Math 171B Homework Assignment #7

Due date: 11pm, June 05, 2022

The first 5 questions are required to be submitted in Gradescope.

1. (10 points) Determine the tangent cone $T(x^*)$ at $x^* = (3, 4, 5)$ for the set $x_1^2 + x_2^2 \leq x_3^2$.

2. (10 points) Find all KKT points of the optimization

$$\begin{align*}
\min & \quad x_1^2 - x_2^2 + 4x_1x_2 \\
\text{s.t.} & \quad 1 - x_1^2 - x_2^2 \geq 0.
\end{align*}$$

Identify the local and global minimizers of them.

3. (10 points) Find all KKT points of the optimization

$$\begin{align*}
\min & \quad x_1^3 + x_2^3 + x_3^3 \\
\text{s.t.} & \quad x_1x_2x_3 - 1 \geq 0.
\end{align*}$$

Identify the local minimizers of them. Does this optimization have a global minimizer?

4. (10 points) Find all KKT points of the optimization

$$\begin{align*}
\min & \quad -x_1^2 - x_2^2 + 4x_1x_2 \\
\text{s.t.} & \quad 1 - x_1^2 \geq 0, 1 - x_2^2 \geq 0.
\end{align*}$$

Identify the local and global minimizers of them.

5. (10 points) Consider the optimization problem

$$\begin{align*}
\min & \quad x_1^3 + x_2^3 - 2x_1 - 3x_2 \\
\text{s.t.} & \quad 3 - x_1 - x_2 \geq 0, x_1 \geq 0, x_2 \geq 0
\end{align*}$$

Find all its local and global minimizers.

The following questions are optional. Do NOT submit them in Gradescope.

6. Consider the optimization problem

$$\begin{align*}
\min_{x \in \mathbb{R}^3} & \quad f(x) \\
\text{s.t.} & \quad x_1 \geq 0, x_2 \geq 0, 1 - x_2 \geq 0,
\end{align*}$$
where the objective \( f \) is second order differentiable. If the origin 0 is a local minimizer, show that there exists \( \lambda_1 \geq 0, \lambda_2 \geq 0 \) such that
\[
\nabla f(0) = \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
\]

Moreover, also show that the second order condition
\[
p^T \left( \nabla^2 f(0) \right) p \geq 0 \quad \forall p \in \mathbb{R}^3 : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} p = 0.
\]

(This is how to prove the KKT conditions for inequality constrained optimization.)

7. For a symmetric matrix \( A \in \mathbb{R}^{n \times n} \), consider the optimization
\[
\left\{ \begin{array}{l}
\min \quad x^T Ax \\
\text{s.t.} \quad 1 - x^T x \geq 0.
\end{array} \right.
\]
If \( x^* \) is an eigenvector of unit length for \( A \), associated to a negative eigenvalue, show that \( x^* \) must be a KKT point.

8. Let \( x^* \) be a local minimizer of the optimization
\[
\left\{ \begin{array}{l}
\min \quad g^T x + x^T H x \\
\text{s.t.} \quad 1 - x^T x \geq 0,
\end{array} \right.
\]
where \( g \in \mathbb{R}^n \) and \( H \in \mathbb{R}^{n \times n} \) is symmetric. Suppose \( \lambda^* \) is the Lagrange multiplier. If \( H + \lambda^* I_n \succeq 0 \), show that \( x^* \) is also a global minimizer.

9. Consider the optimization problem
\[
\left\{ \begin{array}{l}
\min \quad f(x) \\
\text{s.t.} \quad g(x) \geq 0,
\end{array} \right.
\]
where \( f, g \) are two continuous functions. If \( x^* \) is a local minimizer and \( g(x^*) > 0 \), show that \( \nabla f(x^*) = 0 \) and \( \nabla^2 f(x^*) \succeq 0 \). Moreover, if \( g(x^*) > 0, \nabla f(x^*) = 0 \) and \( \nabla^2 f(x^*) \succ 0 \), show that \( x^* \) is a strict local minimizer for the constrained optimization.

10. Consider the inequality constrained optimization
\[
\left\{ \begin{array}{l}
\min \quad f(x) \\
\text{s.t.} \quad c_1(x) \geq 0, \ldots, c_m(x) \geq 0,
\end{array} \right.
\]
where \( f \) and all \( c_i \) are twice differentiable functions. Let \( x^* \) be a KKT point with Lagrange multipliers \( \lambda_i^* \geq 0, \ldots, \lambda_m^* \). If the Hessian
\[
H(x) := \nabla^2 f(x) - \sum_{i=1}^m \lambda_i^* \nabla^2 c_i(x) \succeq 0
\]
is psd everywhere, show that \( x^* \) is also a global minimizer.