EXERCISES ON LIFTS OF POLYTOPES

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1. Lecture 1

The first two exercises lead you through specialized/ad hoc constructions of polytopal lifts for two classes of polytopes that arise in combinatorial optimization. Both constructions are due to Yannakakis.

Problem 1. The *parity polytope* PP_n in dimension n is defined to be the convex hull of all vectors $\mathbf{v} \in \{0,1\}^n$ with an odd number of 1s. The goal of this exercise is to prove that PP_n has a small lift.

(a) Prove that the following inequalities cut out PP_n:

$$0 \leq x_i \leq 1 \ \forall \ i \in [n]$$

$$\sum_{i \in A} x_i - \sum_{i \notin A} x_i \leq |A| - 1 \ \forall \ A \subseteq [n], \ |A| \text{ even}$$

Which of these inequalities are guaranteed to be facets of PP_n?

(b) For $k \in [n]$ odd, let S_k be the convex hull of all 0/1 vectors in $\mathbb{R}^n$ with k 1s. Prove that $\mathbf{x} \in PP_n$ if and only if $\mathbf{x} = \sum_{k \ odd} \alpha_k \mathbf{y}_k$ where $\mathbf{y}_k \in S_k$ and $\sum \alpha_k = 1$, $\alpha_k \geq 0$ for all k.

(c) Prove that PP_n is the projection onto the $\mathbf{x}$-coordinates of the polytope described by the constraints:

$$\sum_{k \ odd} \alpha_k = 1, \qquad x_i = \sum_{k \ odd} z_{ik} \ \ \forall \ \ i \in [n],$$

$$\sum_i z_{ik} = k\alpha_k \ \ \forall \ \ (odd) \ \ k, \qquad 0 \leq z_{ik} \leq \alpha_k \ \ \forall \ \ i, k$$

(d) Determine the size of this lift.

Problem 2. Let $G = ([n], E)$ be a graph with vertex set $[n]$ and edge set E. A set $S \subseteq [n]$ is *stable* if for every $i, j \in S$, $\{i, j\} \notin E$. Let $\chi^S \in \{0,1\}^n$ be the characteristic vector of the stable set S. The *stable set polytope*, $STAB(G)$ is the convex hull of $\{\chi^S \ : \ S \ \text{ stable in } \ G\}$. No complete description of $STAB(G)$ is known and in general, optimizing over $STAB(G)$ is NP-hard.

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(a) Check that the following inequalities are all valid on $STAB(G)$:

$$0 \leq x_i \leq 1 \;\; \forall \;\; i \in [n] \tag{1}$$

$$x_i + x_j \leq 1 \;\; \forall \;\; \{i,j\} \in E \tag{2}$$

$$\sum_{i \in C} x_i \leq (|C|-1)/2 \;\; \forall \;\; C \text{ odd cycle in } G \tag{3}$$

Definition. The polytope in $\mathbb{R}^n$ cut out by inequalities (1)-(2) is called the *fractional stable set polytope* of G and is denoted by $FRAC(G)$.

Definition. The polytope in $\mathbb{R}^n$ cut out by the inequalities (1)-(3) is called the *odd cycle polytope* of G and is denoted as $ODDCYC(G)$.

Definition. A graph G is *t-perfect* if $STAB(G) = ODDCYC(G)$.

Even though there maybe exponentially many inequalities of type (3), one can optimize over $ODDCYC(G)$ in polynomial time (in n) using the ellipsoid algorithm for linear programming. It turns out that we can also prove the same result by exhibiting a small polytopal lift of $ODDCYC(G)$ as follows.

(b) Pick an $\mathbf{x} \in FRAC(G)$ and define $l_{ij} := 1 - x_i - x_j$ for each edge $\{i,j\} \in E$. Since $l_{ij} \geq 0$ for all $\{i,j\} \in E$, interpret l_{ij} as the length of edge $\{i,j\}$. Show that a constraint of type (3) is violated by $\mathbf{x}$ if and only if the shortest odd cycle in G has length less than one.

(c) For every pair of nodes i and j, let e_{ij} and o_{ij} denote the even and odd distances (length of shortest paths) between nodes i and j. Note that $e_{ii} = 0$ but o_{ii} may be non-trivial and so we discard the variables e_{ii}. Consider the polytope $Q(G)$ described by the inequalities:

$$0 \leq x_i \leq 1 \qquad \forall \;\; i \in [n] \tag{4}$$

$$0 \leq o_{ij} \leq 1 - x_i - x_j \qquad \forall \;\; \{i,j\} \in E \tag{5}$$

$$o_{ij} \leq o_{ik} + e_{kj}, \quad e_{ij} \leq o_{ik} + o_{kj} \qquad \forall \;\; \{i,k\} \in E, \quad j \in [n] \tag{6}$$

$$o_{ii} \geq 1 \qquad \forall \;\; i \in [n] \tag{7}$$

Prove that $ODDCYC(G)$ is the projection of $Q(G)$ onto the $\mathbf{x}$-variables. What is the size of $Q(G)$?

Problem 3. Suppose $P = \{\mathbf{x} \in \mathbb{R}^n \; : \; A\mathbf{x} \leq \mathbf{b}\}$ is a polytope. A set $Q \subseteq \mathbb{R}^n \times \mathbb{R}^m_+$ is called a *lift* of P if $Q = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m_+ \; : \; R\mathbf{x} + S\mathbf{y} = \mathbf{t}\}$ and $P = \pi_{\mathbb{R}^n}(Q)$. It can be argued that there is no loss of generality in considering only lifts of P of this form. Note that such a lift is "small" if m is "small". If $A \in \mathbb{R}^{f \times n}$ then P always has the *trivial lift*

$$Q = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^f_+ \; : \; A\mathbf{x} + I\mathbf{y} = \mathbf{b}\}$$

with $n + f$ variables, $n + f$ equations and f inequalities. This lift is small only if f is small.

Can you find lifts of the following polytopes with $m < f$?
(a) The square $[-1,1]^2$ and more generally, the unit cube $[-1,1]^n$?
(b) The standard cross-polytopes in $\mathbb{R}^n$ (dual to the cubes in (a))?
(c) A pentagon? A hexagon?

2. Lecture 2

Problem 4. Let $P = \{(x_1, x_2) : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, x_1 + x_2 \geq \frac{1}{2}\}$. Compute $N(P)$ and decide whether it equals the convex hull of integer points in P.

Problem 5. Lovász and Schrijver proposed a second hierarchy of relaxations for computing the integer hull of a polytope $P \subseteq [0,1]^n$ where the lifts are spectrahedra. This exercise explores this semidefinite programming hierarchy.

(a) Let L be a lattice and $f \in \mathbb{R}^L$. Define two $|L| \times |L|$ matrices W^f and D^f as follows:

$$W^f := (w_{i,j})_{i,j \in L} \quad w_{i,j} := f(i \vee j)$$

$$D^f := diag(f(i))_{i \in L}.$$

Prove that if $f = Zg$, where Z is the Zeta matrix of L, then $W^f = ZD^gZ^t$.

(b) Let H be the cone spanned by the columns of Z. Prove that $f \in H$ if and only if $W^f \succeq 0$.

(c) As in the lecture, let $P \subseteq [0,1]^n$ be a polytope, K be the cone over $\{1\} \times P$ and K^0 be the cone generated by $\{1\} \times (P \cap \{0,1\}^n)$. Define

$$M_+(K) := \left\{ Y \in \mathbb{R}^{(n+1)\times(n+1)} \ : \ \begin{array}{l} (i)\ Y \succeq 0 \\ (ii)\ Y\mathbf{e}_0 = diag(Y) \\ (iii)\ Y\mathbf{e}_j, Y(\mathbf{e}_0 - \mathbf{e}_j) \in K \ \ \forall \ j \in [n] \end{array} \right\}$$

and

$$N_+(K) := \{diag(Y) : Y \in M_+(K)\}.$$

Explain why $M_+(K)$ is also a relaxation of $cone(\mathcal{F})$ where $\mathcal{F}$ is the set of supports of 0/1-vectors in P.

(d) Prove that $K^0 \subseteq N_+(K) \subseteq N(K) \subseteq K$.

As for the N-operator we can define $N_+^1(K) = N_+(K)$ and $N_+^i(K) = N_+(N_+^{i-1}(K))$. By (d), we have $N_+^n(K) = K^0$.

Problem 6. Is $N_+(P)$ strictly contained in $N(P)$ for the polytope P in Problem 4?

3. Lecture 3

Problem 7. Compute the theta bodies, and the spectrahedra that they are projections of, of the ideal $\langle (x+1)x(x-1)(x-2)\rangle$. Draw pictures if you can.

Problem 8. Consider an odd cycle G with $2k+1$ vertices and $k \geq 2$. Argue that I_G is not TH_1-exact. It is known that $STAB(G)$ is described by the following additional inequality to $FRAC(G)$: $\sum x_i \leq k$. Show that $STAB(G) = TH_2(I_G)$. Perhaps try a 5-cycle or 7-cycle as a warm up.

Problem 9. Compute the first theta body of the vanishing ideal of $S = \{(0,0),(1,0),(0,1),(2,2)\}$.

Problem 10. Prove that $Q_k(I) \subseteq TH_k(I)$ for any ideal $I \subset \mathbb{R}[x_1,\ldots,x_n]$.

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