Commutativity of EP Matrices

A solution to Problem 26 - 4 in Image April 2001 volume 26;
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Definition 1 *A matrix $M \in M_n(\mathbb{C})$ is said to be EP if M has the same column space as M^*. Equivalently, M is EP if $M^* = MG$ for some invertible matrix G.*

A slight modification of the original Problem 26-4 is

Theorem 1 *Let A and B be matrices in $M_n(\mathbb{C})$.*

a. If A and B commute, then A is EP only if $A^\dagger B - BA^\dagger = 0$.

b. If A and B are EP, then A and B commute if and only if $A^\dagger$ and $B^\dagger$ commute.

Solution.

The goal is to give a proof based upon fairly automatic use of computer algebra. We use *NCGB* package of *NCAlgebra*. Available from *www.math.ucsd.edu/∼ncalg.*

(a) Problem 26-4 asked whether, under the condition that A and B commute, A is EP *if and only if* $A^\dagger B - BA^\dagger$. Observe that only one implication can always be true, for if B is the identity matrix, then B commutes with both A and its pseudo-inverse regardless of whether A is EP or not.

In the other direction, if A and B commute and A is EP, then we show that $A^\dagger$ and B commute using a noncommutative (NC) Gröbner basis algorithm. Such algorithms in the commutative case are the main tool for solving collections of polynomial equations or at least for reducing them to a " nice " form. For noncommutative polynomials GB's are very experimental. Recall that the input to a Gröbner basis algorithm has two parts. The first part is a list of polynomials, denoted by $\mathcal{S}$, which in our solution of this problem is:

1. The polynomials
$$\left\{(AA^\dagger)^* - AA^\dagger, (A^\dagger A)^* - A^\dagger A, AA^\dagger A - A, A^\dagger AA^\dagger - A^\dagger\right\} \qquad (0.1)$$
which when set to zero define the pseudo-inverse of A.

2. The polynomial
$$\Big\{AB - BA\Big\}. \tag{0.2}$$
3. The fact that A is EP gives
$$\Big\{AG - A^*, GG^{-1} - 1, G^{-1}G - 1\Big\}.$$
4. One way to show that $A^\dagger$ and B commute is to create an additional variable, Q, and we adjoin $Q - A^\dagger B + BA^\dagger$, since we achieve our goal if Q is 0.
5. We adjoin the adjoints of the polynomials above. (NCAlgebra has a convenient command that accomplishes this.)

The second part of the input is an order on monomials which typically is induced by an order on the variables in the problem. In our case the order is degree lexicographic order induced by the following order on variables:
$$(A^\dagger) < (A^\dagger)^* < A < A^* < B < B^* < G < G^{-1} < G^* < (G^*)^{-1} < Q < Q^*. \tag{0.3}$$
Recall that in a Gröbner basis algorithm variables which are high in the order tend to be eliminated before variables which are low in the order.

We use the command `NCProcess`, with options set so that it makes and sorts an NC Gröbner basis by running Mora's algorithm for four iterations and removes some redundant polynomials. This makes a long list of polynomial equations, one of which is $Q = 0$. This shows that part (a) is true. The run took 49 minutes on a 1 GigaHertz Pentium III machine with one gigabyte of RAM.

(b) Suppose A and B commute. Again we use the Gröbner basis package applied to the starting relations:

1. The hypothesis that both A and B are EP is equivalent to the existence of G and H satisfying
$$\Big\{AG - A^*, GG^{-1} - 1, G^{-1}G - 1, BH - B^*, HH^{-1} - 1, H^{-1}H - 1\Big\}.$$
2. Applying part (a) to the pairs A and B which commute, B and A which commute, A and itself, and B and itself allows us to adjoin the relations
$$\Big\{AB^\dagger - B^\dagger A, BA^\dagger - A^\dagger B, AA^\dagger - A^\dagger A, BB^\dagger - B^\dagger B\Big\}.$$

3. The polynomial
$$\left\{AB - BA\right\}. \tag{0.4}$$

4. We adjoin the defining polynomials for the pseudo inverse of A given in equation (0.1) and the corresponding polynomials for B.

5. Once again, our objective is achieved if the additional variable Q satisfying the equation
$$Q - A^{\dagger}B^{\dagger} + B^{\dagger}A^{\dagger}$$
is zero.

6. We adjoin the adjoints of the above equations.

We choose a degree lexicographic order with Q at the top. (We found, when Q is placed at the top of the order, the result to be insensitive to the rest of the order in 4 experiments.) We ran NCProcess with options set to 2 iterations and among the output was $Q = 0$. It follows that $A^{\dagger}$ and $B^{\dagger}$ commute, thus establishing the "if" part of (b). These runs took less than a minute.

The converse is obtained similarly. In the input polynomials we merely replace
$$\left\{AB - BA\right\}$$
by
$$\left\{A^{\dagger}B^{\dagger} - B^{\dagger}A^{\dagger}\right\}$$
and redefine Q by adjoining the polynomial $Q - AB + BA$ instead of $Q - A^{\dagger}B^{\dagger}+B^{\dagger}A^{\dagger}$; we do this since our goal is now to show that A and B commute. Again we choose a degree lexicographic order with Q at the top and find after 3 iterations and about 15 minutes that $Q = 0$. Again we tried some variations on the orders and in each case found that $Q = 0$. ∎

This solves problem Problem 26 - 4 by fairly automatic use of NCAlgebra's noncommutative Gröbner basis package and suggests a problem of our own:

> **Now that some noncommutative computer algebra is available, which of the purely algebraic problems (with a modest number of variables and equations) that have been posed over the past years in Image succumb in an automatic fashion to computer algebra?**

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