DUALITY

Ten Exercises for Bernd Sturmfels' Lectures
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1. Let P be the convex hull of three pairwise touching circles in $\mathbb{R}^2$. Draw the dual convex body P^*. Now repeat the same with four spheres in $\mathbb{R}^3$. Finally, what do you get when you take four pairwise touching circles in $\mathbb{R}^3$?

2. The curve $X = \{(x,y) \in \mathbb{R}^2 \,:\, x^4 + y^4 = 1\}$ is known as the *TV screen*. Determine the irreducible polynomial whose zero set is the dual curve X^*. Explain the results of our computation in terms of two dual norms on $\mathbb{R}^2$.

3. Let X be the variety consisting of all $2 \times 2 \times 2$-tensors of rank one. As a projective variety, X is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^7$. Compute the dual variety X^*. Verify the biduality $(X^*)^* = X$ for this example.

4. Draw a picture of the three-dimensional spectrahedron

$$P \quad = \quad \left\{ (x,y,z) \in \mathbb{R}^3 \,:\, \begin{pmatrix} 1 & x & 0 & x \\ x & 1 & y & 0 \\ 0 & y & 1 & z \\ x & 0 & z & 1 \end{pmatrix} \succeq 0 \right\}.$$

Also, compute the dual convex body P^*. Determine all faces of P and P^*.

5. Examine the trigonometric space curve which has the parametrization $x = \cos(\theta)$, $y = \cos(2\theta)$, $z = \sin(3\theta)$. Compute the convex hull of this curve.

6. Consider the problem of maximizing a linear function $ax+by+cz$ over the spectrahedron in Problem 4. Write down the dual semidefinite program and the critical equations. Derive an explicit formula, in terms of radicals in a, b and c, for the optimal value. Make sure your case distinction is complete.

7. Consider the problem of minimizing the trace of a positive semidefinite 10×10-matrix whose off-diagonal entries are fixed to be random large integers. The optimal matrix has entries in a finite field extension of $\mathbb{Q}$. What ranks are possible for that matrix? What about the degree of the extension?

8. The semidefinite completion problem for a 4-cycle leads to the formula

$$\exists (x,y) \in \mathbb{R}^2 \; : \; \begin{pmatrix} u_{11} & u_{12} & x & u_{14} \\ u_{12} & u_{22} & u_{23} & y \\ x & u_{23} & u_{33} & u_{34} \\ u_{14} & y & y_{34} & u_{44} \end{pmatrix} \succeq 0.$$

This formula defines a convex cone $\mathcal{C}_{\mathcal{L}}$ in $\mathbb{R}^8$. Compute the unique polynomial in the eight unknowns u_{ij} whose zero set is the algebraic boundary of $\mathcal{C}_{\mathcal{L}}$.

9. The polynomial $p(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$ is non-negative on the real line. Find all possible representations of $p(x)$ as a sum of <u>two</u> squares. Does one of these representations involve only rational numbers? Show that the set of all SOS representations of $p(x)$ is a three-dimensional spectrahedron. Draw this spectrahedron and compute its analytic center.

10. For which values of the parameters a and b is the following polynomial non-negative on $\mathbb{R}^2$? In those cases, it is a sum of squares of polynomials.

$$f_{a,b}(x,y) \quad = \quad x^4 + y^4 + a(x^3 + y^2) + b(y^3 + x^2) + (a+b).$$

Draw that convex region $\mathcal{C}$ in the (a,b)-plane. The *fiber* over $(a,b) \in \mathcal{C}$ is the spectrahedron whose points are the SOS representations of $f_{a,b}$. What are the various dimensions of the fibers as (a,b) ranges over the convex set $\mathcal{C}$?