Solutions to the LMIPO Exercises on Duality

Angelica Cueto, Philipp Rostalski and Bernd Sturmfels

March 23, 2010

1. The convex hull P of three pairwise touching circles in $\mathbb{R}^2$ is shown in Figure 1(a). Its boundary consists of three linear pieces (in red) and three quadratic pieces (in blue), so the algebraic boundary $\partial_a P$ is a curve of degree 9. The dual body P^* is the intersection of the dual of each of our three circles. The boundary is the piecewise quadratic curve shown in Figure 1(b), so the algebraic boundary $\partial_a P^*$ is a curve of degree 6.

The convex hull of four pairwise touching spheres in $\mathbb{R}^3$ is shown in Figure 2. Its boundary consists of $4 + 6 + 4 = 14$ distinct surface patches. Namely, there are parts of the four original spheres, next there are the cylinders wrapping around two of the spheres, and finally there are the planes touching three of the spheres. The dual convex body is drawn in green.

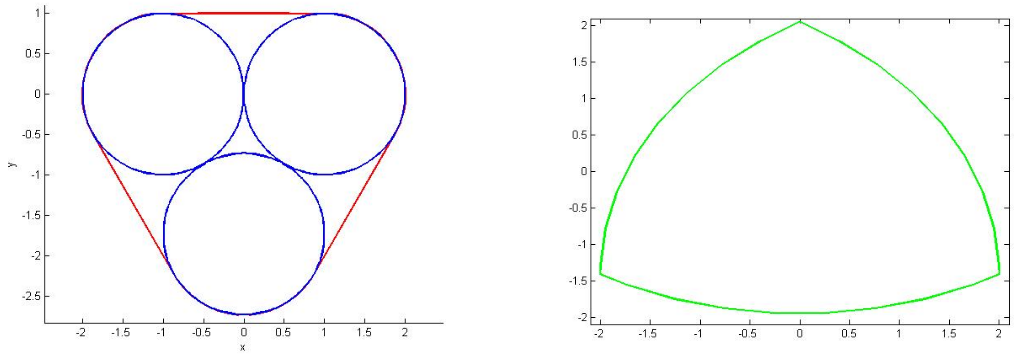
We now discuss the case of four pairwise touching circles in 3-space. A nice symmetric representation of this reducible curve is given by the ideal

$$\langle\, a\,c\,g\,t\ ,\ a^2 + c^2 + g^2 + t^2 - 2ac - 2ag - 2at - 2cg - 2ct - 2gt\,\rangle,$$

where the variety of that ideal is to be taken inside the probability simplex

$$\Delta_3 \ = \ \{\, (a, c, g, t) \in \mathbb{R}^4_{\geq 0} \ :\ a + c + g + t = 1 \,\}.$$

The boundary of this convex body looks combinatorially like a 3-dimensional polytope with 18 vertices, 36 edges and 20 cells. Eight of the 20 cells are actually flat facets. First, there are the planes of the circles themselves. For



(a) The convex hull (b) The dual convex body

Figure 1: Convex hull of three circles in $\mathbb{R}^2$

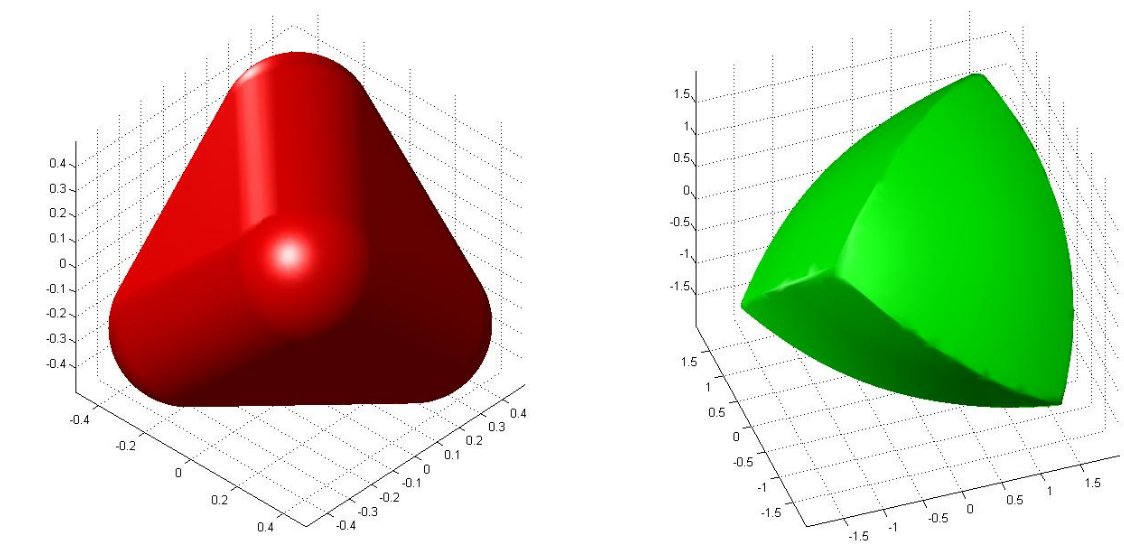


Figure 2: The convex hull of four spheres in $\mathbb{R}^3$ and its dual convex body

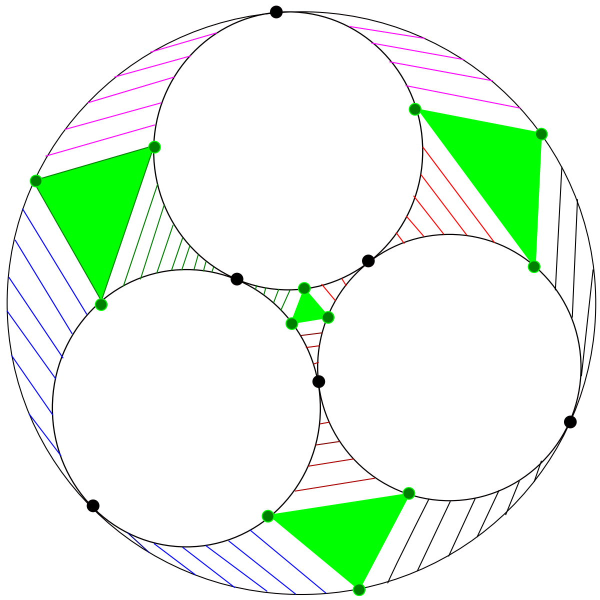


Figure 3: Schlegel diagram of the convex hull of four pairwise touching circles

instance, the facet in the plane $t = 0$ is the disk $\{a^2+c^2+g^2 \leq 2ac+2ag+2cg\}$. Second, there are four triangle facets, formed by the unique planes that are tangent to any three of the circles. The equations of these facet planes are

$$\begin{aligned} P_a &= -a + 2c + 2g + 2t, \qquad & P_c &= 2a - c + 2g + 2t, \\ P_g &= 2a + 2c - g + 2t, \qquad & P_t &= 2a + 2c + 2g - t. \end{aligned}$$

The remaining 12 cells in the boundary of our convex body are quadratic surface patches that arise from the pairwise convex hull of any two of the four circles. This results in 6 quadratic surfaces each of which contributes two triangular cells to the boundary. The equations of these six surfaces are

$$\begin{aligned} Q_{ac} &= a^2 + c^2 + g^2 + t^2 + 2(ac - ag - cg - at - ct - gt)\,, \\ Q_{ag} &= a^2 + c^2 + g^2 + t^2 - 2(ac - ag + cg + at + ct + gt)\,, \\ Q_{ag} &= a^2 + c^2 + g^2 + t^2 - 2(ac + ag - cg + at + ct + gt)\,, \\ Q_{cg} &= a^2 + c^2 + g^2 + t^2 - 2(ac + ag + cg - at + ct + gt)\,, \\ Q_{ct} &= a^2 + c^2 + g^2 + t^2 - 2(ac + ag + cg + at - ct + gt)\,, \\ Q_{gt} &= a^2 + c^2 + g^2 + t^2 - 2(ac + ag + cg + at + ct - gt)\,. \end{aligned}$$

Each circle is subdivided into six arcs of equal length. Three of the nodes arise from intersections with other circles, and the three other nodes are the intersections with the planes P_a, P_c, P_g, P_t. This accounts for all 18 vertices and 24 "edges" that are actually arcs. The other 12 edges of our "polytope" are true edges: they arise from the four triangles. We summarize this description by the *Schlegel diagram* in Figure 3. The green triangles are the four triangular facets. The 12 cells corresponding to the six quadratic surfaces are the 12 ruled cells in the diagram, and they come in pairs according to the six different colors. The six intersection points among the 4 circles are indicated by black dots, whereas the remaining twelve vertices correspond to the twelve green dots with are vertices of our four green triangles.

2. The curve dual X^* to the TV screen $X = \{(x,y) \in \mathbb{R}^2 : x^4 + y^4 = 1\}$ has degree 12. The irreducible polynomial which defines X^* is found to be

$$f(x,y) \;=\; x^{12} + 3x^8y^4 + 3x^4y^8 + y^{12} - 3x^8 + 21x^4y^4 - 3y^8 + 3x^4 + 3y^4 - 1.$$

The interior of the curve X is the unit disk in the L_4-norm on $\mathbb{R}^2$, and the interior of the dual curve X^* is the unit disk in the dual norm which is $L_{4/3}$. The curve $\{f = 0\}$ is the Zariski closure of the $L_{4/3}$-*circle* $\{(x,y) \in \mathbb{R}^2 : |x|^{4/3} + |y|^{4/3} = 1\}$ which is computed by the following line in `Macaulay 2`:

```
R = QQ[a,b,x,y];   eliminate({a,b},ideal(a^4+b^4-1,a^3-x,b^3-y))
```

The following code in `Macaulay 2` computes the dual to any projective variety. Here we run it to produce the homogenization of the polynomial $f(x,y)$:

```
S = QQ[x,y,z,X,Y,Z];  d = 3;

makedual = I -> (e = codim I;
pairing = first sum(d,i->(gens S)_i*(gens S)_{i+d});
J=saturate(I+minors(e+1,submatrix(jacobian(I+ideal(pairing)),{0..d-1},)),
  minors(e,submatrix(jacobian(I),{0..d-1},)));
  eliminate((gens S)_{0..d-1},J));

makedual ideal(x^4+y^4-z^4)
```

3. Let X be the variety consisting of all $2 \times 2 \times 2$-tensors of rank one. As a projective variety, X is the Segre embedding of the product of projective lines $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^7$. Its ideal is minimally generated by nine quadrics:

$$\begin{aligned}\langle\; & x_{101}x_{110} - x_{100}x_{111},\; x_{011}x_{110} - x_{010}x_{111},\; x_{001}x_{110} - x_{000}x_{111},\\ & x_{011}x_{101} - x_{001}x_{111},\; x_{010}x_{101} - x_{000}x_{111},\; x_{011}x_{100} - x_{000}x_{111},\\ & x_{010}x_{100} - x_{000}x_{110}\;, x_{001}x_{100} - x_{000}x_{101},\; x_{001}x_{010} - x_{000}x_{011}\;\rangle.\end{aligned}$$

The dual variety X^* is the hypersurface defined by the *hyperdeterminant*

$$\begin{aligned}& x_{011}^2x_{100}^2 - 2x_{010}x_{011}x_{100}x_{101} - 2x_{001}x_{011}x_{100}x_{110} - 2x_{001}x_{010}x_{101}x_{110}\\ & + 4x_{000}x_{011}x_{101}x_{110} + x_{001}^2x_{110}^2 + 4x_{001}x_{010}x_{100}x_{111} - 2x_{000}x_{011}x_{100}x_{111}\\ & + x_{010}^2x_{101}^2 - 2x_{000}x_{010}x_{101}x_{111} - 2x_{000}x_{001}x_{110}x_{111} + x_{000}^2x_{111}^2.\end{aligned}$$

The transformation from X^* to X is easily done with the above `Macaulay 2` code for computing the dual to any projective variety. For the forward direction from X to X^*, one can also use this code but modifications are needed to ensure timely termination: namely, we recommend replacing the `saturate` command by a single ideal quotient to ensure rapid termination. In any case, it is possible to verify the biduality $(X^*)^* = X$ with a direct computation.

The best conceptual understanding of this example requires an excursion into the theory of A-discriminants, which was introduced in the celebrated 1994 book *Discriminants, Resultants and Multidimensional Resultants* by by Gel'fand, Kapranov and Zelevinsky. In our example, the configuration A consists of the vertices of the standard 3-cube, and our hypersurface X^* is the corresponding A-discriminant. The following `Macaulay 2` code will compute the A-discriminant for any configuration A of lattice points:

```
Adiscr = A -> (
L = entries(A);
d = #L;   n = #L#0;
R = QQ[x_1..x_n,t_0..t_d];
f = (i,j) -> t_(i+1)^(L#(i)#(j));
Tt = transpose(table(d,n,f));
p = homogenize(matrix{apply(Tt,product)}*transpose(matrix{{x_1..x_n}}),t_0);
grad = homogenize(diff(matrix{{t_1..t_d}},p),t_0);
I = ideal(p,grad);
J_0 = I;  k = 0;
for i from 0 to d when J_i!=0 do (
J_(i+1) = eliminate(t_i,J_i); k = k+1;);
for j from 0 to d do J_(k-1) = sub(J_(k-1),t_j=>1); J_(k-1));
```

```
A = matrix{{1,1,1,1,0,0,0,0},{1,1,0,0,1,1,0,0},{1,0,1,0,1,0,1,0},
           {0,0,0,0,1,1,1,1},{0,0,1,1,0,0,1,1},{0,1,0,1,0,1,0,1}}

Adiscr(A)
```

Note that the above 6×8-matrix `A` does indeed represent the 3-cube.

4. The three-dimensional spectrahedron

$$P \quad = \quad \left\{ (x,y,z) \in \mathbb{R}^3 \,:\, \begin{pmatrix} 1 & x & 0 & x \\ x & 1 & y & 0 \\ 0 & y & 1 & z \\ x & 0 & z & 1 \end{pmatrix} \succeq 0 \right\}$$

is shown in Figure 4. It contains no matrices of rank 1, and it contains precisely four matrices of rank 2. Indeed, the ideal of 3×3-minors of the matrix has precisely the following four common zeros:

$$(x,y,z) \quad = \quad \frac{1}{\sqrt{2}}(1,1,-1),\ \frac{1}{\sqrt{2}}(-1,-1,1),\ \frac{1}{\sqrt{2}}(1,-1,1),\ \frac{1}{\sqrt{2}}(-1,1,-1).$$

Those four points lie in the plane $\{y+z=0\}$ in $\mathbb{R}^3$, and our spectrahedron P looks like a veritable pillow. The four coplanar corners form a square whose edges are also edges of P. All other faces of P are exposed points. These 0-dimensional faces come in two protrusions, one above the plane $\{y+z\}$ and one below that plane. The dual convex body P^* appears on the right in Figure 4. The pillow P has four 1-dimensional faces, four singular 0-dimensional faces and two infinite families of smooth 0-dimensional faces. The corresponding dual faces of P^* have dimensions 0, 2 and 0 respectively.

The following alternative approach was presented by Laurent Lessard, who used it sketch Figure 4. Using *Schur complements*, we derive a description of the spectrahedron P by a quadratic matrix inequality as follows:

$$P \;=\; \big\{ (x,y,z) \in \mathbb{R}^3 : \begin{pmatrix} 1-2x^2 & -x(y+z) \\ -x(y+z) & 1-y^2-z^2 \end{pmatrix} \succeq 0 \big\}.$$

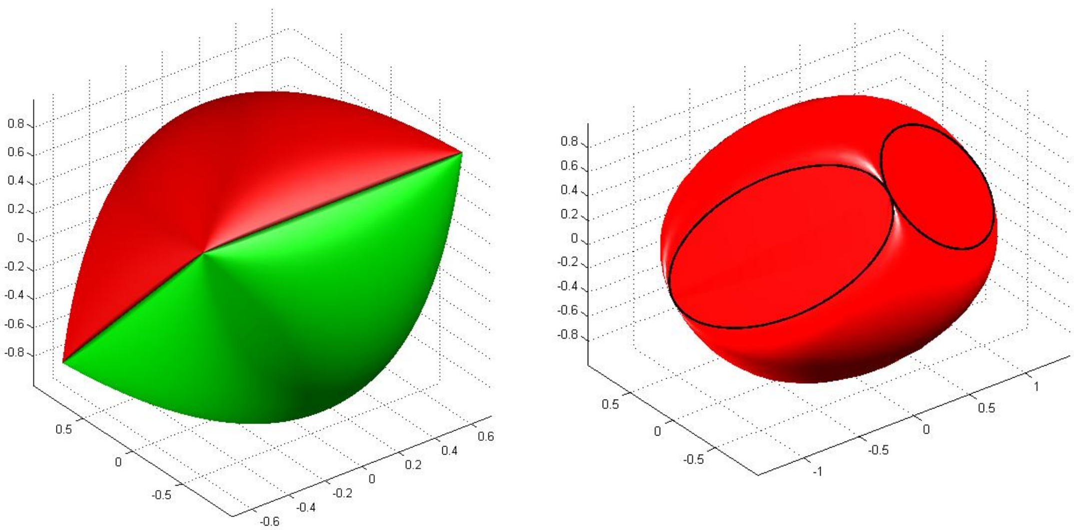


Figure 4: A 3-dimensional spectrahedron and its dual convex body.

Therefore, we can describe P by three polynomial inequalities:

$$1-2x^2 \geq 0 \quad ; \quad 1-y^2-z^2 \geq 0 \quad ; \quad (1-2x^2)(1-y^2-z^2)-x^2(y+z)^2 \geq 0. \quad (1)$$

The middle inequality in (1) describes a cylinder over the circle $y^2 + z^2 \leq 1$ along the x-axis. The first inequality tells us that x must lie in the interval $I = [-1/\sqrt{2}, 1/\sqrt{2}]$. If we intersect this bounded cylinder with the plane $y + z = 0$ we obtain a square which lies in P. This is is precisely the square we described in the previous paragraph.

We know analyze the third inequality in (1) as we vary x along the interval I. For fix $x = x_0$, this inequality is given by a quadratic polynomial in y and z. If $x_0 = 0$, the inequality describes the circle $y^2 + z^2 \leq 1$. As we move x_0 through I towards its boundary, this inequality will correspond to an ellipsis joining the points $(x_0, 1/\sqrt{2}, -1/\sqrt{2})$ and $(x_0, -1/\sqrt{2}, 1/\sqrt{2})$ in the boundary of P. As x_0 approaches the two limiting points (the roots of the polynomial $1 - 2x^2$), this family of ellipses degenerates to two line segments satisfying $y + z = 0$, which are edges of the square contained in P.

5. We consider the trigonometric space curve C which has the parametrization $x = \cos(\theta)$, $y = \cos(2\theta)$, $z = \sin(3\theta)$. This is an algebraic curve of degree six whose implicit representation as the intersection of two surfaces equals

$$2x^2 - y - 1 \;=\; 4y^3 + 2z^2 - 3y - 1 \;=\; 0. \tag{2}$$

To compute the convex hull of this curve, we use the methods described in Sections 3 and 4 of the paper [K. Ranestad and B. Sturmfels: *On the convex hull of a space curve*, `arXiv:0912.2986`]. The edge surface has three irreducible components. Besides the the quartic and the cubic in (2), there is one non-trivial surface of degree 16. The equation of that surface equals

$$\begin{aligned}
&- 419904x^{14}y^2 + 664848x^{12}y^4 - 419904x^{10}y^6 + 132192x^8y^8 - 20736x^6y^{10} \\
&+ 1296x^4y^{12} - 46656x^{14}z^2 + 373248x^{12}y^2z^2 - 69984x^{10}y^4z^2 - 22464x^8y^6z^2 \\
&+ 4320x^6y^8z^2 + 31104x^{12}z^4 + 5184x^{10}y^2z^4 + 4752x^8y^4z^4 + 1728x^{10}z^6 \\
&+ 699840x^{14}y - 46656x^{12}y^3 - 902016x^{10}y^5 + 694656x^8y^7 - 209088x^6y^9 \\
&- 1150848x^{10}y^3z^2 + 279936x^8y^5z^2 + 17280x^6y^7z^2 - 4032x^4y^9z^2 - 98496x^{10}yz^4 \\
&+ 27072x^4y^{11} - 1152x^2y^{13} - 419904x^{12}yz^2 - 25920x^8y^3z^4 - 4608x^6y^5z^4 \\
&- 1728x^8yz^6 - 291600x^{14} - 169128x^{12}y^2 - 256608x^{10}y^4 + 956880x^8y^6 \\
&- 618192x^6y^8 + 148824x^4y^{10} - 13120x^2y^{12} + 256y^{14} + 392688x^{12}z^2 + 671976x^{10}y^2z^2 \\
&+ 1454976x^8y^4z^2 - 292608x^6y^6z^2 - 4272x^4y^8z^2 + 1016x^2y^{10}z^2 \\
&- 116208x^{10}z^4 + 135432x^8y^2z^4 + 18144x^6y^4z^4 + 1264x^4y^6z^4 - 5616x^8z^6 \\
&+ 504x^6y^2z^6 - 1108080x^{12}y + 925344x^{10}y^3 + 215136x^8y^5 - 672192x^6y^7 \\
&+ 331920x^4y^9 - 54240x^2y^{11} + 2304y^{13} + 273456x^{10}yz^2 + 282528x^8y^3z^2 \\
&- 1185408x^6y^5z^2 + 149376x^4y^7z^2 - 368x^2y^9z^2 - 32y^{11}z^2 + 273456x^8yz^4 \\
&- 67104x^6y^3z^4 - 4704x^4y^5z^4 - 64x^2y^7z^4 + 4752x^6yz^6 - 32x^4y^3z^6 + 747225x^{12} \\
&+ 636660x^{10}y^2 - 908010x^8y^4 - 65340x^6y^6 + 291465x^4y^8 - 101712x^2y^{10} \\
&+ 8256y^{12} - 818100x^{10}z^2 - 1405836x^8y^2z^2 - 905634x^6y^4z^2 + 583824x^4y^6z^2 \\
&- 39318x^2y^8z^2 + 368y^{10}z^2 + 193806x^8z^4 - 282996x^6y^2z^4 + 15450x^4y^4z^4 \\
&+ 716x^2y^6z^4 + y^8z^4 + 6876x^6z^6 - 1140x^4y^2z^6 + 2x^2y^4z^6 + x^4z^8 + 507384x^{10}y \\
&- 809568x^8y^3 + 569592x^6y^5 - 27216x^4y^7 - 71648x^2y^9 + 13952y^{11} + 555768x^8yz^2 \\
&+ 869040x^6y^3z^2 + 688512x^4y^5z^2 - 154128x^2y^7z^2 + 4416y^9z^2 - 343224x^6yz^4 \\
&+ 127360x^4y^3z^4 - 1656x^2y^5z^4 - 64y^7z^4 - 4536x^4yz^6 + 48x^2y^3z^6 - 775170x^{10} \\
&- 191808x^8y^2 + 599022x^6y^4 - 245700x^4y^6 + 31608x^2y^8 + 7872y^{10} + 765072x^8z^2 \\
&+ 589788x^6y^2z^2 - 66066x^4y^4z^2 - 234252x^2y^6z^2 + 16632y^8z^2 - 173196x^6z^4 \\
&+ 248928x^4y^2z^4 - 26158x^2y^4z^4 - 32y^6z^4 - 3904x^4z^6 + 804x^2y^2z^6 + 2y^4z^6 \\
&- 2x^2z^8 + 5832x^8y + 98280x^6y^3 - 219456x^4y^5 + 72072x^2y^7 - 8064y^9 - 724032x^6yz^2 \\
&- 515760x^4y^3z^2 - 99672x^2y^5z^2 + 29976y^7z^2 + 225048x^4yz^4 - 76216x^2y^3z^4 + 1912y^5z^4 \\
&+ 1696x^2yz^6 - 32y^3z^6 + 411345x^8 - 66096x^6y^2 - 62532x^4y^4 + 29388x^2y^6 - 11856y^8 \\
&- 365346x^6z^2 + 19812x^4y^2z^2 + 104922x^2y^4z^2 + 24636y^6z^2 + 85090x^4z^4 - 104580x^2y^2z^4 \\
&+ 8282y^4z^4 + 1014x^2z^6 - 144y^2z^6 + z^8 - 39744x^6y + 61992x^4y^3 + 2304x^2y^5 + 576y^7 \\
&+ 305328x^4yz^2 + 86640x^2y^3z^2 + 960y^5z^2 - 73480x^2yz^4 + 16024y^3z^4 - 200yz^6 \\
&- 114966x^6 + 24120x^4y^2 - 5958x^2y^4 + 6192y^6 + 85494x^4z^2 - 39696x^2y^2z^2 \\
&- 11970y^4z^2 - 21610x^2z^4 + 16780y^2z^4 - 94z^6 - 3672x^4y - 11024x^2y^3 + 272y^5 \\
&- 46904x^2yz^2 - 4632y^3z^2 + 9368yz^4 + 15246x^4 - 84x^2y^2 - 1908y^4 - 6892x^2z^2 \\
&+ 2204y^2z^2 + 2215z^4 + 3216x^2y + 168y^3 + 904yz^2 - 664x^2 + 292y^2 - 282z^2 - 96y + 9
\end{aligned}$$

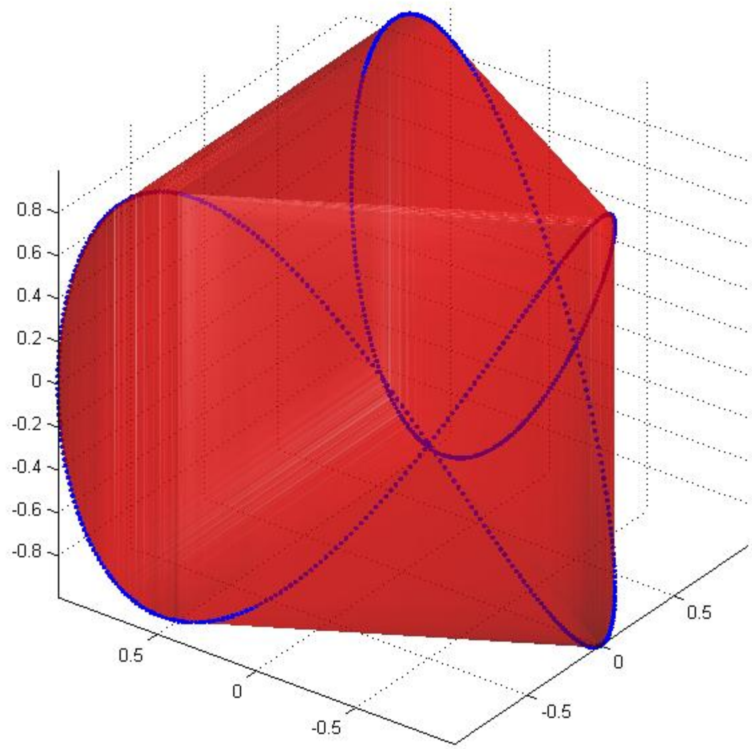


Figure 5: The convex hull of the curve $C = (\cos(\theta), \cos(2\theta), \sin(3\theta))$

In addition to the edge surface, the boundary of $\mathrm{conv}(C)$ also may have 2-dimensional facets, corresponding to planes that are tritangent to C. Our example has two such triangles in its boundary, seen on the top and on the bottom of Figure 5. The planes spanned by the twose triangle facets are

$$5 + 4y - z = 0 \quad \text{and} \quad 5 + 4y + z = 0.$$

The convex hull of this curve is also a projection of a spectrahedron:

$$P \quad = \quad \Bigg\{ (x,y,z) \in \mathbb{R}^3 \, : \exists \, (t,u,v,w) \in \mathbb{R}^4 : \\ M = \begin{pmatrix} 1 & x+iu & y+iv & w+iz \\ x-iu & 1 & x+iu & y+iv \\ y-iv & x-iu & 1 & x+iu \\ w-iz & y-iv & x-iu & 1 \end{pmatrix} \succeq 0 \Bigg\}$$

where $i = \sqrt{-1}$ is the imaginary unit. Note that if you (or your SDP solver) prefer a description of this spectrahedron over the real numbers, you can simply replace $M \succeq 0$ by the following linear matrix inequality of size 8×8:

$$\tilde{M} \;=\; \begin{pmatrix} \mathrm{Re}(M) & \mathrm{Im}(M) \\ -\mathrm{Im}(M) & \mathrm{Re}(M) \end{pmatrix} \succeq 0.$$

Venkat Chandrasekaran presented the following method for deriving an SDP representation of the convex body dual to $\mathrm{conv}(C)$. That dual body corresponds to points $(a,b,c,d) \in \mathbb{R}^4$ such that $a\cos\theta + b\cos(2\theta) + c\sin(3\theta) + d \geq 0$ for some choice of θ. Using the complex change of coordinates $z = exp(i\theta)$, we have $\cos(\theta) = \frac{z+z^{-1}}{2}$, $\cos(2\theta) = \frac{z^2+z^{-2}}{2}$ and $\sin(3\theta) = \frac{z^3-z^{-3}}{2i}$. Therefore, our positivity condition becomes:

$$a\frac{z+z^{-1}}{2} + b\frac{z^2+z^{-2}}{2} + c\frac{z^3-z^{-3}}{2i} + d \;\geq\; 0$$

over the complex unit circle. We can translate this condition into a bilinear form given by a 4×4- positive semidefinite *Hermitian matrix* H:

$$a\frac{z+z^{-1}}{2} + b\frac{z^2+z^{-2}}{2} + c\frac{z^3-z^{-3}}{2i} = \begin{pmatrix} 1 \\ z^{-1} \\ z^{-2} \\ z^{-3} \end{pmatrix}^T \cdot\; H \;\cdot \begin{pmatrix} 1 \\ z^{-1} \\ z^{-2} \\ z^{-3} \end{pmatrix}.$$

The diagonal of the matrix H will determine the point (a,b,c,d):

$$\begin{cases} H_{1,1} + H_{2,2} + H_{3,3} + H_{4,4} = d, \\ H_{1,2} + H_{2,3} + H_{3,4} = \frac{a}{2} \\ H_{1,3} + H_{2,4} = \frac{b}{2} \\ H_{1,4} = \frac{c}{2i} \end{cases}$$

subject to the condition $H \succeq 0$ where H is Hermitian.

6. We regard the given problem as the dual semidefinite program:

$$\text{Maximize } \; ax + by + cz \; \text{ subject to } \; \begin{pmatrix} 1 & x & 0 & x \\ x & 1 & y & 0 \\ 0 & y & 1 & z \\ x & 0 & z & 1 \end{pmatrix} \succeq 0.$$

The corresponding primal semidefinite program equals:

Minimize $\mathrm{trace}(x_{ij})$ s.t. $(x_{ij}) \succeq 0$ and $2x_{12}+2x_{14}+a = 2x_{23}+b = 2x_{34}+c = 0$.

The product of the primal and dual optimal matrices is zero, and their ranks are $(1,3)$ or $(2,2)$. In the former case the optimal value T is determined by

$$(b^2+2bc+c^2)\cdot T^2 - a^2b^2 - a^2c^2 - b^4 - 2b^2c^2 - 2bc^3 - c^4 - 2b^3c \;=\; 0.$$

In the latter case it is given by four corners of the pillow in Exercise 4:

$$(2T^2-a^2+2ab-b^2+2bc-c^2-2ac)\cdot(2T^2-a^2-2ab-b^2+2bc-c^2+2ac) \;=\; 0.$$

These polynomials in T were generated by the following `Macaulay 2` code:

```
R = QQ[ x,y,z, u1,u2,u3,u4,u5,u6,u7, T, a,b,c ];

dm = matrix {{1, x, 0, x},
             {x, 1, y, 0},
             {0, y, 1, z},
             {x, 0, z, 1}};

pm = matrix {{ 2*u1   ,  2*u2 ,   2*u3 , -2*u2-a },
             { 2*u2   ,  2*u4 ,    -b  ,   2*u5  },
             { 2*u3   ,   -b  ,   2*u6 ,    -c   },
             {-2*u2-a ,  2*u5 ,    -c  ,   2*u7  }};

ideal(a*x+b*y+c*z-T)+minors(1,dm*pm)+minors(3,pm)+minors(3,dm);
decompose eliminate({x,y,z,u1,u2,u3,u4,u5,u6,u7}, oo )

ideal(a*x+b*y+c*z-T)+minors(1,dm*pm)+minors(2,pm)+minors(4,dm);
decompose eliminate({x,y,z,u1,u2,u3,u4,u5,u6,u7}, oo )
```

7. Let us start this investigation with some `MATLAB` experiments using the `Yalmip` toolbox; see [J. Löfberg. *YALMIP : A Toolbox for Modeling and Optimization in* `MATLAB`. In Proceedings of the CACSD Conference, Taipei, Taiwan, 2004]. The code following defines a single 10×10 SDP problem with large integers on the off-diagonal elements and variables along the diagonal.

```
x = sdpvar(10,1); % Define a 10 x 1 (vector) variable

% Create a random spectrahedron with large integers on the off-diagonals
aux = randint(10,10,[50,1000]);
A = aux+aux'-diag(diag(2*aux))+diag(x);

% Solve the SDP (you need to have an SDP solver installed)
solvesdp(set(A>0),trace(A));

% Simple rank estimation
rk(i) = sum(svd(double(A))>1e-5);
```

Note that the rank estimation here is done by simply counting all singular values with value larger than some threshold (here: 1e-5). Running the above `MATLAB` code in a loop, with 100,000 different random matrices as input, we obtain the (approximate) rank distribution which is shown in Table 1:

rank	frequency
6	0.005%
7	16.509%
8	79.619%
9	3.867%

Table 1: Rank distribution based on 100,000 samples.

Using more careful mathematical analysis, it can be proved that all four listed ranks will be attained with positive probability. To show this, we begin with the observation that our SDP problem is dual to the problem of maximizing a linear functional over the *elliptope* $\mathcal{E}_{10\times10}$. This is the set of all correlation matrices of size 10×10. According to Corollary 3.2 in [M. Laurent and S. Poljak: On the facial structure of the set of correlation matrices, *SIAM J. Matrix. Anal. Appl.* **17** (1996) 530–547], the exposed points of $\mathcal{E}_{10\times10}$ are matrices of ranks 1, 2, 3 and 4. Each of these four cases is attained with positive probability for some linear functional. By complementarity, the dual solutions will have rank 9, 8, 7 and 6, and these solve our original SDP.

It would be desirable to derive formulas for the algebraic degree of optimizing a linear functional over the elliptope $\mathcal{E}_{n\times n}$, but at present this is an open problem for $n \geq 5$. The case $n = 10$ is out of reach for symbolic computation. That algebraic degree would specify the degree of the field extension of $\mathbb{Q}$ over which the solution matrices to our given SDP are defined.

An upper bound is given by the quantity $\delta(10, 10, r)$, for $r = 6, 7, 8, 9$. This is the algebraic degree introduced in [J. Nie, K. Ranestad and B. Sturmfels: The algebraic degree of semidefinite programming, *Mathematical Programming* **122** (2010) 379–405]. An explicit formula can be found in [H.-C. Graf von Bothmer, K. Ranestad: A general formula for the algebraic degree in semidefinite programming, *Bull.Lond.Math.Soc.* **41** (2009) 193–197].

8. We are interested in the *spectrahedral shadow* $\mathcal{C}_{\mathcal{L}} \subset \mathbb{R}^8$ that is defined by

$$\exists (x,y) \in \mathbb{R}^2 \, : \, \begin{pmatrix} u_{11} & u_{12} & x & u_{14} \\ u_{12} & u_{22} & u_{23} & y \\ x & u_{23} & u_{33} & u_{34} \\ u_{14} & y & u_{34} & u_{44} \end{pmatrix} \succeq 0.$$

The algebraic boundary of $\mathcal{C}_{\mathcal{L}}$ has five irreducible components. The first four are the principal 2×2-minors that are indexed by the edges of the 4-cycle

$$\big(u_{11}u_{22} - u_{12}^2\big)\big(u_{22}u_{33} - u_{23}^2\big)\big(u_{33}u_{44} - u_{34}^2\big)\big(u_{11}u_{44} - u_{14}^2\big) \;=\; 0.$$

The fifth component is defined by the following polynomial of degree eight:

$$\begin{aligned}
&u_{11}^2u_{22}^2u_{34}^4 - 2u_{11}^2u_{22}u_{23}^2u_{34}^2u_{44} - 2u_{11}u_{14}^2u_{22}u_{23}^2u_{33}u_{44} - 2u_{11}u_{12}^2u_{22}u_{33}u_{34}^2u_{44} \\
&- 2u_{11}u_{12}^2u_{23}^2u_{33}u_{44}^2 + 4u_{11}u_{12}^2u_{23}^2u_{34}^2u_{44} + 8u_{11}u_{12}u_{14}u_{22}u_{23}u_{33}u_{34}u_{44} \\
&- 4u_{11}u_{12}u_{14}u_{22}u_{23}u_{34}^3 - 4u_{11}u_{12}u_{14}u_{23}^3u_{34}u_{44} - 2u_{11}u_{14}^2u_{22}^2u_{33}u_{34}^2 + u_{14}^4u_{22}^2u_{33}^2 \\
&+ 4u_{11}u_{14}^2u_{22}u_{23}^2u_{34}^2 + u_{12}^4u_{33}^2u_{44}^2 - 4u_{12}^3u_{14}u_{23}u_{33}u_{34}u_{44} - 2u_{12}^2u_{14}^2u_{22}u_{33}^2u_{44} \\
&+ 4u_{12}^2u_{14}^2u_{22}u_{33}u_{34}^2 + 4u_{12}^2u_{14}^2u_{23}^2u_{33}u_{44} - 4u_{12}u_{14}^3u_{22}u_{23}u_{33}u_{34} + u_{11}^2u_{23}^4u_{44}^2.
\end{aligned}$$

We obtain this by eliminating x and y from the 3×3-minors of the matrix. This is the $m = 4$ instance of the homogeneous cycle polynomial Γ_m which is conjectured to have degree $m \cdot 2^{m-3}$. For more information see Section 4.2 in the paper [B. Sturmfels and C. Uhler: *Multivariate Gaussians, Semidefinite Matrix Completion, and Convex Algebraic Geometry*, `arXiv:0906.3529`].

9. The polynomial $p(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$ is non-negative on the real line. The sums of squares representations of $p(x)$ have the form

$$p(x) = (1, x, x^2, x^3) \cdot M \cdot (1, x, x^2, x^3)^T,$$

where
$$M = \begin{pmatrix} 1 & 1/2 & u & v \\ 1/2 & 1-2u & 1/2-v & w \\ u & 1/2-v & 1-2w & 1/2 \\ v & w & 1/2 & 1 \end{pmatrix} \succeq 0. \tag{3}$$

The number of squares is the rank of the matrix M, so we need to find all vectors (u, v, w) such that $\mathrm{rank}(M) \leq 2$ and $M \succeq 0$. There are ten complex solutions of which four are real. Precisely one solution is rational, namely,

$$u = -1/2\,, \quad v = -1\,, \quad w = -1/2.$$

The Cholesky factorization of that rank 2 matrix yields the representation

$$p(x) = \frac{7}{4}x^2(x+1)^2 + \frac{1}{4}(x-1)^2 + (2x^2+3x+2)^2.$$

The linear matrix inequality (3) defines a 3-dimensional spectrahedron. This spectrahedron is shown in Figure 6. Its boundary consists of SOS representations with rank at most 3, whereas the interior of the convex body corresponds to representations with four summands. Each one of the rank 2 real solutions gives a corner of this convex body, as seen in Figure 6. The rightmost corner is the rational SOS representation of $p(x)$ given above.

The algebraic boundary of our spectrahedron is the quartic hypersurface defined by the vanishing of the determinant of the matrix in (3), which equals

$$\begin{aligned}\det(M) = {} & v^4 - 2uv^2w + u^2w^2 + 2u^3 - 2u^2v + 2uv^2 - v^3 - uvw + 2v^2w \\ & - 2vw^2 + 2w^3 - u^2 - 5/4v^2 + 7/2uw - w^2 - u + 3/4v - w + 5/16.\end{aligned}$$

The analytic center of this spectrahedron is the unique point $(\hat{u}, \hat{v}, \hat{w})$ at which the determinant of M is maximized. That point has coordinates

$$\hat{u} = \hat{w} = -0.85720945... \quad \text{and} \quad \hat{v} = -0.25999478....$$

It has degree 3 over $\mathbb{Q}$, since its algebraic representation is the prime ideal

$$\langle u - w,\ 4v - 12w^2 - 8w + 3\,,\ 36w^3 + 12w^2 - 15w + 1 \rangle \ \subset\ \mathbb{Q}[u, v, w].$$

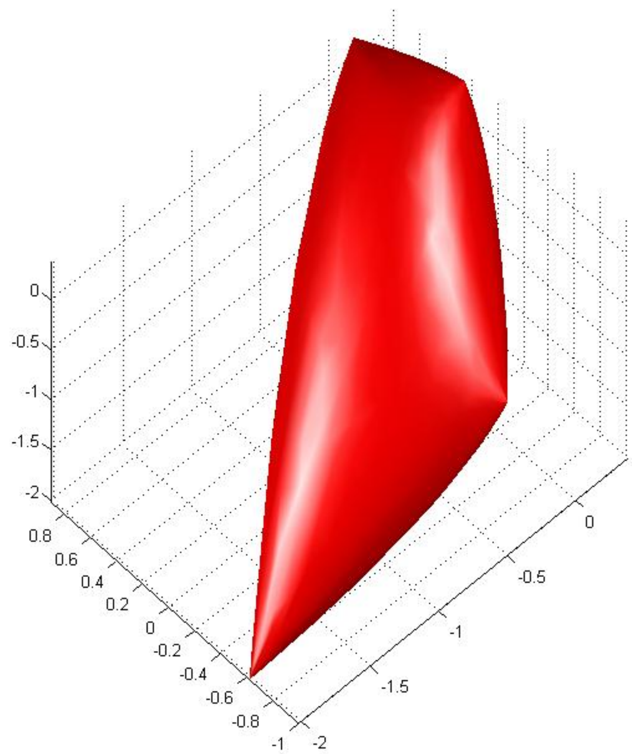


Figure 6: The spectrahedron of all possible SOS representations of $p(x)$.

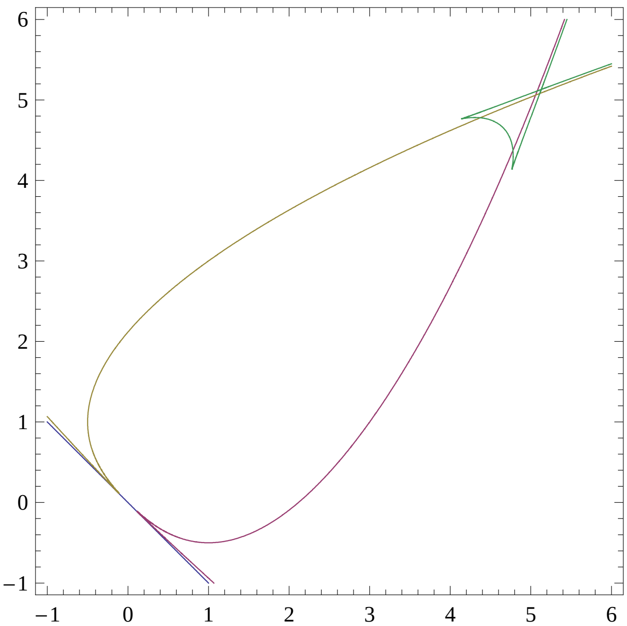


Figure 7: The discriminant of $f_{a,b}(x,y)$ defines a curve in the (a,b)-plane

10. This problem is meant as an advertisement for our host's most recent paper [J. Nie: *Discriminants and Nonnegative Polynomials*, `arXiv:1002.2230`].

Given any particular choice of real parameters a and b, the polynomial

$$f_{a,b}(x,y) \quad = \quad x^4 + y^4 + a(x^3 + y^2) + b(y^3 + x^2) \,+\, a + b$$

is non-negative on $\mathbb{R}^2$ if and only if it is a sum of squares. This follows from Hilbert's classical results on sums of squares. This condition defines a closed convex region $\mathcal{C}$ in the (a,b)-plane. It is non-empty because $(0,0) \in \mathcal{C}$. Its algebraic boundary $\partial_a(\mathcal{C})$ is derived from the A-*discriminant* of a generic bivariate polynomial with support $A = \{(4,0),(0,4),(3,0),(0,2),(0,3),(2,0),(0,0)\}$. This A-discriminant is a homogeneous irreducible polynomial of degree 24 in the nine coefficients. What we are interested in here is the *specialized discriminant* which is obtained from A-discriminant by substituting the vector of coefficients $(1,1,a,a,b,b,a+b)$ corresponding to our polynomial $f_{a,b}$. The specialized discriminant is a homogeneous irreducible polynomial of degree 24 in the two unknowns a and b, but it is no longer irreducible. In fact, it is the product of four irreducible factors, which have degrees 13, 5, 5 and 1.

The factor of degree 13 equals

$$
\begin{aligned}
&2916a^{11}b^2 + 19683a^9b^4 + 19683a^8b^5 + 2916a^7b^6 + 2916a^6b^7 + 19683a^5b^8 \\
&+ 19683a^4b^9 + 2916a^2b^{11} - 11664a^{12} - 104976a^{10}b^2 - 136080a^9b^3 - 27216a^8b^4 \\
&- 225504a^7b^5 - 419904a^6b^6 - 225504a^5b^7 - 27216a^4b^8 - 136080a^3b^9 \\
&- 104976a^2b^{10} - 11664b^{12} + 93312a^{11} + 217728a^{10}b + 76032a^9b^2 \\
&+ 1133568a^8b^3 + 1976832a^7b^4 + 891648a^6b^5 + 891648a^5b^6 + 1976832a^4b^7 \\
&+ 1133568a^3b^8 + 76032a^2b^9 + 217728ab^{10} + 93312b^{11} - 241920a^{10} \\
&- 1368576a^9b - 2674944a^8b^2 - 1511424a^7b^3 - 4729600a^6b^4 - 9369088a^5b^5 \\
&- 4729600a^4b^6 - 1511424a^3b^7 - 2674944a^2b^8 - 1368576ab^9 - 241920b^{10} \\
&+ 663552a^9 + 2949120a^8b + 10539008a^7b^2 + 17727488a^6b^3 + 9981952a^5b^4 \\
&+ 9981952a^4b^5 + 17727488a^3b^6 + 10539008a^2b^7 + 2949120ab^8 + 663552b^9 \\
&- 2719744a^8 - 8847360a^7b - 14974976a^6b^2 - 36503552a^5b^3 - 56360960a^4b^4 \\
&- 36503552a^3b^5 - 14974976a^2b^6 - 8847360ab^7 - 2719744b^8 + 4587520a^7 \\
&+ 25821184a^6b + 52035584a^5b^2 + 50724864a^4b^3 + 50724864a^3b^4 + 52035584a^2b^5 \\
&+ 25821184ab^6 + 4587520b^7 - 6291456a^6 - 31457280a^5b - 94371840a^4b^2 \\
&- 138412032a^3b^3 - 94371840a^2b^4 - 31457280ab^5 - 6291456b^6 + 16777216a^5 \\
&+ 50331648a^4b + 67108864a^3b^2 + 67108864a^2b^3 + 50331648ab^4 + 16777216b^5 \\
&- 16777216a^4 - 67108864a^3b - 100663296a^2b^2 - 67108864ab^3 - 16777216b^4.
\end{aligned}
$$

The factors of degree 5 are

$$
\begin{aligned}
&-256a^2+27a^5-512ab-144a^3b+27a^4b-256b^2+128ab^2-144a^2b^2+128b^3+4a^2b^3-16b^4, \\
&256a^2-128a^3+16a^4+512ab-128a^2b+256b^2+144a^2b^2-4a^3b^2+144ab^3-27ab^4-27b^5.
\end{aligned}
$$

Finally, the linear factor of the specialized discriminant equals $a + b$.

The relevant pieces of these four curves in the (a, b)-plane are shown in Figure 7. The line $a + b = 0$ is seen in the lower left, the degree 13 curve is the swallowtail in the upper right, and the two quintic curves form the upper-left and lower-right boundary of the enclosed convex region $\mathcal{C}$. The region $\mathcal{C}$ is the set of points (a, b) for which $f_{a,b}(x, y)$ is non-negative on $\mathbb{R}^2$.

We know that the SOS representation of $f_{a,b}(x, y)$ correspond to real positive semidefinite matrices $M \in \mathbb{R}^{6\times 6}$ satisfying 15 linear constraints given by the 15 coefficients of a polynomial of degree four in two unknowns:

$$
f_{a,b}(x, y) \; = \; (1, x, y, x^2, xy, y^2) \cdot M \cdot (1, x, y, x^2, xy, y^2)^T \quad ; \quad M \succeq 0.
$$

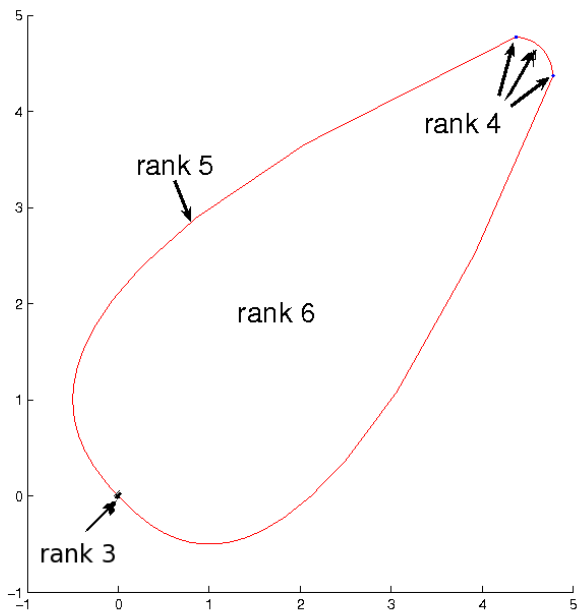


Figure 8: The region in the (a,b)-plane where $f_{a,b}(x,y)$ is SOS.

This formula expresses the convex region $\mathcal{C}$ is a *spectrahedral shadow*, namely, it is the projection of an 8-dimensional spectrahedron. The *fiber* of this projection over a point $(a,b) \in \mathcal{C}$ is the spectrahedron whose points are the SOS representations M of $f_{a,b}$. This is analogous to the fiber in Problem 9.

If (a,b) lies in the interior of $\mathcal{C}$ then the fiber is a 6-dimensional spectrahedron. If (a,b) lies in the boundary $\partial\mathcal{C}$ then the fiber consists of a single point. The ranks of these unique matrix is indicated in Figure 8. Notice that $\partial\mathcal{C}$ has three singular points, at which the rank drops from 5 to 3 or 4.

Fiber bundles of spectrahedra, such as those seen in Problems 9 and 10, are very important for polynomial optimization. They deserve further study.