y \subset Z_i\}\,.$$

We claim that each subset X_i is closed. Indeed, let Z_i be the zero set of the functions $\{f_\alpha\} \subset \mathcal{P}(V \oplus W)$ and define

$$X_y^{(i)} = \{x \in X \,:\, f_\alpha(x, y) = 0 \text{ for all } \alpha\}$$

for $y \in Y$. Then $X_y^{(i)}$ is closed in X and $X_i = \bigcap_{y \in Y} X_y^{(i)}$, which shows that X_i is closed. From the irreducibility of X it now follows that either $X = X_1$ or else $X = X_2$. Hence either $X \times Y = Z_1$ or else $X \times Y = Z_2$. ⧫

Lemma A.1.15. *Suppose $f : X \longrightarrow Y$ is a regular map between affine algebraic sets. Suppose X is irreducible. Then $\overline{f(X)}$ is irreducible.*

Proof. Suppose $g, h \in \mathcal{O}[Y]$ and $g(f(x))h(f(x)) = 0$ for all $x \in X$. Then since $\mathcal{O}[X]$ is an integral domain, either $f^*g = 0$ or $f^*h = 0$. Hence either g or h vanishes on $\overline{f(X)}$. This proves that $\mathcal{O}[\overline{f(X)}]$ is an integral domain, and hence $\overline{f(X)}$ is irreducible. ⧫

A.1.6 Transcendence Degree and Dimension

Let X be an irreducible affine algebraic set. The algebra $\mathcal{O}[X]$ is finitely generated over $\mathbb{C}$ and has no zero divisors. The following result (the *Noether Normalization Lemma*) describes the structure of such algebras:

Lemma A.1.16. *Let k be a field and $B = k[x_1, \ldots, x_n]$ a finitely generated commutative algebra over k without zero divisors. Then there exist $y_1, \ldots, y_r \in B$ such that*

(1) *the set $\{y_1, \ldots, y_r\}$ is algebraically independent over k;*

(2) *every $b \in B$ is integral over the subring $k[y_1, \ldots, y_r]$.*

The integer r is uniquely determined by properties (1) *and* (2) *and is called the* transcendence degree *of B over k. A set $\{y_1, \ldots, y_r\}$ with properties* (1) *and* (2) *is called a* transcendence basis *for B over k.*

Proof. If $\{x_1, \ldots, x_n\}$ is algebraically independent over k, then we can set $y_i = x_i$. Otherwise, there is a nontrivial relation

$$\textstyle\sum_J a_J\, x^J = 0\,, \tag{A.5}$$

By relabeling the generators x_i if necessary, we may assume that x_1 occurs in (A.5) with a nonzero coefficient. Fix an integer d with $d > j_k$ for all $J = (j_1, \ldots, j_n)$ such that $a_J \neq 0$. Set $M = (1, d, d^2, \ldots, d^{n-1}) \in \mathbb{N}^n$ and introduce the variables

$$y_2 = x_2 - x_1^d, \quad y_3 = x_3 - x_1^{d^2}, \ldots, \quad y_n = x_n - x_1^{d^{n-1}}. \tag{A.6}$$

Then the monomials x^J become

$$x^J = x_1^{J\cdot M} + \sum_{k<J\cdot M} f_k(y_2, \ldots, y_n) x_1^k\,,$$

where $J \cdot M = j_1 + j_2 d + \cdots + j_n d^{n-1}$ and f_k is a polynomial. By the choice of d we see that distinct indices J with $a_J \neq 0$ give distinct values of $J \cdot M$. Thus relation (A.5) can be written as

$$\textstyle\sum_J\, a_J\, x_1^{J\cdot M} \,+\, f(x_1, y_2, \ldots, y_n) = 0\,, \tag{A.7}$$

with f a polynomial whose degree in x_1 is less than the degree of the first summation in (A.7). Dividing (A.7) by the coefficient $a_J \neq 0$ for which $J \cdot M$ is the largest, we see that x_1 satisfies a monic polynomial with coefficients in $k[y_2, \ldots, y_n]$. Hence by (A.6) $x_2, \ldots, x_n$ also satisfy monic polynomials with coefficients in $k[y_2, \ldots, y_n]$. Thus B is integral over $k[y_2, \ldots, y_n]$.

If $\{y_2, \ldots, y_n\}$ is algebraically independent, then we are done. Otherwise, we repeat the procedure above until we arrive at an algebraically independent set. The number r is the cardinality of any maximal algebraically independent set in B (see Lang [1971, Chap. X, §1, Theorem 1]); hence it is uniquely determined by B. ♦

Definition A.1.17. Let X be an affine algebraic set. When X is irreducible, $\dim X$ is the transcendence degree of the algebra $\mathcal{O}[X]$. If X is reducible, then $\dim X$ is the maximum of the dimensions of the irreducible components of X.

With this notion of dimension available, we can obtain a very useful *ascending chain property* for algebraic sets.

Theorem A.1.18.

(1) *Let M, N be irreducible affine algebraic sets, such that $M \subseteq N$. Then* $\dim M \leq \dim N$. *Furthermore, if* $\dim M = \dim N$ *then* $M = N$.

(2) *Let $X_1 \subset X_2 \subset \cdots$ be an increasing chain of irreducible affine algebraic subsets of an algebraic set X. Then there exists an index p so that $X_j = X_p$ for $j \geq p$.*

Proof. (1): The Noether normalization lemma implies that $\dim M \leq \dim N$. Suppose that the dimensions are equal. To show $M = N$, it suffices to show that the restriction homomorphism $\sigma : \mathcal{O}[N] \longrightarrow \mathcal{O}[M]$ is injective (cf. Corollary A.1.4).

Suppose $u \in \mathrm{Ker}(\sigma)$. Take a transcendence basis $S = \{f_1, \ldots, f_r\}$ for $\mathcal{O}[M]$, where $r = \dim M$. Since $M \subseteq$, the map σ is surjective. Thus there exist $\bar{f}_i \in \mathcal{O}[N]$ such that $\sigma(\bar{f}_i) = f_i$ for $i = 1, \ldots, r$. Clearly, $\bar{S} = \{\bar{f}_1, \ldots, \bar{f}_r\}$ is algebraically independent. Since $\dim N = \dim M = r$, it follows that $\bar{S}$ is a transcendence basis for $\mathcal{O}[N]$. Thus by the Noether normalization lemma there are $b_i \in \mathcal{O}[\bar{f}_1, \ldots, \bar{f}_r]$ such that

$$u^n + b_{n-1}u^{n-1} + \cdots + b_1 u + b_0 = 0.$$

Choose n minimal. Applying the homomorphism σ, we get $\sigma(b_0) = 0$. But this implies that $b_0 = 0$, by the algebraic independence of S. Thus $n = 1$ and $u = 0$.

(2): By (1) we have $\dim X_j \leq \dim X_{j+1} \leq \dim X$. Hence there exists an index p such that $\dim X_j = \dim X_p$ for $j \geq p$. Then $X_j = X_p$ for $j \geq p$ by the last part of (1). ♦

A.1.7 Exercises

1. Prove that the ring of polynomials $k[X]$ in one variable is integrally closed in its quotient field $k(X)$ for any coefficient field k. (HINT: Use the unique factorization property of $k[X]$.)

2. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a bijective map that is not a polynomial. Show that f is continuous in the Zariski topology but that it is not a regular map.

3. For any subset $X \subset \mathbb{C}^n$ let $\mathcal{I}_X$ be the ideal of all polynomials vanishing on X. For any ideal $\mathcal{J} \subset \mathcal{P}(\mathbb{C}^n)$ let $V(\mathcal{J}) = \{x \in \mathbb{C}^n : f(x) = 0 \text{ for all } f \in \mathcal{J}\}$ be the zero set of $\mathcal{J}$.

   (a) Show that the Zariski closure of X is $V(\mathcal{I}_X)$.

   (b) If $\mathcal{I}, \mathcal{J}$ are ideals in $\mathcal{P}(\mathbb{C}^n)$, show that $V(\mathcal{I}\mathcal{J}) = V(\mathcal{I} \cap \mathcal{J})$.

4. Suppose f is an irreducible polynomial (this means that f cannot be factored as a product of polynomials of strictly smaller degree). Show that the zero set of f is irreducible.

5. Let $f_i(t) = g_i(t)/h_i(t)$ be rational functions on $\mathbb{C}^n$ for $i = 1, \ldots, r$. Let $X \subset \mathbb{C}^r$ be the Zariski closure of the set

$$\{[f_1(t), \ldots, f_r(t)] : t \in \mathbb{C}^n \text{ and } h_1(t) \cdots h_r(t) \neq 0\}.$$

   Prove that V is irreducible. (HINT: Show that the ideal $\mathcal{I}_X$ is prime.)

6. Let $X = \{(x_1, x_2) \in \mathbb{C}^2 : x_1 x_2 = 0\}$.

   (a) Show that the irreducible components of X are $X_1 = \{(z, 0) : z \in \mathbb{C}\}$ and $X_2 = \{(0, z) : z \in \mathbb{C}^2\}$ and hence that X is reducible.

   (b) Show that X is connected (in the Zariski topology). (HINT: Suppose $X = U \cup V$ with U and V Zariski-open, $U \cap V = \emptyset$, and $U \neq \emptyset$. Use (a) to argue that $U \cap X_i = X_i$ for $i = 1, 2$ and hence $V = \emptyset$.)

7. (a) Show that the Zariski-closed subsets of $\mathbb{C}$ are precisely the finite sets.

   (b) Show that the Zariski topology on $\mathbb{C} \times \mathbb{C}$ is *not* the product of the Zariski topologies on each factor.

A.2 Maps of Algebraic Sets

A.2.1 Rational Maps

Let A be a commutative ring with 1 and without zero divisors. Then A is embedded in its *quotient field* $\mathrm{Quot}(A)$. The elements of this field are the formal expressions $f = g/h$, where $g, h \in A$ and $h \neq 0$, with the usual algebraic operations on fractions. For example, let X be an irreducible algebraic set. The algebra $A = \mathcal{O}[X]$ has no zero divisors, so it has a quotient field. We denote this field by $\mathrm{Rat}(X)$ and call it the field of *rational functions* on X.

We may view the elements of $\mathrm{Rat}(X)$ as functions, as follows: If $f \in \mathrm{Rat}(X)$, then we say that f is *defined at a point* $x \in X$ if there exist $g, h \in \mathcal{O}[X]$ with $f = g/h$ and $h(x) \neq 0$. In this case we set $f(x) = g(x)/h(x)$. The *domain* $\mathcal{D}_f$ of f is the subset of X at which f is defined. It is a dense open subset of X, since it contains the principal open set X^h.

A map f from X to an algebraic set Y is called *rational* if $f^*(\varphi)$ is a rational function on X for all $\varphi \in \mathcal{O}[Y]$. Suppose $Y \subset \mathbb{C}^n$ and y_i is the restriction to Y of the linear coordinate function x_i on $\mathbb{C}^n$. Set $f_i = f^*(y_i)$. Then f is rational if and only if $f_i \in \mathrm{Rat}(X)$ for $i = 1, \ldots, n$. The domain of a rational map f is defined as

$$\mathcal{D}_f = \bigcap_{\varphi \in \mathcal{O}[Y]} \mathcal{D}_{f^*(\varphi)}\,.$$

By Lemma A.1.11 we know that $\mathcal{D}_f = \bigcap_{i=1}^n \mathcal{D}_{f^*(y_i)}$ is a dense open subset of X.

Lemma A.2.1. *Suppose X is irreducible and $f : X \longrightarrow Y$ is a rational map. If $\mathcal{D}_f = X$ then f is a regular map.*

Proof. Let $\varphi \in \mathcal{O}[Y]$. Let $\mathcal{I} \subset \mathcal{O}[X]$ be the set of all functions h such that the function $x \mapsto h(x)\varphi(f(x))$ is regular on X. Then $\mathcal{I}$ is a nonzero ideal in $\mathcal{O}[X]$, since f is a rational map. Suppose $\mathcal{I}$ is a proper ideal. Then it is contained in some maximal ideal. In this case Theorem A.1.3 implies that all $h \in \mathcal{I}$ vanish at some point $x_0 \in X$. Thus $x_0 \notin \mathcal{D}_f$, a contradiction. Hence $1 \in \mathcal{I}$ and $f^*(\varphi) \in \mathcal{O}[X]$ for all $\varphi \in \mathcal{O}[Y]$. ♦

A.2.2 Extensions of Homomorphisms

The points of an affine algebraic set X correspond to the homomorphisms of the algebra $\mathcal{O}[X]$ into the base field $\mathbb{C}$. We will use this correspondence to obtain the key results concerning regular maps. The translation from geometric to algebraic language leads us to two main problems: When do homomorphisms of an algebra extend to homomorphisms of a larger algebra? If an extension exists, when is it unique? We shall solve these algebraic problems in this section and give the geometric applications in later sections.

We will need some properties of ring extensions. Let $A \subset B$ be commutative rings with 1, and assume that B has no zero divisors. An element $b \in B$ is said to be *integral over* A if b satisfies a monic polynomial

$$b^n + a_1 b^{n-1} + \cdots + a_{n-1} b + a_n = 0 \tag{A.8}$$

with coefficients $a_i \in A$.

Lemma A.2.2. *An element $b \in B$ is integral over A if and only if there exists a finitely generated A-submodule $C \subset B$ such that $b \cdot C \subset C$.*

Proof. Let b satisfy (A.8). Then $A[b] = A \cdot 1 + A \cdot b + \cdots + A \cdot b^{n-1}$ is a finitely-generated A-submodule, so we may take $C = A[b]$. Conversely, suppose C exists as stated and is generated by nonzero elements $\{x_1, \ldots, x_n\}$ as an A-module. Since $bx_i \in C$, there are elements $a_{ij} \in A$ so that

$$bx_i - \textstyle\sum_{j=1}^n a_{ij}\, x_j = 0 \quad \text{for } i = 1, \ldots, n\,.$$

Since B has no zero divisors, this system of equations implies that $\det[b\delta_{ij} - a_{ij}] = 0$. This determinant is a monic polynomial in b, with coefficients in A. Hence b is integral over A. ♦

Let A be an algebra with 1 over $\mathbb{C}$. Given $0 \neq a \in A$, we set

$$\mathrm{Hom}(A, \mathbb{C})^a = \{\varphi \in \mathrm{Hom}(A, \mathbb{C}) \,:\, \varphi(a) \neq 0\}.$$

For example, if $A = \mathcal{O}[X]$ for some affine algebraic set X, then $\mathrm{Hom}(A, \mathbb{C})$ is naturally identified with X, by Corollary A.1.4, and $\mathrm{Hom}(A, \mathbb{C})^a$ corresponds to the principal open set X^a.

Theorem A.2.3. *Let B be a commutative algebra over $\mathbb{C}$ with 1 and no zero divisors. Suppose that $A \subset B$ is a subalgebra such that $B = A[b_1, \ldots, b_n]$ for some elements $b_i \in B$. Then given $0 \neq b \in B$, there exists $0 \neq a \in A$ such that every $\varphi \in \mathrm{Hom}(A, \mathbb{C})^a$ extends to $\psi \in \mathrm{Hom}(B, \mathbb{C})^b$. If B is integral over A and $b = 1$, then one may take $a = 1$.*

Proof. We start with the case $B = A[u]$ for some element $u \in B$. Let $b = f(u)$, where $f(X) = a_n X^n + \cdots + a_0$ with $a_i \in A$. For $g(X) \in A[X]$, denote by $g^\varphi(X) \in \mathcal{O}[X]$ the polynomial obtained by applying φ to the coefficients of $g(X)$. If u is transcendental over A, then for any $\lambda \in \mathbb{C}$ we can define an extension ψ of φ by

$$\psi(g(u)) = g^\varphi(\lambda) \quad \text{for } g \in A[X]. \tag{A.9}$$

This extension satisfies $\psi(b) = f^\varphi(\lambda)$. Since $b \neq 0$, the polynomial $f(X) \neq 0$. Take for $a \in A$ any nonzero coefficient of $f(X)$. If we assume that $\varphi(a) \neq 0$, then $f^\varphi(X) \neq 0$ and any $\lambda \in \mathbb{C}$ with $f^\varphi(\lambda) \neq 0$ will serve to define the desired extension of φ.

Now assume that u is algebraic over A. Then b is also algebraic over A, so there are nonzero polynomials $p(X) = a_m X^m + \cdots + a_0$ and $q(X) = c_n X^n + \cdots + c_0$ (where a_i and c_i in A), whose degrees are minimal and which satisfy $p(u) = 0$ and $q(b) = 0$. We have $a_m \neq 0$ and $c_0 \neq 0$ (otherwise, the polynomial $q(X)/X$ would annihilate b). We will prove that the element $a = a_m c_0$ has the desired property. Note that if u is integral over A and $b = 1$ then $a = 1$.

We first observe that for any $g(X) \in A[X]$ there is an integer $d \geq 0$ and polynomials $h(X), r(X) \in A[X]$ that satisfy $\deg r(X) < m$ and

$$(a_m)^d g(X) = p(X)h(X) + r(X)$$

(by the Euclidean division algorithm). In particular, if $g(u) = 0$ then $r(X) = 0$, since $r(u) = 0$ and $\deg r(X) < m$. Thus $(a_m)^d g(X)$ is divisible by $p(X)$ in $A[X]$. Suppose $\varphi \in \mathrm{Hom}(A, \mathbb{C})^a$ and λ is a root of $p^\varphi(X)$. We have $\varphi(a_m)^d g^\varphi(\lambda) = 0$. But $\varphi(a_m) = \varphi(a)/\varphi(c_0) \neq 0$. Hence $g^\varphi(\lambda) = 0$, and so formula (A.9) determines an extension $\psi \in \mathrm{Hom}(B, \mathbb{C})$ of φ. For this extension

$$0 = \psi(q(b)) = q^\varphi(\psi(b))\,.$$

However, 0 is not a root of $q^\varphi(X)$ since $\varphi(c_0) \neq 0$. Thus $\psi(b) \neq 0$, and the theorem is proved when B has a single generator over A.

Let $n \geq 1$. We assume the theorem is true for all algebras A and all algebras B with $n-1$ generators over A. We shall show that it is true for the case of n generators. Let $B = A[b_1, \ldots, b_n]$ and let $0 \neq b \in B$ be given. Set $\tilde{A} = A[b_1, \ldots, b_{n-1}]$ and $u = b_n$. Then $B = \tilde{A}[u]$. By the proof just given there exists $\tilde{a} \in \tilde{A}$ so that every complex homomorphism $\tilde{\varphi}$ of $\tilde{A}$ satisfying $\tilde{\varphi}(\tilde{a}) \neq 0$ extends to a complex homomorphism ψ of B with $\psi(b) \neq 0$. Now we invoke the induction hypothesis, with B replaced by $\tilde{A}$ and b replaced by $\tilde{a}$. This gives a nonzero element $a \in A$ so that every complex homomorphism φ of A satisfying $\varphi(a) \neq 0$ extends to a

complex homomorphism $\tilde{\varphi}$ of $\tilde{A}$ with $\tilde{\varphi}(\tilde{a}) \neq 0$. We complete the induction step by combining these two extension processes. ♦

Corollary A.2.4. *Let B be a finitely generated commutative algebra over $\mathbb{C}$ having no zero divisors. Given $0 \neq b \in B$, there exists $\psi \in \mathrm{Hom}(B, \mathbb{C})$ such that $\psi(b) \neq 0$.*

Proof. In Theorem A.2.3 take $A = \mathbb{C}$ and $\varphi(\lambda) = \lambda$ for $\lambda \in \mathbb{C}$. ♦

Next we consider the uniqueness of the extensions in Theorem A.2.3. Let $A \subset B$ be a subalgebra, and identify $\mathrm{Quot}(A)$ with the subfield of $\mathrm{Quot}(B)$ generated by A.

For example, if $A = \mathcal{O}[X]$ for an irreducible variety X, and $B = \mathcal{O}[X^f]$ for some nonzero $f \in A$, then $B = A[b] \subset \mathrm{Quot}(A)$, where $b = 1/f$. In this example, every $\psi \in \mathrm{Hom}(B, \mathbb{C})$ such that $\psi(b) \neq 0$ is given by evaluation at a point $x \in X^f$, and hence ψ is uniquely determined by its restriction to A. The unique restriction property in this example characterizes the general case in which $B \subset \mathrm{Quot}(A)$, as follows:

Theorem A.2.5. *Let B be a finitely generated algebra over $\mathbb{C}$ with no zero divisors. Let $A \subset B$ be a finitely generated subalgebra. Assume that there exists a nonzero element $b \in B$ so that every element of $\mathrm{Hom}(B, \mathbb{C})^b$ is uniquely determined by its restriction to A. Then $B \subset \mathrm{Quot}(A)$.*

Proof. As in Theorem A.2.3, it suffices to consider the case $B = A[u]$ for some element $u \in B$. Let $b = f(u)$, where $f(X) = a_n X^n + \cdots + a_0$ with $a_i \in A$. We first show that if u is not algebraic over A, then the unique extension property does not hold. Indeed, in this case every $\lambda \in \mathbb{C}$ determines an extension $\psi \in \mathrm{Hom}(B, \mathbb{C})$ of $\varphi \in \mathrm{Hom}(A, \mathbb{C})$ by (A.9). The condition $\psi(b) \neq 0$ implies that $f^{\varphi}(X) \neq 0$, so there exist infinitely many choices of λ such that $\psi(b) \neq 0$, as in the proof of Theorem A.2.3. Hence ψ is not uniquely determined by φ.

Now assume that u is algebraic over A. Let $p(X)$ and $q(X)$ be as in the proof of Theorem A.2.3. We shall prove that $m = \deg(p) = 1$. This will imply the theorem, since then $u = -a_0/a_1$. Suppose to the contrary that $m > 1$. Set $K = \mathrm{Quot}(A)$ and define

$$r(X) = a_m \left(X + \frac{a_{m-1}}{m a_m} \right)^m = a_m X^m + a_{m-1} X^{m-1} + \sum_{j=0}^{m-2} d_j X^j$$

in $K[X]$. Then $p(X) \neq r(X)$, since $p(X)$ is irreducible over A. Hence there exists some $j_0 \leq m-2$ such that $a_{j_0} \neq d_{j_0}$. Set

$$a = (a_m)^{m-1}(a_{j_0} - d_{j_0})c_0 \,.$$

Then $0 \neq a \in A$, and so by Corollary A.2.4 there exists $\varphi \in \mathrm{Hom}(A, \mathbb{C})$ such that $\varphi(a) \neq 0$. Set $\alpha = \varphi(a_m) \neq 0$ and $\beta = \varphi(a_{m-1})/(m\alpha)$. Then $r^{\varphi}(X) = \alpha(X + \beta)^m$, while the condition $\varphi(a_{j_0}) \neq \varphi(d_{j_0})$ implies that

$$p^{\varphi}(X) \neq \alpha(X + \beta)^m \,. \tag{A.10}$$

We claim that if $m \geq 2$, then $p^\varphi(X)$ must have at least two distinct roots. For if not, then $p^\varphi(X)$ would be the mth power of a linear polynomial and hence would have to be $\alpha(X+\beta)^m$, contradicting (A.10).

As in the proof of Theorem A.2.3, each root λ of $p^\varphi(X)$ determines an extension $\psi \in \mathrm{Hom}(B, \mathbb{C})$ of φ by (A.9). In particular, $\psi(u) = \lambda$, so distinct roots of $p^\varphi(X)$ determine distinct extensions of φ. Also, $0 = \psi(q(b)) = q^\varphi(\psi(b))$. But 0 is not a root of $q^\varphi(X)$, since $\varphi(c_0) \neq 0$. Thus $\psi(b) \neq 0$. So we conclude that if $m \geq 2$, then there exists $\psi_1, \psi_2 \in \mathrm{Hom}(B, \mathbb{C})$ with $\psi_1(u) \neq \psi_2(u)$, $\psi_i(b) \neq 0$, but $\psi_1 = \psi_2$ on A, contradicting the unique restriction assumption. ♦

A.2.3 Image of a Dominant Map

Let X and Y be affine algebraic sets and let $f : X \longrightarrow Y$ be a regular map.

Definition A.2.6. The map f is *dominant* if $f(X)$ is dense in Y.

The definition of dominance is equivalent to the property that $f^* : \mathcal{O}[Y] \longrightarrow \mathcal{O}[X]$ is injective. Dominant maps have the following remarkable property (which does not hold for smooth maps of differentiable manifolds):

Theorem A.2.7. *Assume that X and Y are irreducible affine algebraic sets and $f : X \longrightarrow Y$ is a dominant map. Let $M \subset X$ be a nonempty open set. Then $f(M)$ contains a nonempty open subset of Y.*

Proof. Set $B = \mathcal{O}[X]$ and $A = f^*(\mathcal{O}[Y])$. Since X is irreducible, it follows that B has no zero divisors. We may assume that $M = X^b$ for some $0 \neq b \in B$. Let $a \in A$ be as in Theorem A.2.3, and let $a = f^*(\bar{a})$, where $\bar{a} \in \mathcal{O}[Y]$. We claim that

$$f(X^b) \supset Y^{\bar{a}}. \tag{A.11}$$

To prove this, view the points of Y as the homomorphisms from A to $\mathbb{C}$. For $y \in Y^{\bar{a}}$, the corresponding homomorphism φ satisfies $\varphi(a) = \bar{a}(y) \neq 0$. Hence by Theorem A.2.3, there is an extension ψ of φ to B such that $\psi(b) \neq 0$. In geometric language, this means that ψ is given by evaluation at a point $x \in X^b$. The extension property means that $g(f(x)) = g(y)$ for all $g \in \mathcal{O}[Y]$. Since f is dominant, this implies that $f(x) = y$, proving (A.11). ♦

Theorem A.2.8. *Let $f : X \longrightarrow Y$ be a regular map between affine algebraic sets. Then $f(X)$ contains an open subset of $\overline{f(X)}$.*

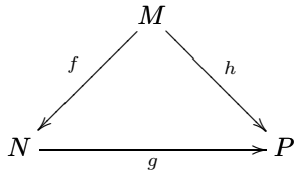
Proof. Let $X_1, \ldots, X_r$ be the irreducible components of X. Then

$$\overline{f(X)} = \overline{f(X_1)} \cup \cdots \cup \overline{f(X_r)}\,.$$

The theorem now follows from Theorem A.2.7. ♦

A.2.4 Factorization of a Regular Map

Let M, N, and P be irreducible affine varieties. Suppose we have regular maps $f : M \longrightarrow N$ and $h : M \longrightarrow P$. If there is a map g that satisfies the commutative diagram



(A.12)

then h is constant on the fibers of f. Indeed, $f(m) = f(m')$ implies that $h(m) = g(f(m)) = h(m')$. Furthermore, if f is surjective, then any such map g is uniquely determined by f and h.

We now weaken the fiber and surjectivity conditions with the aim of obtaining a rational map g that satifies the diagram (A.12) in the sense of rational maps.

Theorem A.2.9. *Assume that f and h are dominant and that there is a nonempty open subset U of M so that $f(m) = f(m')$ implies $h(m) = h(m')$ for $m, m' \in U$. Then there exists a rational map $g : N \longrightarrow P$ such that $h = g \circ f$.*

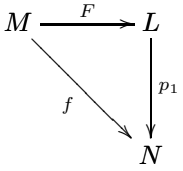
Proof. Consider first the case in which $M = P$ and h is the identity map. We may assume that $U = M^b$ for some $0 \neq b \in \mathcal{O}[M]$. We claim that the conditions of Theorem A.2.5 are satisfied by $A = f^*(\mathcal{O}[N])$ and $B = \mathcal{O}[M]$. Indeed, every homomorphism $\psi : B \longrightarrow \mathbb{C}$ such that $\psi(b) \neq 0$ is given by evaluation at some $x \in M^b$ (cf. Corollary A.1.4). Hence $\psi(f^*p) = p(f(x))$ for $p \in \mathcal{O}[N]$. Since $f(M)$ is dense in N and f is injective on M^b, it follows that x is uniquely determined by the homomorphism $p \mapsto p(f(x))$.

Let x_i, for $i = 1, \ldots, k$, be the coordinate functions defined by some embedding $M \subset \mathbb{C}^k$. Applying Theorem A.2.5, we obtain $q_i \in \mathrm{Rat}(N)$ so that $f^*(q_i) = x_i$ where defined. The rational map g defined by $g(n) = (q_1(n), \ldots, q_k(n))$ for $n \in N$ then satisfies $g \circ f(m) = m$ for m in a dense subset of M, as required.

We now reduce the general case to the one just treated. Let $F : M \longrightarrow N \times P$ with $F(m) = (f(m), h(m))$, and take the projection maps

$$\pi_1 : N \times P \longrightarrow N \quad \text{and} \quad \pi_2 : N \times P \longrightarrow P\,.$$

Let $L = \overline{F(M)}$ and let $p_i = \pi_i|_L$ for $i = 1, 2$. Then we have the commutative diagram



The set L is irreducible and p_1 is a dominant map that is injective on an open set. Hence there is a rational map $k : N \longrightarrow L$ such that $k \circ p_1$ is the identity map on a dense open set. We take $g = p_2 \circ k$ to obtain the desired map. ♦

A.2.5 Exercises

1. Let X and Y be affine algebraic sets and $\varphi : X \longrightarrow Y$ a regular map.
   (a) Show that the graph $\Gamma_\varphi = \{(x, \varphi(x)) : x \in X\}$ of φ is closed in $X \times Y$.
   (b) Show that the projection map $\pi : \Gamma_\varphi \longrightarrow X$ onto the first coordinate is an isomorphism of affine algebraic sets.
2. Let $\varphi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be the map $\varphi(x, y) = (xy, y)$.
   (a) Show that φ has dense image but is not surjective.
   (b) Show that x is not integral over $\mathbb{C}[xy, y]$.
3. Let $X = \{(x, y) \in \mathbb{C}^2 : y^2 = x^3\}$.
   (a) Show that every element of $\mathcal{O}[X]$ can be written uniquely in the form $P(x) + Q(x)y$, where P and Q are polynomials.
   (b) Let $\varphi : \mathbb{C} \longrightarrow X$ be the map $\varphi(t) = (t^2, t^3)$. Show that φ is bijective and regular, but φ^{-1} is not regular.

A.3 Tangent Spaces

A.3.1 Tangent Space and Differentials of Maps

Let $X \subset \mathbb{C}^n$ be an algebraic set. Let $\mathcal{O}[X]^*$ denote the linear maps from $\mathcal{O}[X]$ (viewed as a $\mathbb{C}$ vector space) to $\mathbb{C}$.

Definition A.3.1. A *tangent vector* at a point $x \in X$ is an element $v \in \mathcal{O}[X]^*$ such that

$$v(fg) = v(f)g(x) + f(x)v(g) \tag{A.13}$$

for all $f, g \in \mathcal{O}[X]$. The set of all tangent vectors at x the *tangent space* $T(X)_x$ of X at x.

Clearly $T(X)_x$ is a linear subspace of $\mathcal{O}[X]^*$. We observe that if v is a tangent vector at x, then $v(1) = v(1 \cdot 1) = 2v(1)$. Hence $v(1) = 0$, so v vanishes on the space of constant functions on X.

We now give an alternate description of the tangent space in terms of maximal ideals. Let $\mathfrak{m}_x \subset \mathcal{O}[X]$ be the maximal ideal of all functions that vanish at x. Then $f - f(x) \in \mathfrak{m}_x$ for any $f \in \mathcal{O}[X]$, and $v(f) = v(f - f(x))$. Hence v is determined by its restriction to $\mathfrak{m}_x$. However, by (A.13) we see that $v(\mathfrak{m}_x^2) = 0$, so v naturally defines an element $\tilde{v} \in (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$. Conversely, given any $\tilde{v} \in (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$, we define a linear functional v on $\mathcal{O}[X]$ by $v(f) = \tilde{v}(f - f(x))$. To verify that v satisfies (A.13), we observe that if $f, g \in \mathcal{O}[X]$, then $(f - f(x))(g - g(x)) \in \mathfrak{m}_x^2$. Hence

$$0 = v(fg) - f(x)v(g) - g(x)v(f),$$

since $v(c) = 0$ for $c \in \mathbb{C}$. This shows that there is a natural isomorphism

$$T(X)_x \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^*. \tag{A.14}$$

Let X and Y be algebraic sets and $\varphi : X \longrightarrow Y$ a regular map. Then the induced map $\varphi^* : \mathcal{O}[Y] \longrightarrow \mathcal{O}[X]$ is an algebra homomorphism. If $v \in T(X)_x$ then the linear functional $f \mapsto v(\varphi^*(f))$, for $f \in \mathcal{O}[Y]$, is a tangent vector at $y = \varphi(x)$. Indeed, for $f, g \in \mathcal{O}[Y]$ we have

$$v(\varphi^*(fg)) = v(\varphi^*(f)\varphi^*(g)) = v(\varphi^*(f))g(y) + f(y)v(\varphi^*(g)).$$

We denote this element of $T(Y)_y$ by $d\varphi_x(v)$. At each point $x \in X$ we thus have a linear map $d\varphi_x : T(X)_x \longrightarrow T(Y)_{\varphi(x)}$ that we call the *differential* of φ at x.

Examples

1. If $X = \mathbb{C}^n$, then $\mathcal{O}[X] = \mathcal{P}(\mathbb{C}^n) = \mathbb{C}[x_1, \ldots, x_n]$. For $u \in \mathbb{C}^n$ and $f \in \mathcal{P}(\mathbb{C}^n)$ define $D_u f(x) = (\partial/\partial t) f(x + tu)|_{t=0}$ (the *directional derivative* of f along the line $t \mapsto x + tu$, $t \in \mathbb{C}$). The linear functional $v(f) = D_u f(a)$ is a tangent vector at $a \in \mathbb{C}^n$. If $u = (u_1, \ldots, u_n)$, then we can express v in terms of the partial derivative operators $\partial/\partial x_i$:

$$v = \textstyle\sum_{i=1}^n u_i \left(\partial/\partial x_i\right)_a \ ,$$

where we write $(\partial/\partial x_i)_a$ for the tangent vector $f \mapsto (\partial f/\partial x_i)(a)$ with $a \in \mathbb{C}^n$.

We claim that $T(\mathbb{C}^n)_a$ has basis $(\partial/\partial x_1)_a, \ldots, (\partial/\partial x_n)_a$ and hence has dimension n. Indeed, let a have coordinates $(a_1, \ldots, a_n)$. Then it is clear by the Taylor expansion of a polynomial that the ideal $\mathfrak{m}_a$ is generated by the independent linear functions $x_i - a_i$, for $1 \le i \le n$. Thus $v \in T(\mathbb{C}^n)_a$ can be uniquely expressed as $v(f) = D_u f(a)$, where $u = (u_1, \ldots, u_n) \in \mathbb{C}^n$ and $u_i = v(x_i - a_i)$. When convenient, we will identify $T(\mathbb{C}^n)_a$ with $\mathbb{C}^n$ by the map $v \mapsto u$.

2. Suppose $X \subset \mathbb{C}^n$ is an algebraic set and $a \in X$. Since $\mathcal{O}[X] = \mathcal{P}(\mathbb{C}^n)/\mathcal{I}_X$, for every $v \in T(X)_a$ there exists $\tilde{v} \in T(\mathbb{C}^n)_a$ with $\tilde{v}(\mathcal{I}_X) = 0$ and $\tilde{v}(f) = v(f + \mathcal{I}_X)$. Conversely, any $\tilde{v}$ with this property induces a tangent vector v to X at a, with $v(f + \mathcal{I}_X) = \tilde{v}(f)$. Hence

$$T(X)_a = \{\tilde{v} \in T(\mathbb{C}^n)_a \ : \ \tilde{v}(\mathcal{I}_X) = 0\} \, .$$

Let $\{f_1, \ldots, f_r\}$ be a generating set of polynomials for the ideal $\mathcal{I}_X$. The defining equation for a derivation shows that $\tilde{v}(\mathcal{I}_X) = 0$ if and only if $\tilde{v}(f_i) = 0$ for $i = 1, \ldots, r$. Hence if we set $u_j = \tilde{v}(x_j - a_j)$ then $\tilde{v} \in T(X)_a$ if and only if

$$\sum_{j=1}^n u_j \frac{\partial f_i(a)}{\partial x_j} = 0 \quad \text{for } i = 1, \ldots, r \, . \tag{A.15}$$

This is a set of r linear equations for $u = (u_1, \ldots, u_n) \in \mathbb{C}^n$. In particular, we see that $\dim T(X)_a = n - \operatorname{rank}(J(a))$, where $J(a)$ is the $r \times n$ matrix $[\partial f_i(a)/\partial x_j]$ of partial derivatives.

3. Let X be an irreducible affine algebraic set. Define

$$m(X) = \min_{a \in X} \dim T(X)_a \, .$$

Let $X_0 = \{a \in X : \dim T(X)_a = m(X)\}$. The points of X_0 are called *smooth*. These are the points at which the matrix $J(a)$ defined above has maximum rank $d = n - m(X)$. Because this condition can be described by the nonvanishing of some $d \times d$ minor of $J(a)$, it follows that X_0 is Zariski-dense in X when X is irreducible. If $X_0 = X$ then X is said to be *smooth*.

4. If X is a reducible algebraic set with irreducible components X_i, then we say that X is smooth if each X_i is smooth. We define $m(X) = \max_i m(X_i)$ in this case.

A.3.2 Vector Fields

Recall that a *derivation* of an algebra A is a linear map $D : A \longrightarrow A$ such that $D(ab) = D(a)b + aD(b)$. If A is commutative and D and D' are derivations of A, then any linear combination of D and D' with coefficients in A is a derivation, and the *commutator* $[D, D'] = DD' - D'D$ is a derivation, as we check by the obvious calculation. Thus the derivations of a commutative algebra A form a Lie algebra $\mathrm{Der}(A)$, which is also an A-module (see Section 4.1.1). In the case $A = \mathcal{O}[X]$, where X is an affine algebraic set, a derivation of A is usually called a *vector field* on X. We denote by $\mathrm{Vect}(X)$ the Lie algebra of all vector fields on X.

Given $L \in \mathrm{Vect}(X)$ and $x \in X$, we define $L_x f = (Lf)(x)$ for $f \in \mathcal{O}[X]$. Then $L_x \in T(X)_x$, by the definition of tangent vector. Conversely, if we have a correspondence $x \mapsto L_x \in T(X)_x$ such that the functions $x \mapsto L_x(f)$ are regular for every $f \in \mathcal{O}[X]$, then L is a vector field on X.

Example

Let $0 \neq f \in \mathcal{O}[X]$ and consider a vector field L on the principal open set X^f. Recall that $\mathcal{O}[X^f]$ is generated by the restrictions to X^f of functions in $\mathcal{O}[X]$ together with f^{-1}. However, since L is a derivation, we have $L(f)f^{-1} + fL(f^{-1}) = L(ff^{-1}) = L(1) = 0$, so that $L(f^{-1}) = -f^{-2}L(f)$. Hence L is completely determined by its action on $\mathcal{O}[X]$. For example, if $X = \mathbb{C}^n$, then

$$L = \sum_{i=1}^{n} \varphi_i \frac{\partial}{\partial x_i} \quad \text{with } \varphi_i = L(x_i) \in \mathbb{C}[x_1, \ldots, x_n, 1/f]\,. \tag{A.16}$$

A.3.3 Dimension

We now show that the notion of dimension defined geometrically via the tangent space coincides with the algebraic definition in terms of transcendence degree.

Theorem A.3.2. *Let X be an algebraic set. Then $m(X) = \dim X$.*

Proof. If X has irreducible components X_i, then $m(X) = \max(m(X_i))$, so we may assume $X \subset \mathbb{C}^n$ is irreducible. Set $K = \mathrm{Rat}(X)$. We prove first that

$$\dim X = \dim_K \mathrm{Der}(K)\,. \tag{A.17}$$

For this, we apply the Noether normalization lemma to $B = \mathcal{O}[X]$ to obtain a transcendence basis $\{u_1, \ldots, u_d\}$, where $d = \dim X$. Let $A = \mathbb{C}[u_1, \ldots, u_d]$ and

$k = \mathrm{Quot}(A) \subset K$. A derivation D of k is determined by an arbitrary choice of d rational functions $Du_i \in k$, since $\{u_1, \ldots, u_d\}$ is algebraically independent. On the other hand, we claim that every derivation D of k extends uniquely to a derivation of K. To establish this, we use the fact that the field K is a finite extension of k, since B is algebraic over A. Hence the theorem of the primitive element (Lang [1971, Ch VII, §6, Theorem 14]) furnishes an element $b \in K$ so that $K = k(b)$. Let $f(X) \in k[X]$ be the minimal polynomial for b. Then $f'(b) \neq 0$, so we may define $Db = -f^D(b)/f'(b) \in K$, where for any polynomial $g(X) = \sum_i a_i X^i \in k[X]$ we let

$$g^D(X) = \textstyle\sum_i D(a_i)\, X^i\,.$$

If D can be extended to a derivation of K, it must act by

$$D(g(b)) = g^D(b) + g'(b)Db \quad \text{for all } g(X) \in k[X]\,. \tag{A.18}$$

To prove that such an extension exists, we must verify that the right side of (A.18) does not depend on the representation of an element of K as $g(b)$. But if $g(b) = h(b)$ for some $h(X) \in k[X]$, then the polynomial $\varphi(X) = g(X) - h(X)$ is of the form $\psi(X)f(X)$ for some $\psi(X) \in k[X]$. Hence $\varphi'(b) = \psi(b)f'(b)$ and $\varphi^D(b) = \psi(b)f^D(b)$ since $f(b) = 0$. It follows that

$$\begin{aligned} g'(b)Db - h'(b)Db &= \psi(b)f'(b)Db = -\psi(b)f^D(b) \\ &= -g^D(b) + h^D(b) \end{aligned}$$

by the definition of Db. Thus (A.18) is unambiguous and defines the desired extension. This completes the proof of (A.17).

Let $f_1, \ldots, f_r$ be a set of generators for $\mathfrak{I}_X$. A derivation D of K is uniquely determined by the n functions $\xi_j = Dx_j \in K$, where x_j are the linear coordinate functions from $\mathbb{C}^n$. It must also satisfy $D(f_i) = 0$ for $i = 1, \ldots, r$. Thus by the chain rule we have

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}\, \xi_j = 0, \quad \text{for } i = 1, \ldots, r\,. \tag{A.19}$$

By definition of $m(X)$, system (A.19) has rank $n - m(X)$ over K. Hence the solution space to (A.19) has dimension $m(X)$ over K. Combining this fact with (A.17) completes the proof. ⧫

A.3.4 Differential Criterion for Dominance

We begin with a general criterion for a map to be dominant.

Proposition A.3.3. *Let X and Y be affine algebraic sets and $\psi : X \longrightarrow Y$ a regular map. Assume Y is irreducible and $\dim Y = m$. Suppose there exists an algebraically independent set $\{u_1, \ldots, u_m\} \subset \mathcal{O}[Y]$ such that the set*

$$\{\psi^* u_1, \ldots, \psi^* u_m\} \subset \mathcal{O}[X]$$

is also algebraically independent. Then $\psi(X)$ is dense in Y.

Proof. Suppose there exists $0 \neq f \in \mathcal{O}[Y]$ such that $\psi^*(f) = 0$. Since $\dim Y = m$, f is algebraic over the field $\mathbb{C}(u_1, \ldots, u_m)$. Thus there are rational functions $a_j \in \mathbb{C}(x_1, \ldots, x_m)$ so that

$$\sum_{j=0}^{d} a_j(u_1, \ldots, u_m)\, f^j = 0\,.$$

We choose the functions a_j with d as small as possible ($d \geq 1$ since $f \neq 0$). Take $0 \neq \gamma \in \mathbb{C}[u_1, \ldots, u_m]$ such that $\gamma a_j(u_1, \ldots, u_m) \in \mathbb{C}[u_1, \ldots, u_m]$. Then

$$\sum_{j=0}^{d} \psi^*(\gamma)\, a_j(\psi^* u_1, \ldots, \psi^* u_m)\, \psi^*(f^j) = 0\,.$$

Since $\psi^*(f^j) = \psi^*(f)^j = 0$ for $j \geq 1$, we obtain the relation

$$\psi^*(\gamma)\, a_0(\psi^* u_1, \ldots, \psi^* u_m) = 0. \tag{A.20}$$

But we are given that $\psi^* : \mathbb{C}[u_1, \ldots, u_m] \longrightarrow \mathbb{C}[\psi^* u_1, \ldots, \psi^* u_m]$ is an isomorphism, so we have $\psi^*(\gamma) \neq 0$. Thus (A.20) implies that $a_0(\psi^* u_1, \ldots, \psi^* u_m) = 0$, and hence $a_0(u_1, \ldots, u_m) = 0$. This contradicts the definition of d. We conclude that $\psi^* : \mathcal{O}[Y] \longrightarrow \mathcal{O}[X]$ is injective. Since Y is irreducible, this implies that $\psi(X)$ is dense in Y. ♦

Let X and Y be irreducible affine algebraic sets and let $\psi : X \longrightarrow Y$ be a regular map. We have the following criterion for ψ to be dominant:

Theorem A.3.4. *Suppose there exists a smooth point $p \in X$ such that $\psi(p)$ is a smooth point of Y and $d\psi_p : T(X)_p \longrightarrow T(Y)_{\psi(p)}$ is bijective. Then $\psi(X)$ is dense in Y.*

We first prove the following lemma, whose statement and proof is similar to the corresponding result for C^∞ manifolds (the local triviality of the tangent bundle):

Lemma A.3.5. *Let $X \subset \mathbb{C}^n$ be closed and irreducible and let $p \in X$ be a smooth point of X. Then there exists a open subset $U \subset X$ with $p \in U$ and regular maps $w_j : U \longrightarrow \mathbb{C}^n$ for $j = 1, \ldots, m = \dim X$ such that*

$$T(X)_q = \bigoplus_{j=1}^{m} \mathbb{C} w_j(q) \quad \textit{for all } q \in U\,.$$

Proof. Since p is a smooth point, we have

$$T(X)_p = \{v \in \mathbb{C}^n \,:\, (d\varphi)_p\,(v) = 0 \quad \text{for all } \varphi \in \mathcal{I}_X\}$$

by (A.15). Hence there exist $\varphi_1, \ldots, \varphi_{n-m} \in \mathcal{I}_X$ such that

$$T(X)_p = \{v \in \mathbb{C}^n \,:\, (d\varphi_i)_p\,(v) = 0 \quad \text{for } i = 1, \ldots, n-m\}\,.$$

This implies that $(d\varphi_1)_p \wedge \cdots \wedge \left(d\varphi_{(n-m)}\right)_p \neq 0$, so there is a open subset U_1 of $\mathbb{C}^n$ such that $p \in U_1$ and

$$(d\varphi_1)_q \wedge \cdots \wedge \left(d\varphi_{(n-m)}\right)_q \neq 0 \quad \text{for } q \in U_1\,. \tag{A.21}$$

For any point $q \in U_1 \cap X$ we have

$$T(X)_q \subset W_q = \{v \in \mathbb{C}^n \,:\, (d\varphi_i)_q (v) = 0 \quad \text{for } i = 1, \ldots, n-m\}\,.$$

Now $\dim T(X)_q \geq m$ since X is irreducible. Since $\dim W_q = m$ by (A.21), we conclude that

$$T(X)_q = \{v \in \mathbb{C}^n \,:\, (d\varphi_i)_q (v) = 0, \quad i = 1, \ldots, n-m\} \tag{A.22}$$

for all $q \in U_1 \cap X$.

Fix a basis $\{e_1 \ldots, e_n\}$ for $\mathbb{C}^n$ such that

$$\det[(d\varphi_i)_p (e_j)]_{1 \leq i,j \leq n-m} \neq 0\,.$$

Let $V = \bigoplus_{i=1}^{n-m} \mathbb{C}e_i$ and $W = \bigoplus_{i=n-m+1}^{n} \mathbb{C}e_i$. We then write $x \in \mathbb{C}^n$ as

$$x = \begin{bmatrix} y \\ z \end{bmatrix} \quad \text{with } y \in V \text{ and } z \in W\,.$$

For $q \in \mathbb{C}^n$ we write the $n \times n$ matrix $J_q = [(d\varphi_i)_q (e_j)]_{1 \leq i,j \leq n}$ in block form as

$$J_q = \begin{bmatrix} A_q & B_q \\ C_q & D_q \end{bmatrix}$$

with A_q of size $(n-m) \times (n-m)$. In terms of the decomposition $\mathbb{C}^n = V \oplus W$ we have $A_q : V \longrightarrow V$ and $B_q : W \longrightarrow V$.

Since $\det A_p \neq 0$ there exists a Zariski-open subset $U_2 \subset U_1$ with $p \in U_2$ and $\det A_q \neq 0$ for all $q \in U_2$. We define regular maps $w_j : U_2 \longrightarrow \mathbb{C}^n$ by

$$w_j(q) = \begin{bmatrix} -A_q^{-1} B_q e_{n-m+j} \\ e_{n-m+j} \end{bmatrix} \quad \text{for } j = 1, \ldots, m\,.$$

Clearly $\{w_1(q), \ldots, w_m(q)\}$ is linearly independent. Also,

$$J_q w_j(q) = \begin{bmatrix} -A_q A_q^{-1} B_q e_{n-m+j} + B_q e_{n-m+j} \\ * \end{bmatrix} = \begin{bmatrix} 0 \\ * \end{bmatrix}.$$

Set $U = U_2 \cap X$. Then for all $q \in U$ we have

$$(d\varphi_i)_q (w_j(q)) = 0 \quad \text{for } i = 1, \ldots, n-m\,.$$

Thus $\{w_1(q), \ldots, w_m(q)\}$ is a basis for $T(X)_q$ by (A.22). ♦

Proof of Theorem A.3.4 Let $p \in X$ satisfy the conditions of the theorem. Take U and $w_1, \ldots, w_m$ be as in Lemma A.3.5. Since $\{d\psi_p(w_1(p)), \ldots, d\psi_p(w_m(p))\}$ is a basis for $T(Y)_{\psi(p)}$, there exist functions $u_1, \ldots, u_m$ in $\mathcal{O}[Y]$ such that

$$d\psi_p(w_j(p)) u_i = \delta_{ij} \quad \text{for } i, j = 1, \ldots, m\,.$$

Now for $q \in X$ we can write $d\psi_q(w_j(q))u_i = d(\psi^* u_i)_q(w_j(q))$. Thus there is a open subset $V \subset U$ containing p such that

$$\det[d(\psi^* u_i)_q(w_j(q))]_{1\le i,j\le m} \neq 0 \quad \text{for } q \in V\,.$$

This implies that $\{\psi^* u_1, \ldots, \psi^* u_m\}$ is algebraically independent, so $\{u_1, \ldots, u_m\}$ is algebraically independent. The theorem follows from Proposition A.3.3. ♦

Proposition A.3.6. *Let $\varphi : X \longrightarrow Y$ be a dominant regular map of irreducible affine algebraic sets. For $y \in Y$ let $F_y = \varphi^{-1}\{y\}$. Then there is a nonempty open set $U \subset X$ such that $\dim X = \dim Y + \dim F_{\varphi(x)}$ and $\dim F_{\varphi(x)} = \dim \mathrm{Ker}(d\varphi_x)$ for all $x \in U$.*

Proof. Let $d = \dim X - \dim Y$, $S = \varphi^* \mathcal{O}[Y]$, and $R = \mathcal{O}[X]$. Set $k = \mathrm{Quot}(S)$ and let $B \subset \mathrm{Quot}(R)$ be the subalgebra generated by k and R (the rational functions on X with denominators in $S \setminus \{0\}$). Since B has transcendence degree d over k, Lemma A.1.17 furnishes an algebraically independent set $\{f_1, \ldots, f_d\} \subset R$ such that B is integral over $k[f_1, \ldots, f_d]$. Taking the common denominator of a set of generators of the algebra B, we obtain $f = \varphi^* g \in S$ such that R_f is integral over $S_f[f_1, \ldots, f_d]$, where $R_f = \mathcal{O}[X^f]$ and $S_f = \varphi^* \mathcal{O}[Y^g]$. By Theorem A.2.7 we can take g so that $\varphi(Y^g) = X^f$.

Define $\psi : X^f \longrightarrow Y^g \times \mathbb{C}^d$ by $\psi(x) = (\varphi(x), f_1(x), \ldots, f_d(x))$. Then $\psi^* \mathcal{O}[Y^g \times \mathbb{C}^d] = S_f[f_1, \ldots, f_d]$, and hence $\mathcal{O}[X^f]$ is integral over $\psi^* \mathcal{O}[Y^g \times \mathbb{C}^d]$. By Theorem A.2.3 every homomorphism from $S_f[f_1, \ldots, f_d]$ to $\mathbb{C}$ extends to a homomorphism from R_f to $\mathbb{C}$. Hence ψ is surjective. Let $\pi : Y^g \times \mathbb{C}^d \longrightarrow Y^g$ by $\pi(y, z) = y$. Then $\varphi = \psi^* \pi$ and $F_y = \psi^{-1}(\{y\} \times \mathbb{C}^d)$. If W is any irreducible component of F_y then $\mathcal{O}[W]$ is integral over $\psi^* \mathcal{O}[\{y\} \times \mathbb{C}^d]$, and hence $\dim W = d$.

We have $d\varphi_x = d\pi_{\psi(x)} \circ d\psi_x$. The integrality property implies that every derivation of $\mathrm{Quot}(\psi^*(Y^g \times \mathbb{C}^d))$ extends uniquely to a derivation of $\mathrm{Rat}(X^f)$, as in the proof of Theorem A.3.2. Hence $d\psi_x$ is bijective for x in a nonempty dense open set U by Lemma A.3.5. For such x, $\mathrm{Ker}(d\varphi_x) = \mathrm{Ker}(d\pi_{\psi(x)})$ has dimension d. ♦

A.4 Projective and Quasiprojective Sets

A.4.1 Basic Definitions

Let V be a complex vector space. The *projective space* $\mathbb{P}(V)$ associated with V is the set of lines through 0 (one-dimensional subspaces) in V. For $x \in V \setminus \{0\}$, $[x] \in \mathbb{P}(V)$ will denote the line through x. The map $p : V \setminus \{0\} \longrightarrow \mathbb{P}(V)$ given by $p(x) = [x]$ is surjective, and $p(x) = p(y)$ if and only if $x = \lambda y$ for some $\lambda \in \mathbb{C}^\times$. We denote $\mathbb{P}(\mathbb{C}^{n+1})$ by $\mathbb{P}^n$ and for $x = (x_0, \ldots, x_n) \in \mathbb{C}^{n+1}$ we call $\{x_i\}$ the *homogeneous coordinates* of $[x]$.

If $f(x_0, \ldots, x_n)$ is a homogeneous polynomial in $n+1$ variables and $0 \neq x \in \mathbb{C}^{n+1}$, then $f(x) = 0$ if and only if f vanishes on line $[x]$. Hence f defines a subset

$$A_f = \{[x] \in \mathbb{P}^n \,:\, f(x) = 0\}$$

in projective space. The *Zariski topology* on $\mathbb{P}^n$ is obtained by taking as closed sets the intersections

$$X = \bigcap_{f \in S} A_f \,,$$

where S is any set of homogeneous polynomials on $\mathbb{C}^{n+1}$. Any such set X will be called a *projective algebraic set* . The set

$$p^{-1}(X) \cup \{0\} = \{x \in \mathbb{C}^{n+1} \,:\, f(x) = 0 \text{ for all } f \in S\}$$

is closed in $\mathbb{C}^{n+1}$ and is called the *cone over* X.

Every closed set in projective space is definable as the zero locus of a finite collection of homogeneous polynomials, and the descending chain condition for closed sets is satisfied (this is proved by passing to the cones over the sets and using the corresponding results for affine space). Just as in the affine case, this implies that every closed set is a finite union of irreducible closed sets, and any nonempty open subset of an irreducible closed set M is dense in M.

We define a covering of projective space by affine spaces, as follows: For $i = 0, \ldots, n$ let $\mathbb{U}_i^n = \{[x] \in \mathbb{P}^n \,:\, x_i \neq 0\}$. Each $\mathbb{U}_i^n$ is an open set in $\mathbb{P}^n$, and every point of $\mathbb{P}^n$ lies in $\mathbb{U}_i^n$ for some i. For $[x] \in \mathbb{U}_i^n$ define the *inhomogeneous coordinates* of $[x]$ to be $y_j = x_j/x_i$ for $j \neq i$. The map

$$\pi_i([x]) = (y_0, \ldots, \widehat{y_i}, \ldots, y_n) \quad \text{(omit } y_i\text{)}$$

is a bijection between $\mathbb{U}_i^n$ and $\mathbb{C}^n$. It is also a topological isomorphism (where $\mathbb{U}_i^n$ has the relative Zariski topology from $\mathbb{P}^n$ and $\mathbb{C}^n$ carries the Zariski topology). To see this, let $U = \{y \in \mathbb{C}^n \,:\, f(y) \neq 0\}$, with f a polynomial of degree k, and set $g = \big(x_i(\pi_i^* f)\big)^k$. Then g is a homogeneous polynomial of degree k on $\mathbb{C}^{n+1}$, and

$$\pi_i^{-1}(U) \cap \mathbb{U}_i^n = \{[x] \,:\, g(x) \neq 0 \text{ and } x_i \neq 0\} \,.$$

Thus we have a covering by $\mathbb{P}^n$ by the $n+1$ open sets $\mathbb{U}_i^n$, each homeomorphic to the affine space $\mathbb{C}^n$.

Lemma A.4.1. *Let $X \subset \mathbb{P}^n$. Suppose that for all $i = 0, 1, \ldots, n$, $X \cap \mathbb{U}_i$ is the set of zeros of homogeneous polynomials $f_{ij}(y_0, \ldots, \widehat{y_i}, \ldots, y_n)$, where $\{y_k\}$ are the inhomogeneous coordinates on $\mathbb{U}_i$. Then X is closed in $\mathbb{P}^n$.*

Proof. Let d_{ij} be the degree of f_{ij}. Define $g_{ij}(x) = x_i f_{ij}(x_0, \ldots, \widehat{x_i}, \ldots, x_n)$ for $x \in \mathbb{C}^{n+1}$. Then g_{ij} is a homogeneous polynomial of degree $d_{ij}+1$. For $x_i \neq 0$ we have

$$g_{ij}(x_0, \ldots, \widehat{x_i}, \ldots, x_n) = x_i^{d_{ij}+1} f_{ij}(y_0, \ldots, \widehat{y_i}, \ldots, y_n) \,.$$

Hence $X \cap \mathbb{U}_i = \{[x] \,:\, g_{ij}(x) = 0 \quad \text{for all } j\}$. Since $g_{ij}(x) = 0$ when $x_i = 0$ and the sets $X \cap \mathbb{U}_i$ cover X, it follows that $X = \{[x] \,:\, g_{ij}(x) = 0 \quad \text{for all } i, j\}$. Thus X is closed in $\mathbb{P}^n$. ⧫

A *quasiprojective algebraic set* is a subset $M \subset \mathbb{P}^n$ defined by a finite set of equalities and inequalities on the homogeneous coordinates of the form

$$f_i(x) = 0 \quad \text{for all } i = 1, \ldots, k \text{ and } \; g_j(x) \neq 0 \quad \text{for some } \; j = 1, \ldots, l\,,$$

where f_i and g_j are homogeneous polynomials on $\mathbb{C}^{n+1}$. In topological terms, M is the intersection of the closed set $Y = \{[x] \in \mathbb{P}^n \,:\, f_i(x) = 0 \text{ for all } i = 1, \ldots, k\}$ and the open set $Z = \{[x] \in \mathbb{P}^n \,:\, g_j(x) \neq 0 \text{ for some } j\}$.

Examples

1. Any projective algebraic set is quasiprojective, since a closed subset of $\mathbb{P}^n$ always has a finite set of defining equations.

2. The set $\mathbb{U}_i^n$ introduced above is the quasiprojective algebraic set in $\mathbb{P}^n$ defined by $\{x_i \neq 0\}$.

3. Suppose $M \subset \mathbb{U}_i^n$ for some i and $X = \pi_i(M) \subset \mathbb{C}^n$ is an affine algebraic set. Since π_i is a homeomorphism, M is a (relatively) closed subset of the open set $\mathbb{U}_i^n$ and hence is quasiprojective.

A.4.2 Products of Projective Sets

Consider now the problem of putting the structure of a quasi-projective algebraic set on the product of two such sets. We begin with the basic case of projective spaces. Let x and y be homogeneous coordinates on $\mathbb{P}^m$ and $\mathbb{P}^n$. Denote the space of complex matrices of size $r \times s$ by $M_{r,s}$ and view $\mathbb{C}^r = M_{1,r}$ as row vectors. We map $\mathbb{C}^{m+1} \times \mathbb{C}^{n+1} \longrightarrow M_{m+1,\,n+1}$ by $(x, y) \mapsto xy^t$, where y^t is the transpose of y. The image of $(\mathbb{C}^{m+1} \setminus \{0\}) \times (\mathbb{C}^{n+1} \setminus \{0\})$ consists of all rank one matrices; hence it is defined by the vanishing of all minors of size greater than 1. These minors are homogeneous polynomials in the matrix coordinates z_{ij} of $z \in M_{m+1,\,n+1}$. Passing to projective space, we have thus obtained an embedding

$$\mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}(M_{m+1,\,n+1}) = \mathbb{P}^{mn+m+n} \tag{A.23}$$

with closed image. We take this as the structure of a projective algebraic set on $\mathbb{P}^m \times \mathbb{P}^n$.

Now let $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ be projective algebraic sets. The image of $X \times Y$ under the map (A.23) is all points $[xy^t]$ such that

$$0 \neq x \in \mathbb{C}^{m+1}, \;\; 0 \neq y \in \mathbb{C}^{n+1}, \text{ and } f_i(x) = 0,\, g_i(y) = 0 \text{ for } i = 1, \ldots k\,,$$

where f_i and g_i are homogeneous polynomials on $\mathbb{C}^m$ and $\mathbb{C}^n$ that define X and Y, respectively. To see that such sets are projective, consider a homogeneous polynomial $f(x)$ on $\mathbb{C}^{m+1}$ of degree r and a point $[y] \in \mathbb{P}^n$. We shall describe the rank-one matrices $z \in M_{m+1,\,n+1}$ of the form $z = xy^t$ that satisfy $f(x) = 0$ by giving homogeneous equations in the entries z_{ij} of z. Since $y \neq 0$, the equation $f(x) = 0$ is equivalent to the system of equations

$$y_i^r f(x) = 0 \quad \text{for } i = 0, \ldots, n+1\,.$$

But given that $z_{ij} = x_i y_j$, we can write $y_i^r f(x) = f(y_i x) = \varphi_i(z)$, where $\varphi_i(z) = f(z_{1i}, \ldots, z_{n+1,i})$ is a homogeneous polynomial of degree r. Thus the desired equations are

$$\varphi_i(z) = 0 \quad \text{for } i = 0, \ldots, n+1\,.$$

Interchanging the roles of x and y, we can likewise express homogeneous equalities in y in terms of z, for fixed $[x] \in \mathbb{P}^m$. It follows that the image of $X \times Y$ in $\mathbb{P}^{m+n+mn}$ is closed.

A similar argument for equations of the form $f(x) \neq 0$ shows that the image of $X \times Y$ under the map (A.23) is quasiprojective if X and Y are quasiprojective.

Lemma A.4.2. *Let X be a quasi-projective algebraic set and let*

$$\Delta = \{(x, x) : x \in X\} \subset X \times X$$

be the diagonal. Then Δ is closed.

Proof. Since $X \times X$ is again quasiprojective, it suffices to prove that the diagonal is closed in $\mathbb{P}^n \times \mathbb{P}^n$. In terms of the embedding $\mathbb{P}^n \times \mathbb{P}^n \hookrightarrow \mathbb{P}(M_{n+1,\,n+1})$, the diagonal consists of the lines $[z]$ where $z = xx^t$ and $x \in \mathbb{C}^{n+1}$. We claim that the diagonal can be described by the homogeneous equation $z^t = z$ and hence is closed. Indeed, if $z = xy^t$, with $x, y \in \mathbb{C}^{n+1} \setminus \{0\}$, and $z^t = z$, then $x_i y_j = x_j y_i$ for all i, j. Pick j so that $y_j \neq 0$. Then $x_i = (x_j y_j^{-1}) y_i$ for all i, so $[x] = [y]$ in $\mathbb{P}^n$. ♦

A.4.3 Regular Functions and Maps

We encounter the basic difference between affine and projective algebraic sets when we consider the notion of a *regular function*. In the affine case we have global functions (polynomials) on an ambient vector space whose restrictions define the regular functions. By contrast, a polynomial $f(x)$ on $\mathbb{C}^{n+1}$, even if homogeneous, does not define a function on $\mathbb{P}^n$ except in the trivial case that f is homogeneous of degree zero (constant). A way around this difficulty is to replace polynomials by rational functions and to *localize* the notion of regularity, as follows.

Let M be an irreducible affine set. Suppose $U \subset M$ is an open subset. Define the *regular functions on U* to be the restrictions to U of rational functions $f \in \mathrm{Rat}(M)$ such that $\mathcal{D}_f \supset U$. The set $\mathcal{O}_M(U)$ of all such functions obviously is a commutative algebra over $\mathbb{C}$. For $U = M$ we have $\mathcal{O}_M(M) = \mathcal{O}[M]$, by Lemma A.2.1, so the terminology is consistent. Replacing U by a point $x \in M$, we define the *local ring* $\mathcal{O}_x$ at x to consist of all rational functions on M that are defined at x. Clearly $\mathcal{O}_x$ is a subalgebra of $\mathrm{Rat}(M)$, and $\mathcal{O}_x = \bigcup_{x \in V} \mathcal{O}_M(V)$, where V runs over all open sets containing x.

This notion of regular function has two key properties:

(restriction) If $U \subset V$ are open subsets of M and $f \in \mathcal{O}_M(V)$, then $f|_U \in \mathcal{O}_M(U)$.

(locality) Suppose $f : U \longrightarrow \mathbb{C}$ and for every $x \in U$ there exists $\varphi \in \mathcal{O}_x$ with $\varphi = f$ on some open neighborhood of x. Then $f \in \mathcal{O}_M(U)$.

(We leave the proof of these properties as an exercise.) Thus all regularity properties of functions can be expressed in terms of the local rings. To carry over these constructions to an arbitrary quasiprojective algebraic set, we need the following covering lemma:

Lemma A.4.3. *Suppose X is a quasiprojective algebraic set. There is a finite open covering $X = \bigcup_{\alpha\in A} U_\alpha$ with the following properties:*

(1) *For $\alpha \in A$ there is an irreducible affine algebraic set M_α and a homeomorphism $\varphi_\alpha : U_\alpha \longrightarrow M_\alpha$.*

(2) *For all $\alpha, \beta \in A$ the maps $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ are regular.*

Proof. We have $X = U \cap Y$, where U is open and Y is closed in $\mathbb{P}^n$. Set $V_i = \pi_i(\mathbb{U}_i^n \cap X)$ and $M_i = \pi_i(Y \cap \mathbb{U}_i^n)$, where $\pi_i : \mathbb{U}_i^n \longrightarrow \mathbb{C}^n$ is the map defined by inhomogeneous coordinates. Then V_i is an open subset of the affine algebraic set M_i, and so it is a finite union of principal open sets V_{ij}, by Lemma A.1.7. Decomposing the sets M_i into irreducible components and intersecting them with V_{ij}, we obtain a collection of irreducible affine sets M_α (cf. Lemma A.1.13). We take the maps φ_α to be restrictions of the maps π_i^{-1}. The regularity of $\varphi_\beta \circ \varphi_\alpha^{-1}$ follows from the formula

$$(\pi_i \circ \pi_j^{-1})(y) = (y_1/y_i, \ldots, \underbrace{1}_{i\text{th}}, \ldots, \underbrace{1/y_i}_{j\text{th}}, \ldots, y_n/y_i)\,,$$

for $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$. ♦

Let X be a quasiprojective algebraic set. We define the *local ring* $\mathcal{O}_x$ at $x \in X$ by carrying over the local rings of the affine open sets U_α via the maps φ_α:

$$\mathcal{O}_x = \varphi_\alpha^*(\mathcal{O}_{\varphi_\alpha(x)}) \quad \text{for } x \in U_\alpha\,.$$

If $x \in U_\alpha \cap U_\beta$ then $\mathcal{O}_x$ is the same, whether we use φ_α or φ_β, by the last statement in Lemma A.4.3. For any open set $U \subset X$ we can now define the ring $\mathcal{O}_X(U)$ of *regular functions* on U using the local rings, just as in the affine case: a continuous function $f : U \longrightarrow \mathbb{C}$ is *regular* if for each $x \in U$ there exists $g \in \mathcal{O}_x$ so that $f = g$ on an open neighborhood of x. One then verifies that the restriction and locality properties hold for the rings $\mathcal{O}_X(U)$.

Definition A.4.4. Let X and Y be quasiprojective algebraic sets. A map $\varphi : X \longrightarrow Y$ is *regular* if φ is continuous and if for all open sets $U \subset Y$ one has $\varphi^*(\mathcal{O}(U)) \subset \mathcal{O}(\varphi^{-1}(U))$.

When X and Y are affine, this notion of regularity agrees with the earlier definition.

Lemma A.4.5. *Let X, Y, and Z be quasiprojective algebraic sets. Then a map $z \mapsto (f(z), g(z))$ from Z to $X \times Y$ is regular if and only if the component maps $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ are regular.*

Proof. By Lemma A.4.3, it is enough to check when X, Y, and Z are affine. In this case, it follows immediately from the property $\mathcal{O}[X \times Y] = \mathcal{O}[X] \otimes \mathcal{O}[Y]$. ♦

Proposition A.4.6. *Suppose X and Y are quasiprojective algebraic sets and let $\varphi : X \longrightarrow Y$ be regular. Then $\Gamma_\varphi = \{(x, \varphi(x)) : x \in X\}$ (the graph of φ) is closed in $X \times Y$.*

Proof. It is enough to consider the case that X and Y are affine, by Lemma A.4.1. A point $(x, y) \in X \times Y$ is in Γ_φ if and only if

$$g(\varphi(x)) - g(y) = 0 \qquad \text{for all } g \in \mathcal{O}[Y].$$

Now for $g \in \mathcal{O}[Y]$, the function $f(x, y) = g(\varphi(x)) - g(y)$ is a regular function on $X \times Y$, so Γ_φ is the zero set of a family of regular functions on $X \times Y$. ♦

Corollary A.4.7. *Let X, Y be quasiprojective algebraic sets and $\varphi : X \times Y \longrightarrow X$ a regular map. Then $\{(x, y) \in X \times Y : \varphi(x, y) = x\}$ is closed in $X \times Y$.*

Proof. Use the same argument as for Proposition A.4.5. ♦

We denote by $\mathcal{O}[X] = \mathcal{O}_X(X)$ the ring of functions that are everywhere regular on X (the notation is consistent by Lemma A.2.1). If X is affine, then $\mathcal{O}[X]$ separates the points of X. When X is not an affine algebraic set, however, $\mathcal{O}[X]$ may not contain many functions. Here is the extreme case.

Theorem A.4.8. *Let X be an irreducible projective algebraic set. Then $\mathcal{O}[X] = \mathbb{C}$.*

Proof. Assume $X \subset \mathbb{P}^n$ and let $Y \subset \mathbb{C}^{n+1}$ be the cone over X. Then the irreducibility of X implies that Y is irreducible. The ideal $\mathcal{I}_Y$ is generated by homogeneous polynomials, so the algebra $\mathcal{O}[Y]$ is graded.

Let $f \in \mathcal{O}[X]$. We can consider f as a rational function on Y that is homogeneous of degree 0. The assumption that f is regular everywhere on X means that for all $0 \neq y \in Y$ there exist $p, q \in \mathcal{O}[Y]$, each homogeneous of the same degree, with $q(y) \neq 0$ and $f = p/q$ on a neighborhood of y. Let $\mathcal{I}$ be the ideal in $\mathcal{P}(\mathbb{C}^{n+1})$ generated by $f_1, \ldots, f_k$ together with all denominators q that occur this way, and let $f_1, \ldots, f_s$ be homogeneous polynomials that generate $\mathcal{I}$. The only common zero of $\{f_i\}$ on $\mathbb{C}^{n+1}$ is $y = 0$. Thus by Corollary A.1.5 there is an integer r so that $x_i^r \in \mathcal{I}$ for $i = 0, 1, \ldots, n$, where $\{x_i\}$ are coordinates on $\mathbb{C}^{n+1}$. Hence $\mathcal{I}$ contains the space $\mathcal{P}^m(\mathbb{C}^{n+1})$, where $m = r(n+1)$. This implies that if g is a homogeneous polynomial of degree m, then fg is a homogeneous rational function of degree m, and $fg|_Y = h|_Y$, where h is a homogeneous polynomial of degree m.

Let $V \subset \mathcal{O}[Y]$ be the restrictions to Y of the homogeneous polynomials of degree m. It is a finite-dimensional space, and we have just shown that the operator T of multiplication by f maps V into V. Let λ be an eigenvalue of T, and let $0 \neq h \in \mathcal{O}[Y]$ be the corresponding eigenvector. Then $(f - \lambda)h = 0$ in the field $\mathrm{Rat}(Y)$. Hence $f = \lambda$ is a constant. ♦

Corollary A.4.9. *If X is an irreducible projective algebraic set that is also isomorphic to an affine algebraic set, then X is a single point.*

Proof. By Theorem A.4.8 $\mathcal{O}[X] = \mathbb{C}$. But for every affine algebraic set, the functions in $\mathcal{O}[X]$ separate the points of X. Hence X must consist of one point. ♦

Theorems A.2.7 and A.2.8 are also valid when X and Y are quasiprojective algebraic sets (this follows by covering X and Y by irreducible open affine algebraic sets and restricting the map to the sets of the covering). Furthermore, if f is a rational map between affine algebraic sets, then the open set $\mathcal{D}_f$ is a quasiprojective algebraic set, and $f : \mathcal{D}_f \longrightarrow Y$ is a regular map in this new sense. Thus Theorem A.2.9 is also valid for quasiprojective algebraic sets.

If X is quasiprojective and $x \in X$, we define $\dim_x(X) = \dim T(U_\alpha)_x$, where $x \in U_\alpha$ as in Lemma A.4.3. It is easy to see that $\dim_x(X)$ only depends on the local ring $\mathcal{O}_x$ (cf. Theorem A.3.2). We set

$$\dim X = \min_{x \in X} \{ \dim_x(X) \} .$$

It is clear from this definition of dimension that Theorem A.1.18 holds for quasiprojective algebraic sets. Just as in the affine case, a point $x \in X$ is called *smooth* if $\dim_x(X) = \dim X$. When X is irreducible, the smooth points form a dense open set. If every point of X is smooth then X is said to be *smooth* or *nonsingular*.

An extension of method of proof of Theorem A.4.8 can be used to prove an important theorem that gives a characterization of projective algebraic sets among quasiprojective sets. We will use this result only once in the text (Theorem 11.4.9) and will leave it to the interested reader to read the proof in, say, Shafarevich [1994, Ch. I §5.2 Theorem 3]. The argument is at precisely the same level as the rest of this appendix.

Theorem A.4.10. *Let X, Y be quasiprojective sets with X projective. Let $p(x, y) = y$ for $(x, y) \in X \times Y$. If $C \subset X \times Y$ is closed then $p(C)$ is closed in Y.*

Here is an example of the power of this result.

Corollary A.4.11. *Let X be projective and $f : X \to Y$ be a regular map with Y quasiprojective. Then $f(X)$ is closed in Y.*

Proof. Proposition A.4.6 implies that Γ_f is closed in $X \times Y$. Since $f(X)$ is the projection in the second factor of Γ_f it is closed by Theorem A.4.10. ♦

Another result that will be needed in the proof of Theorem 11.4.9 is the following (the connectedness assumption can be shown to be redundant):

Proposition A.4.12. *Suppose $X \subset \mathbb{C}^m$ is a smooth irreducible affine variety that is also connected as a C^∞ manifold. If $X = \bigcup_{n=1}^\infty X_n$, where each X_n is a Zariski-closed subset, then $X = X_n$ for some n.*

Proof. Assume to the contrary that $X \neq X_n$ for every n. Then X_n is a proper subset of X that is defined by the vanishing of a finite number of nonconstant regular functions on X. The assumption $X = \bigcup_{n=1}^\infty X_n$ thus implies that there is a countable

set of nonconstant regular functions $f_1, f_2, \ldots$ so that X is the union of the zero sets of these functions. But since f_k is holomorphic and X is connected, the zero set of f_k is nowhere dense in X in the relative metric topology from $\mathbb{C}^n$. We conclude that X is a countable union of nowhere dense sets, which contradicts the Baire Category theorem. ⧫