The first 5 questions are required to be submitted in Gradescope.

1. (10 points) Determine the range of all values for $a$ such that $x_1^2 + x_2^2 + 2ax_1x_2 + 2x_1 - 2x_2$ is bounded from below.

2. (10 points) Determine the range of all values for $a$ such that the origin is a local minimizer of the function $x_1x_2^2 + x_1^2 + 4x_2^2 + 2ax_1x_2$.

3. (10 points) Apply the modified steepest descent method to minimize the function $f(x) := x_1^2 + 3x_2^2 + 2x_1 + 2x_2$. Use the initial point $x^{(0)} = (1, 1)$ and the iterative formula $x^{(k+1)} = x^{(k)} - \alpha_k \nabla f(x^{(k)})$, where $\alpha_k$ is the minimizer of $f(x^{(k)} - \alpha \nabla f(x^{(k)}))$. Compute the first two iterative points $x^{(1)}, x^{(2)}$.

4. (10 points) Apply the Newton’s method to minimize the function $f(x) := x_1^4 + x_2^4 + x_1^2x_2^2 + x_1^2 + x_2^2 + x_1 + x_2$. Use the initial point $x^{(0)} = (1, 1)$. Write down the Newton’s iterative formula. Compute the first two iterative points $x^{(1)}, x^{(2)}$.

5. (10 points) Consider the following function $f(x) := x_1^2 + 2x_1x_2 + 2x_2^2 + x_1 + 2x_2$. At the initial point $x^{(0)} = (1, 1)$, let $p^{(0)} = -\nabla f(x^{(0)})$ be the negative gradient direction. Compute the biggest steplength $\alpha \in (0, 2)$ such that $\frac{f(x^{(0)}) - f(x^{(0)} + \alpha p^{(0)})}{-\alpha \nabla f(x^{(0)})^T p^{(0)}} \geq 0.9$. 


The following questions are optional. Do NOT submit them in Gradescope.

6. Is the function \( q(x) := x_1^2 + x_2^2 + 4x_3^2 + 2x_1x_2 - 4x_1x_3 - x_2x_3 + x_1 + x_2 + x_3 \) bounded from below or not? If it is, find a global minimizer. If it not, find a direction \( p \) such that \( q(\alpha p) \to -\infty \) as \( \alpha \to +\infty \).

7. Let \( A \in \mathbb{R}^{n \times n} \) be symmetric, \( b \in \mathbb{R}^n \) and \( c \in \mathbb{R} \). Show that if
\[
x^T Ax + 2b^T x + c \geq 0 \quad \forall x \in \mathbb{R}^n,
\]
then the block matrix
\[
\begin{bmatrix}
A & b \\
b^T & c
\end{bmatrix} \succeq 0.
\]

8. Show that for all values \( a, b \in \mathbb{R} \), the quartic function
\[
x_1^4 + 2x_1^2x_2^2 + 2x_2^4 + ax_1^2 + bx_2^2
\]
is bounded from below.

9. Let \( H \in \mathbb{R}^{n \times n} \) be a symmetric positive definite matrix. Find the vector \( g \in \mathbb{R}^n \) such that the minimum value of \( g^T x + \frac{1}{2} x^T H x \) is maximum.

10. Consider the quadratic function \( f(x) = g^T x + \frac{1}{2} x^T A x \), where \( A \) is a symmetric matrix. If \( A \) has a negative eigenvalue \( \lambda < 0 \), with the eigenvector \( v \), show that \( \lim_{\alpha \to \infty} f(x_0 + \alpha v) = -\infty \) for all initial points \( x_0 \).