\[ P_{12} \quad 2ny \, dx = (y^2 - 3x^2) \, dy = 0 \]

We see \( M = 2ny \) and \( N = y^2 - 3x^2 \)

\[ \frac{2M}{\partial y} = 2n \quad \frac{\partial N}{\partial x} = -6n \]

\[ \therefore \quad \frac{2M}{\partial y} - \frac{\partial N}{\partial x} = -8n \]

\[ \Rightarrow \quad \frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = -8n = \frac{-y}{2ny} \]

\[ \Rightarrow \quad \text{Pure function of } y \]

So

\[ \frac{1}{\mu} \frac{dy}{dy} = - \frac{y}{2ny} \]

\[ \mu \frac{dy}{y} = -y \]

\[ \ln |\mu| = -y \ln |y| \]

\[ \Rightarrow \quad \mu = -y \]
Multiplying by \( \mu \) the differential equation becomes:

\[
2n y^{-3} \frac{dx}{dy} + y^{-4} (y^2 - 3x^2) dy = 0
\]

Now,

\[
M(x, y) = 2ny^{-3}
\]

\[
N(x, y) = y^{-4} (y^2 - 3x^2)
\]

\[
\frac{2M}{\partial y} = (-3) 2n y^{-4} = -\frac{6n}{y^4}
\]

\[
\frac{2N}{\partial x} = -\frac{6n}{y^4}
\]

So the equation is exact as required.

So now we need to find \( \Psi(x, y) \) so that

\[
\frac{\partial \Psi}{\partial x} = M(x, y) = 2ny^{-3}
\]

\[
\frac{\partial \Psi}{\partial y} = N(x, y) = y^{-4} (y^2 - 3x^2)
\]
\[ \sin \theta \quad \frac{dy}{dx} = 2n^2 y^{-3} \]

Integrating with respect to \(x\):

\[ \phi(x, y) = n^2 y^{-3} + g(y) \]

\[ \frac{dy}{dy} = -3n^2 y^{-4} + g'(y) \]

We also have:

\[ \frac{dy}{dy} = y^{-4} (y^2 - 3n^2) = y^{-2} - \frac{3n^2}{y^2} \]

So we get:

\[ g'(y) = y^{-2} \]

\[ \Rightarrow g(y) = -y^{-1} + c \]

\[ \phi(x, y) = n^2 y^{-3} - y^{-1} + c \]

Solution is:

\[ \phi(x, y) = 0 \]

\[ n^2 y^{-3} - y^{-1} + c = 0 \]
Problem 2 is
\[ y'' + y' + 4y = e^{-2t} \text{ e}^{4t} \]

We will use variation of parameters.

First, we check the homogeneous part

\[ y^2 + y + 4 = 0 \]
\[ (y + 2)^2 = 0 \implies y = -2 \text{ with multiplicity } 2 \]

So
\[ y_1(t) = e^{-2t}, \quad y_2(t) = te^{-2t} \]

Now we want to understand \( v_1(t) \) and \( v_2(t) \). From Lecture 12,

\[
v_1(t) = \int -f(t) \frac{y_2(t)}{y_1(t) y_2'(t) - y_2(t) y_1'(t)} \, dt
\]
\[ = \int \frac{f(t) y_2(t)}{y_1(t) y_2'(t) - y_2(t) y_1'(t)} \, dt \]

\[
v_2(t) = \int f(t) \frac{y_1(t)}{y_1(t) y_2'(t) - y_2(t) y_1'(t)} \, dt
\]
\[ = \int f(t) \, dt \]

Here \( f(t) = e^{-2t} \text{ e}^{4t} \).
We need to figure out

\[ y_1(t) y_2'(t) - y_2(t) y_1'(t) \]

So

\[ y_2'(t) = e^{-2t} - 2t e^{-2t} \]

\[ y_1'(t) = -2e^{-2t} \]

So

\[ y_1(t) y_2'(t) - y_2(t) y_1'(t) = e^{-2t} (e^{-2t} - 2t e^{-2t}) - t e^{-2t} (-2 e^{-2t}) \]

\[ = e^{-4t} - 2te^{-4t} + 2te^{-4t} \]

\[ = e^{-4t} \]

So substituting

\[ v_1(t) = -\int \frac{t e^{-2t} e^{-2t} e^{-4t}}{e^{-4t}} dt \]

\[ = -\int t e^{4t} \]

\[ = -\frac{t^2}{2} e^{4t} + \frac{t^2}{2} e^{4t} + C \]
\[ v_2(t) = \int \frac{e^{-2t} e^{-2t} \ln t}{e^{-4t}} \, dt = \int \ln t \, dt = t \ln t - t + c_2. \]

So a particular solution is

\[
y_p(t) = e^{-2t} \left( \frac{t^2}{4} - \frac{t^2}{2} \ln t + c_1 \right) + te^{-2t} \left( t \ln t - t + c_2 \right)
\]

Since we only need a particular solution, we can set \( c_1 \) and \( c_2 = 0 \).

Either way, both answers are correct.

A general solution is

\[
y(t) = c_1 e^{-2t} + c_2 + te^{-2t} + e^{-2t} \left( \frac{t^2}{4} - \frac{t^2}{2} \ln t \right) + t e^{-2t} \left( t \ln t - t \right)
\]
P2 (iii). \[ f^2 y'' + 7ty' - 7y = 0 \]

This is a Cauchy–Euler equation.

We have a guess of the form \[ y = t^r \]

and we get a characteristic equation of the form

\[ r(r-1) + 7r - 7 = 0 \]

\[ r^2 - r + 7r - 7 = 0 \]

\[ r^2 + 6r - 7 = 0 \]
\[ y = -\frac{6 \pm \sqrt{36 + 28}}{2} \]

\[ = -\frac{6 \pm \sqrt{64}}{2} \]

\[ = -\frac{6 \pm 8}{2} \]

\[ = -7, 1 \]

So solutions are -7 and 1

and so \( y_1(t) = e^{-7t} \) and \( y_2(t) = y \)

So a general solution is

\[ y(t) = c_1 e^{-7t} + c_2 t \]
Problem 1. \( y'' - 2ty' + ty = 0 \)

\[ y_1(t) = e^{-t^2} \]

We need to find \( y_2(t) \).

Recall from Lecture 13 (pages 3 and 4) that given a differential equation of the form

\[ y'' + p(t)y' + q(t)y = 0 \]

and a solution \( y_1(t) \), we can find the solution \( y_2(t) \) by the formula,

\[ y_2(t) = y_1(t) \int e^{-\int p(t) \, dt} \frac{e^{\int q(t) \, dt}}{(y_1(t))^2} \]

9th hour problem \( p(t) = -2t \), \( q(t) = 1 \)

\[ -\int p(t) \, dt = -\int (-2t) \, dt \]

\[ \int q(t) \, dt = \int 1 \, dt \]

So \( e^{-\int p(t) \, dt} = e^{-\int (-2t) \, dt} = e^{\int 1 \, dt} = e \)

\[ y_2(t) = 1 - 2t^2 \int \frac{e^{t^2}}{(1-2t)^2} \, dt \]
\[ y_2(x) = 1 - 2x \int \frac{e^{2x}}{1 + ye^2 - ye^{-x}} dx \]

This integral can't be done using elementary functions.