Effective Equidistribution of Horospherical Flows in Infinite Volume

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Background: $\mathrm{PSL}_2(\mathbb{R})$ acting on $\mathbb{H}$

- $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbb{H}$, the upper half-plane, by Möbius transformations.
- There is a natural simply transitive action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathrm{T}^1(\mathbb{H})$: for $(z, v) \in \mathrm{T}^1(\mathbb{H})$, $g \in \mathrm{PSL}_2(\mathbb{R})$,

$$g \cdot (z, v) = \left(\frac{az + b}{cz + d}, \frac{v}{(cz + d)^2} \right).$$

- This lets us identify $\mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{T}^1(\mathbb{H})$.

Acting on the Hyperbolic surface

The geodesic flow is implemented by the diagonal subgroup

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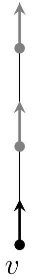

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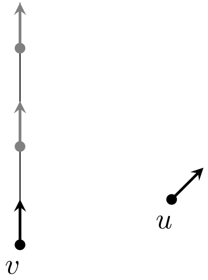

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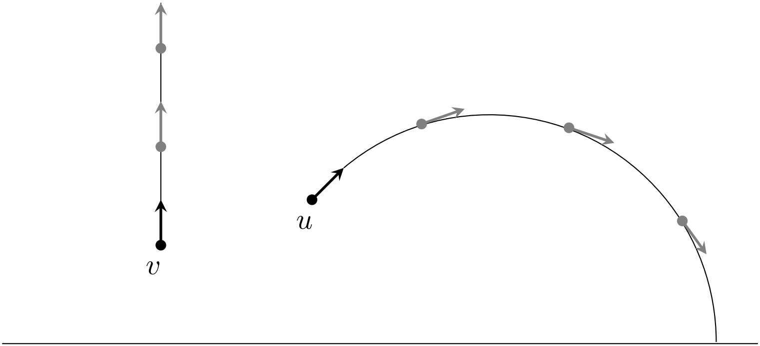

Acting on the Hyperbolic surface

The horocycle subgroup

$$U = \{g \in G \, : \, a_{-s} g a_s \to e \text{ as } s \to +\infty\}$$
$$= \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \, : \, t \in \mathbb{R} \right\}.$$

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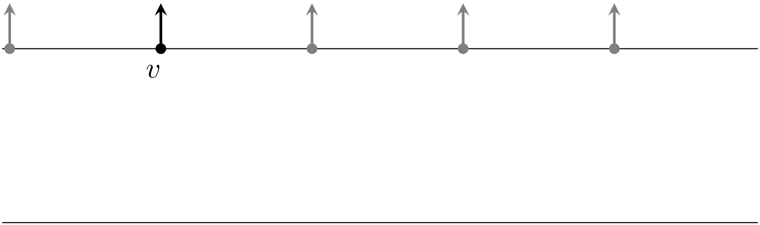

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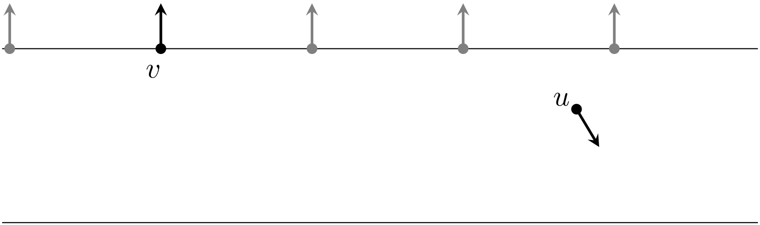

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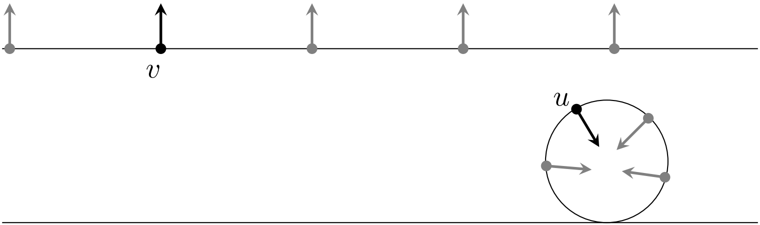


Acting on the Hyperbolic surface

- Let Γ be a lattice (finite covolume discrete subgroup) in $G = \mathrm{PSL}_2(\mathbb{R})$, such as $\mathrm{PSL}_2(\mathbb{Z})$.
- The unit tangent bundle $\mathrm{T}^1(\mathbb{H}/\Gamma)$ of $\mathbb{H}/\Gamma$ may be identified with the homogeneous space G/Γ. G acts on G/Γ by left multiplication

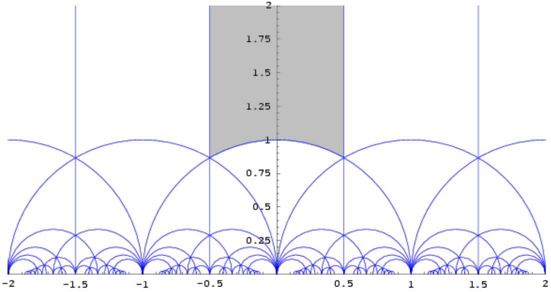


Figure: Fundamental domain of $\mathrm{PSL}_2(\mathbb{Z})$ (Anastasios Taliotis)

Equidistribution in finite volume

Theorem (Dani and Smillie, 1984)

Let Γ *be a lattice in* $G = \mathrm{PSL}_2(\mathbb{R})$. *For every* $x \in G/\Gamma$, *we have one of the following:*

- Ux *is periodic.*
- *For any* $f \in C_c(G/\Gamma)$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(u_t x)dt = m(f) := \int_{G/\Gamma} f dm$$

where m *denotes the normalized Haar measure on* G/Γ.

However, this theorem does not tell us the rate of equidistribution – it is not *effective*.

Equidistribution in finite volume

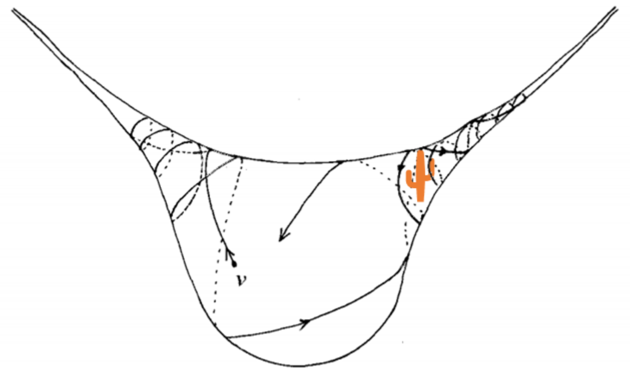


Figure: An orbit in direction v (Sullivan)

Effective Versions

- There are many effective generalizations, e.g.:

Theorem (McAdam, '18 (roughly stated))

For any $x \in X := \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$*, there exist constants* $\gamma, C > 0$ *such that for all* $f \in C_c^\infty(X)$ *and* $T > C$*,*

$$\left| \frac{1}{m(B_U(T))} \int_{B_U(T)} f(ux)dm(u) - m(f) \right| \ll_f T^{-\gamma},$$

unless there is an explicit algebraic obstruction.

- Here, $B_U(T)$ denotes the ball of radius T in U.

Additional results

Equidistribution results:

- Dani (1982): $G = \mathrm{SL}_2(\mathbb{R})$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and U is horospherical.
- Ratner (1991): G a Lie group, $\Gamma \subseteq G$ a lattice, and U is generated by one parameter unipotent subgroups.

Effective equidistribution results (U is horospherical):

- Sarnak (1981): $G = \mathrm{PSL}_2(\mathbb{R})$ and Γ a lattice, closed orbits.
- Burger (1990): $G = \mathrm{PSL}_2(\mathbb{R})$ and Γ cocompact.
- Strömbergsson (2013): $G = \mathrm{PSL}_2(\mathbb{R})$ and Γ non-cocompact.
- Sarnak, Ubis (2015): $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.
- Katz (2019): G semisimple linear group without compact factors and Γ a lattice

Acting on a Hyperbolic manifold

- For $n \geq 2$, let G be the identity component of the special orthogonal group $\mathrm{SO}(n,1)$.
- G can be considered as the group of orientation preserving isometries of n-upper half-space $\mathbb{H}^n$.
- Let U denote the horospherical subgroup

$$U = \{g \in G \,:\, a_{-s} g a_s \to e \text{ as } s \to +\infty\} \\ = \left\{u_{\mathbf{t}} \,:\, \mathbf{t} \in \mathbb{R}^{n-1}\right\}.$$

- Let Γ be a discrete subgroup of G (not necessarily a lattice).
- Any complete hyperbolic (constant negative curvature) n-manifold can be presented as $\mathbb{H}^n/\Gamma$, and G/Γ is the space of positively oriented frames on $\mathbb{H}^n/\Gamma$.

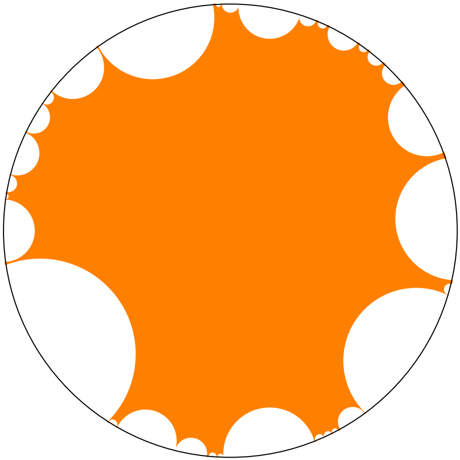
The convex core

- The *limit set* of Γ, $\Lambda = \Lambda(\Gamma) \subset \partial\mathbb{H}^n$, is the set of accumulation points of Γo for some $o \in \mathbb{H}^n$.
  - When Γ is not a lattice, Λ is a fractal set.
  - When it is a lattice, $\Lambda = \partial(\mathbb{H}^n)$.
- The convex core of $\mathbb{H}^n/\Gamma$ is the convex submanifold given by

$$\text{hull}(\Lambda)/\Gamma = \text{hull}\,\Lambda \subset \mathbb{H}^n/\Gamma,$$

  where $\text{hull}(\Lambda)$ is the smallest convex subset containing all geodesics connecting two points in Λ.

The convex hull in the Poincaré disc model



A limit set example

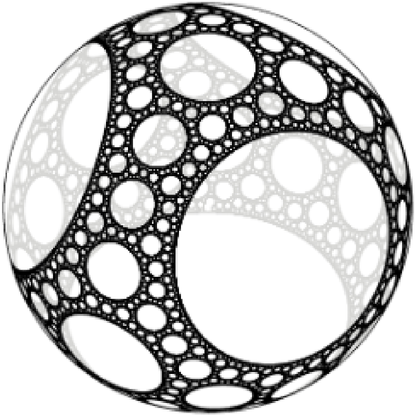


Figure: Limit set (McMullen, Mohammadi, Oh)

Convex cocompact and geometrically finite

- Γ is called *geometrically finite* (GF) if the unit neighborhood of the convex core has finite volume.
- May be thought of as $\mathbb{H}^n/\Gamma$ having a finite-sided fundamental domain. Examples include quasifuchsian groups, or cutting a compact n-manifold along a totally geodesic hyperplane.
- Γ is called *convex cocompact* if the convex core is compact.
- In this case, there are no cusps.
- Schottky groups without parabolic elements are examples (finitely generated by hyperbolic elements satisfying certain conditions, "ping pong" construction), with the convex core being a handle body in this case.

Equidistribution in Infinite Volume

Theorem (Hopf ratio ergodic thm, Hopf, 1937, Hochman, 2010)

Let μ be a locally finite U-invariant ergodic measure on G/Γ. Let $f_1, f_2 \in L^1(G/\Gamma)$ such that $\mu(f_2) \neq 0$. Then, for μ-a.e. x

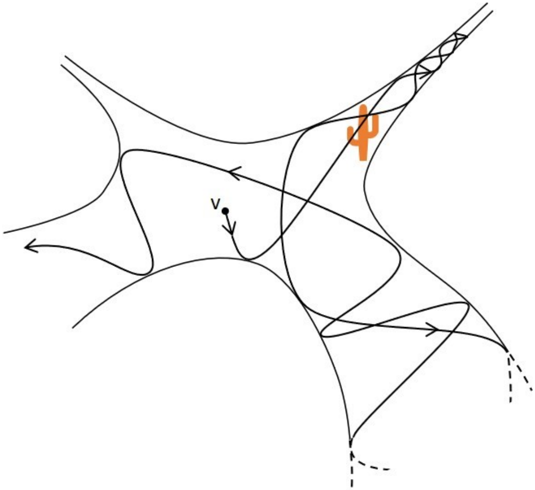
$$\lim_{T\to\infty} \frac{\int_{-T}^{T} f_1(ux)dt}{\int_{-T}^{T} f_2(ux)dt} = \frac{\mu(f_1)}{\mu(f_2)}.$$

When the Haar measure is infinite, it follows that for a.e. x,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(u_t x)dt = 0.$$

The time an orbit spends in any compact set is sub-linear.
"There is not enough recurrence."

Equidistribution in Infinite Volume



Equidistribution in Infinite Volume

- The correct normalization will be given by the *Patterson-Sullivan* measure, and the limit will be given by the *Burger-Roblin (BR)* measure, which is a natural, geometrically defined measure.
- If $x^- \in \Lambda(\Gamma)$, the PS measure governs the return times of $u_{\mathbf{t}} x$ to the convex core.
- When Γ is a lattice and $n = 2$, the Haar measure is the only U-ergodic measure not supported on a closed U orbit.
- If Γ is geometrically finite, the BR measure is the natural analogue of the Haar measure.

Our Equidistribution Theorem

Theorem (Tamam and W.)

Let $\Gamma \subseteq G$ *be a convex cocompact subgroup of* G *and let* $\Omega \subset G/\Gamma$ *be a compact set. There exists* $\kappa = \kappa(\Gamma) > 0$ *such that for any* $x \in \Omega$ *such that* $x^- \in \Lambda(\Gamma)$, $\psi \in C_c^\infty(G/\Gamma)$, *and* $r > r_0(x, \operatorname{supp}\psi) > 0$,

$$\left| \frac{1}{\mu_x^{\mathrm{PS}}(B_U(r))} \int_{B_U(r)} \psi(u_{\boldsymbol{t}} x) d\boldsymbol{t} - m^{\mathrm{BR}}(\psi) \right| \ll r^{-\kappa},$$

where the implied constant depends on Γ, Ω, *and* ψ.

Remark

- The dependence of r_0 on x and $\operatorname{supp}\psi$ is explicit.
- We also prove this theorem for Γ geometrically finite, but additional assumptions are necessary.

Prior results in infinite volume

Equidistribution:

- Schapira (2005): $G = \mathrm{PSL}_2(\mathbb{R})$ and Γ is geometrically finite.
- Mohammadi and Oh (2010): $G = \mathrm{SO}(n,1)^\circ$ and Γ is geometrically finite.

Effective equidistribution:

- Edwards (2019): $G = \mathrm{PSL}_2(\mathbb{R})$ and Γ is geometrically finite.

Application: Γ orbits on $\mathbb{R}^{n+1}$

- Γ acts on $V := \mathbb{R}^{n+1} \setminus \{0\}$ by matrix multiplication.

Proposition (Tamam-W.)

Let Γ be convex cocompact. For any $\varphi \in C_c(V)$ and every $v \in V$ with "$v^- \in \Lambda(\Gamma)$," as $T \to \infty$, we have that

$$\frac{1}{T^{\delta_\Gamma/2}} \sum_{\gamma \in \Gamma, \|\gamma\| \leq T} \varphi(v\gamma) \asymp \int_V \varphi(u) \frac{d\overline{\nu}(u)}{(\|v\|\|u\|)^{\delta_\Gamma/2}}.$$

- δ_Γ is the Hausdorff dimension of $\Lambda(\Gamma)$.
- $\overline{\nu}$ is the pushforward of a measure that appears in the product structure of m^{BR}.
- We also prove a quantitative ratio theorem, and allow for geometrically finite Γ.

History: Γ orbits on $\mathbb{R}^{n+1}$

- Ledrappier (1999): Proved a similar ergodic theorem for lattices in $\mathrm{PSL}_2(\mathbb{R})$ acting on $\mathbb{R}^2$.
- Maucourant and Weiss (2012): A quantitative version of Ledrappier's theorem.
- Maucourant and Schapira (2014): proved an asymptotic version for convex cocompact Γ in $\mathrm{SL}_2(\mathbb{R})$.
  - Showed there cannot be convergence of the same form as in Maucourant-Weiss.
  - Proved convergence with an additional averaging.
- Many additional works studying the finite volume setting in broad generality, e.g. works by Gorodnik-Weiss, Gorodnik-Nevo, Nogueira, and more.

The PS and BR measures

- A Γ-*invariant conformal density of dimension* δ is a family of finite measures $\{\mu_x : x \in \mathbb{H}^n\}$ on $\partial(\mathbb{H}^n)$ such that

$$\gamma_* \mu_x = \mu_{\gamma x} \quad \text{and} \quad \frac{d\mu_x}{d\mu_y}(\xi) = e^{-\delta \beta_\xi(x,y)}.$$

- The Patterson-Sullivan density, denoted $\{\nu_x : x \in \mathbb{H}^n\}$, is a Γ-invariant conformal density with dimension equal to the Hausdorff dimension of Λ. It is unique up to scaling.

- We can use a weighted stereographic projection to define the PS measure on a horosphere from this conformal density. This will give an infinite measure.

A limit set example revisited

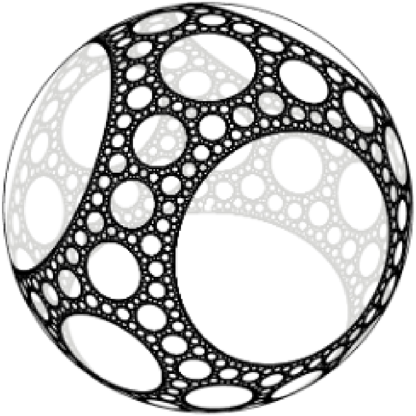


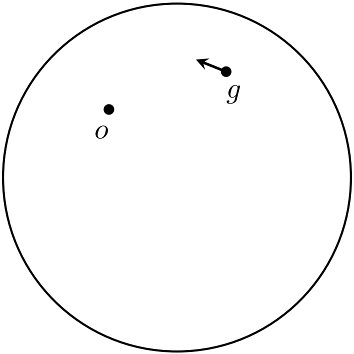
Figure: Limit set (McMullen, Mohammadi, Oh)

Hopf Parametrization

$\mathrm{T}^1(\mathbb{H}^2)$ is homeomorphic to

$$(\partial(\mathbb{H}^2) \times \partial(\mathbb{H}^2) - \{(\xi, \xi) : \xi \in \partial(\mathbb{H}^2)\}) \times \mathbb{R}$$

via $g \mapsto (g^+, g^-, s = \beta_{g^+}(o, \pi(g)))$, for fixed $o \in \mathbb{H}^2$.



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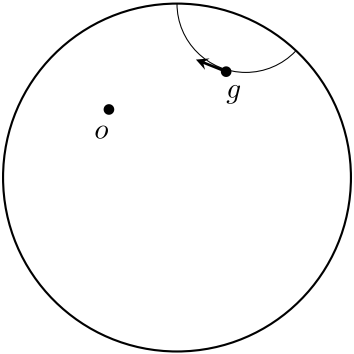

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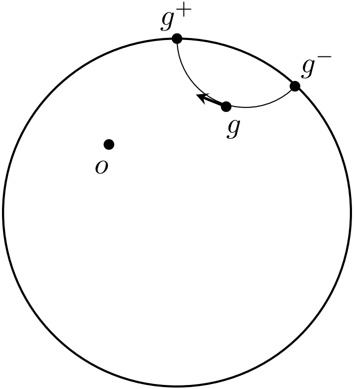

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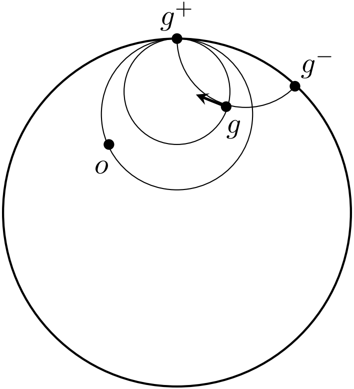

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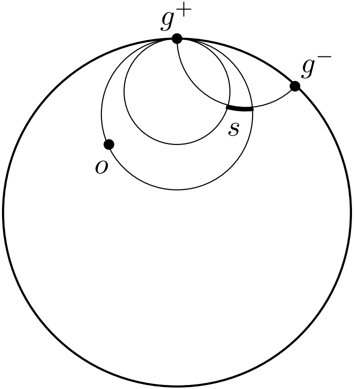

The PS and BR measures

- In the convex cocompact case, for $x^{\pm} \in \Lambda(\Gamma)$,

$$\mu_x^{\mathrm{PS}}(B_U(T)) \asymp T^{\delta_\Gamma}.$$

- When there are cusps, there is an additional term scaling by the distance into the cusp.

- The BR measure is defined geometrically using weighted stereographic projection from the PS density and the Lebesgue measure on $\partial(\mathbb{H}^n)$ (with a product structure).

- The support is $\{g\Gamma \in G/\Gamma : g^- \in \Lambda(\Gamma)\}$.

Theorem revisited

Theorem (Tamam and W.)

Let $\Gamma \subseteq G$ be a convex cocompact subgroup of G and let $\Omega \subset G/\Gamma$ be a compact set. There exists $\kappa = \kappa(\Gamma) > 0$ such that for any $x \in \Omega$ such that $x^- \in \Lambda(\Gamma)$, $\psi \in C_c^\infty(G/\Gamma)$, and $r > r_0(x, \operatorname{supp} \psi) > 0$,

$$\left| \frac{1}{\mu_x^{\mathrm{PS}}(B_U(r))} \int_{B_U(r)} \psi(u_{\boldsymbol{t}} x) d\boldsymbol{t} - m^{\mathrm{BR}}(\psi) \right| \ll r^{-\kappa},$$

where the implied constant depends on Γ, Ω, and ψ.

Main ingredients in the proof: Exponential mixing

- The BMS measure is a finite measure that is closely related to the BR measure.
- It is supported on the convex core of Γ.
- Our theorem requires exponential mixing of the A action for m^{BMS}:

Assumption (Exponential Mixing)

There exist $\kappa = \kappa(\Gamma) > 0$ and $s_0 = s_0(\Gamma)$ such that for $s > s_0$ and $\psi, f \in C_c^\infty(G/\Gamma)$,

$$\left| \int_{G/\Gamma} \psi\left(a_s x\right) f\left(x\right) dm^{\mathrm{BMS}}\left(x\right) - m^{\mathrm{BMS}}\left(\psi\right) m^{\mathrm{BMS}}\left(f\right) \right| \ll e^{-\kappa s},$$

where the implied constant depends only on f, ψ, and Γ.

Main ingredients in the proof: Exponential mixing

- For Γ convex co-compact, the exponential mixing was proved by Winter in 2016, building on work of Stoyanov and Dolgopyat. (See also Sarkar-Winter (2020).)
- For Γ GF such that $L^2(G/\Gamma)$ has a spectral gap, exponential mixing was proved by Mohammadi and Oh in 2015.
- When the critical exponent of Γ, $\delta_\Gamma > n-2$, there is such a spectral gap.
- When there is a cusp of rank $n-1$, $\delta_\Gamma > \frac{n-1}{2}$. In particular, for $n = 2, 3$, there is a spectral gap.
- It is conjectured to be true for all n.

Main ingredients in the proof: quantitative nondivergence

- For Γ GF, we consider ϵ-*Diophantine* points $x \in G/\Gamma$.
- "x does not travel into the cusps too quickly", a necessary assumption.

Theorem (Tamam-W.)

There exists $\beta > 0$ satisfying the following: for any ϵ-Diophantine element $x \in X$, there exists $T_0 = T_0(x) > 0$ such that for every $R \geq 0$, $T > T_0$, $s \leq T^\epsilon$, and $x_0 = a_{-\log s}x$, we have

$$\frac{\mu^{\mathrm{PS}}_{x_0}\left(B_U(T/s)x_0 \cap C_R\right)}{\mu^{\mathrm{PS}}_{x_0}(B_U(T/s)x_0)} - 1 \ll e^{-\beta R},$$

where the implied constant depends on Γ.

- Here, C_R is an explicit compact set arising from the thick-thin decomposition of the convex core.

Main ingredients in the proof: "Friendliness"

- A key difficulty in higher dimensions is understanding the PS measure.
- In particular, can a large portion of the measure of a ball be concentrated near its boundary?
- Das, Fishman, Simmons, and Urbański proved in 2015 that the PS density is "friendly" if Γ is GF and all cusps have rank $n-1$. Using this, we proved that for such Γ:

Proposition

There exists $\alpha = \alpha(\Gamma) > 0$ such that for any $x \in G/\Gamma$ with $x^+ \in \Lambda(\Gamma)$, $0 < \xi < \eta$,

$$\frac{\mu_x^{\mathrm{PS}}(B_U(\xi+\eta))}{\mu_x^{\mathrm{PS}}(B_U(\eta))} - 1 \ll_\Gamma \left(\frac{\xi}{\eta}\right)^\alpha.$$

Proof of the theorem

Using the exponential mixing, the relation between the measures, and "Margulis' thickening trick" one can show:

Theorem (Tamam-W.)

There exist $\kappa = \kappa(\Gamma)$ and $s_0 = s_0(\Gamma)$ which satisfy the following. Let $\Omega \subseteq G$ be a compact set, $0 < r < 1$ be smaller than the injectivity radius of Ω, $\psi \in C_c^\infty(G/\Gamma)$, and $f \in C_c^\infty(B_U(r))$. Then, for any $x \in \Omega$, $x \in \mathrm{supp}(m^{\mathrm{BMS}})$, and $s > s_0$ we have

$$\left| \int_U \psi(a_s u_{\boldsymbol{t}} x) f(\boldsymbol{t}) d\mu_{Ux}^{\mathrm{PS}}(\boldsymbol{t}) - \mu_{Ux}^{\mathrm{PS}}(f) m^{\mathrm{BMS}}(\psi) \right| \ll e^{-\kappa s},$$

where the implied constant depends on f, ψ and Γ.

Proof of the theorem

Using the previous theorem and the relation between the measures one can show:

Theorem (Tamam-W.)

There exist $\kappa = \kappa(\Gamma)$ and $s_0 = s_0(\Gamma)$ which satisfy the following. Let $\Omega \subseteq G$ be a compact set, $0 < r < 1$ be smaller than the injectivity radius of Ω, $\psi \in C_c(G/\Gamma)$, and $f \in C_c^\infty(B_U(r))$. Then, for any $x \in \Omega$, $x \in \operatorname{supp}(m^{\mathrm{BMS}})$, and $s > s_0$ we have

$$\left| e^{(n-1-\delta_\Gamma)s} \int_{B_U(r)} \psi(a_s u_{\boldsymbol{t}} x) f(\boldsymbol{t}) d\boldsymbol{t} - \mu_x^{\mathrm{PS}}(f) m^{\mathrm{BR}}(\psi) \right| \ll e^{-\kappa s}.$$

where the implied constant depends on ψ, f, and Γ.

Proof of the theorem

For

$$s_0 := \frac{\epsilon}{2}\log T, \quad x_0 := a_{-s_0}x, \quad \text{and} \quad T_0 = T^{1-\frac{\epsilon}{2}}$$

we have

$$\frac{1}{\mu_x^{\mathrm{PS}}(B_U(T))}\int_{B_U(T)} \psi(u_{\mathbf{t}}x)d\mathbf{t} = \frac{e^{(n-1-\delta)s_0}}{\mu_{x_0}^{\mathrm{PS}}(B_U(T_0))}\int_{B_U(T_0)} \psi(a_{s_0}u_{\mathbf{t}}x_0)d\mathbf{t}$$

and

$$\mu_{x_0}^{\mathrm{PS}}\left(B_U(T_0)x_0 \cap C_R^c\right) \ll \mu_{x_0}^{\mathrm{PS}}(B_U(T_0)x_0)e^{-\beta R}.$$

By decomposing

$$B_U(T_0) = (B_U(T_0)\cap C_R) \cap (B_U(T_0)\cap C_R^c),$$

we can get a bound on $B_U(T_0)\cap C_R^c$ and use the nondivergence statement on $B_U(T_0)\cap C_R$ to get the estimate we want.

Thank You
for Your Attention!