Math 171B Homework Assignment #1

Due Time: 11pm, April 10, 2022, PDT.

The first 5 questions are required to be submitted in Gradescope.

1. (10 points) For the symmetric matrix

\[
A = \begin{bmatrix}
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]

find an orthogonal matrix \(Q\) such that \(Q^T AQ = D\) is diagonal. What are the eigenvalues?

2. (10 points) For the matrix

\[
A = \begin{bmatrix}
1 & -1 & -1 \\
-1 & 2 & 1 \\
-1 & 1 & 3
\end{bmatrix}
\]

determine whether or not \(A\) is psd (or pd)?

If it is, find a square matrix \(L\) such that \(A = LL^T\). If it is not, explain reasons.

3. (10 points) Find the singular value decomposition for the matrix

\[
A = \begin{bmatrix}
0 & 2 \\
-3 & 0
\end{bmatrix}
\]

4. (10 points) Among all vectors \(x := (x_1, x_2)\in \mathbb{R}^2\) with \(\|x\|_1 = 1\), find the vector \(x\) such that the norm \(\left\| \begin{bmatrix} x_1 - 2 \\ x_2 - 2 \end{bmatrix} \right\|_2\) is minimum.

5. (10 points) Find the range of real number \(t\) such that the angle between the vectors \((1, t)\) and \((1, 1)\) is between \(0^\circ\) and \(60^\circ\).

The following questions are optional. Do NOT submit them in Gradescope.

6. For \(x := (x_1, x_2)\in \mathbb{R}^2\), define the function

\[
\|x\| := \sqrt{x_1^2 - 2x_1x_2 + 2x_2^2}.
\]

Show that \(\|x\|\) is a norm function on \(\mathbb{R}^2\) and give a formula for the dual norm \(\|x\|_*\).

7. If \(A \in \mathbb{R}^{n \times n}\) is symmetric positive definite, show that its inverse \(A^{-1}\) is also positive definite.

If \(B \in \mathbb{R}^{n \times n}\) is invertible, show that \(B^TB\) is positive definite.

8. If \(A \in \mathbb{R}^{n \times n}\) is symmetric, show that its biggest singular value equals the maximum absolute value of its eigenvalues.

9. For each \(B \in \mathbb{R}^{m \times n}\), show that its biggest singular value \(\sigma_1\) can be expressed as

\[
\sigma_1 = \max_{0 \neq x \in \mathbb{R}^m \atop 0 \neq y \in \mathbb{R}^n} \frac{x^T By}{\|x\|_2 \|y\|_2}.
\]

10. Let \(C \in \mathbb{R}^{n \times n}\) be symmetric positive definite. For all \(x, y \in \mathbb{R}^n\), show that

\[
|x^T C y| \leq \sqrt{x^T C x} \sqrt{y^T C y}.
\]