UCSD Tutorial on positive polynomials, Mihai Putinar

Warm-up problems for the lectures on *Hermitian sums of squares*

Notation.

$$z = x + iy \in \mathbf{C}^n,$$

$$\Sigma_h^2 = \mathrm{co}\{|Q(z)|^2,\ Q \in \mathbf{C}[z]\},$$

$$\Sigma^2 = \mathrm{co}\{|R(x,y)|^2,\ R \in \mathbf{R}[x,y]\}.$$

1. Let $P(z,\overline{z}) = 1 - \frac{4}{3}|z|^2 + a|z|^4$.
Prove that P is a sum of squares if $a \geq \frac{4}{9}$.
Show that $P(z,\overline{z}) > 0$ whenever $|z| \leq 1$, $a > \frac{1}{3}$, yet

$$P \notin \Sigma_h^2 + (1 - |z|^2)\Sigma_h^2.$$

2. Let $z \in \mathbf{C}^2$ be the complex coordinate. Prove that for all $N > 0$

$$\|z\|^N (|z_1|^2 - |z_2|^2)^2 \notin \Sigma_h^2 + (1 - \|z\|^2)\mathbf{C}[z,\overline{z}],$$

although the polynomial is non-negative on the unit sphere.

3. Prove that the polynomial

$$p(z,\overline{z}) = (|z_1 z_2|^2 - |z_3|^4)^2 + |z_1|^8, \quad z \in \mathbf{C}^3,$$

cannot be written as a quotient of sums of hermitian squares.

4. The polynomial

$$p(z, \overline{z}) = 1 + bz^2 + \overline{b}\overline{z}^2 + c|z|^2 + |z|^4$$

is non-negative if $c \geq 2|b| - 2$, while it is a quotient of sums of hermitian squares if and only if one of the following constraints are satisfied:

$$c > 2|b|^2 - 2,$$

$$b = 0, c > -2,$$

$$|b| = 1, c = 0.$$

5. *[Some of the statements in the problem are highly non-trivial. Try only to understand the complexity of the questions]*

Consider the polynomial $P(z, \overline{z}) = 1 - |\zeta z_1 z_2|^2$, where $z \in \mathbf{C}^2$ and ζ is a complex parameter. Let $N = N(\zeta)$ denote the minimal number of hermitian squares of rational functions necessary to write P on the unit sphere.

If $N = 1$, then $\zeta = 0$.

If $N = 2$, then $\zeta = 0, |\zeta|^2 = 1, |\zeta|^2 = 2$, or $|\zeta|^2 = 3$.

We have $N(\zeta) \to \infty$ when $|\zeta| \to 2$.

7. Let $P(z, \overline{z}) = 1 + \lambda(z^4 + \overline{z}^4) + |z|^6$. Prove that P is a quotient of sums of hermitian squares of polynomials only if $\lambda = 0$.

8. Let $A = (A_1, ..., A_n)$ be an n-tuple of commuting, complex $d \times d$ matrices. Assume that

$$I = A_1^* A_1 + ... + A_n^* A_n.$$

Prove that every $A_k,\ 1 \leq k \leq n$, is normal.

9. Let A, B be two $d \times d$ commuting matrices. Are they normal if

$$A^{*2N} A^{2N} + B^{*2N} B^{2N} = I$$

for some large N?