Problem 1. Find the Inverse Laplace transform of the following:

\[ F(s) = \frac{s^2 - 26s - 47}{(s - 1)(s + 2)(s + 5)} \]

Answer. Partial fraction expansion is

\[ F(s) = \frac{A_1}{s - 1} + \frac{A_2}{s + 2} + \frac{A_3}{s + 5} \]

So we have

\[ s^2 - 26s - 47 = A_1(s + 2)(s + 5) + A_2(s - 1)(s + 5) + A_3(s - 1)(s + 2) \]

Some computation gives us \( A_1 = -4 \), \( A_2 = -1 \) and \( A_3 = 6 \). So the Inverse Laplace transform is

\[ \mathcal{L}^{-1}\{F(s)\} = -4e^t - e^{-2t} + 6e^{-5t} \]

Problem 2. Find the Inverse Laplace transform of the following:

\[ \frac{2s + 5}{(s - 1)(s^2 + 2s + 1)} \]

Answer. Partial fraction expansion is

\[ F(s) = \frac{A_1}{s - 1} + \frac{A_2}{s + 1} + \frac{A_3}{(s + 1)^2} \]

So we have

\[ 2s + 5 = A_1(s + 1)^2 + A_2(s - 1)(s + 1) + A_3(s - 1) \]

Some computation gives us \( A_1 = \frac{7}{4} \), \( A_2 = -\frac{7}{4} \) and \( A_3 = \frac{3}{2} \). So the Inverse Laplace transform is

\[ \mathcal{L}^{-1}\{F(s)\} = \frac{7}{4}e^t - \frac{7}{4}e^{-t} + \frac{3}{2}te^{-t} \]
Problem 3. Find the Inverse Laplace transform of the following:

\[
\frac{2 - 5s}{(s - 6)(s^2 + 11)}
\]

Answer. Partial fraction expansion is

\[
F(s) = \frac{A_1}{s - 6} + \frac{A_2 s + A_3}{s^2 + 11}
\]

So we have

\[
2 - 5s = A_1(s^2 + 11) + (A_2 s + A_3)(s - 6)
\]

Some computation gives us \(A_1 = -\frac{28}{47}, A_2 = \frac{28}{47}\) and \(A_3 = -\frac{67}{47}\). So the Inverse Laplace transform is

\[
\mathcal{L}^{-1}\{F(s)\} = -\frac{28}{47}e^{6t} + \frac{28}{47} \cos(\sqrt{11}t) - \frac{67}{47} \sqrt{11} \sin(\sqrt{11}t)
\]

Problem 4. Solve the following IVP:

\[
y'' + 4y = 4t^2 - 4t + 10, \quad y(0) = 0, \quad y'(0) = 3
\]

Answer. We apply Laplace transform to the ODE to get

\[
\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{4t^2 - 4t + 10\} = \frac{8}{s^3} - \frac{4}{s^2} + \frac{10}{s}
\]

Denoting \(\mathcal{L}\{y\}\) by \(Y(s)\), and recalling that \(\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 3\), we get

\[
s^2Y(s) - 3 + 4Y(s) = \frac{8}{s^3} - \frac{4}{s^2} + \frac{10}{s} = \frac{10s^2 - 4s + 8}{s^3}
\]

Rewriting this, we have

\[
(s^2 + 4)Y(s) = \frac{10s^2 - 4s + 8}{s^3} + 3 = \frac{3s^3 + 10s^2 - 4s + 8}{s^3}
\]

So, we have

\[
Y(s) = \frac{3s^3 + 10s^2 - 4s + 8}{s^3(s^2 + 4)}
\]

So we have \(y(t) = \mathcal{L}^{-1}\{Y\}\), i.e.

\[
y(t) = \mathcal{L}^{-1}\left\{\frac{3s^3 + 10s^2 - 4s + 8}{s^3(s^2 + 4)}\right\}
\]

Partial fraction expansion is

\[
F(s) = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{A_3}{s^3} + A_4 s + B_4
\]
So we have

\[ 3s^3 + 10s^2 - 4s + 8 = A_1 s^2(s^2 + 4) + A_2 s(s^2 + 4) + A_3 (s^2 + 4) + (A_4 s + B_4) s^3 \]

Some computation gives us \( A_1 = 2, A_2 = -1, A_3 = 2, A_4 = -2 \) and \( B_4 = 4 \). So the Inverse Laplace transform is

\[ y(t) = 2 - t + t^2 - 2 \cos 2t + 2 \sin 2t \]

**Problem 5.** Solve the following IVP:

\[ 2y'' + 3y' - 2y = te^{-2t}, \quad y(0) = 0, \quad y'(0) = -2 \]

**Answer.** We apply Laplace transform to the ODE to get

\[ 2\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{te^{-2t}\} = \frac{1}{(s+2)^2} \]

Denoting \( \mathcal{L}\{y\} \) by \( Y(s) \), and recalling that \( \mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) \) and \( \mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) + 2 \), we get

\[ 2s^2Y(s) + 4 + 3sY(s) - 2Y(s) = \frac{1}{(s+2)^2} \]

Rewriting this, we have

\[ (2s^2 + 3s - 2)Y(s) = -4 + \frac{1}{(s+2)^2} = \frac{-4s^2 - 16s - 15}{(s+2)^2} \]

So, we have

\[ Y(s) = \frac{-4s^2 - 16s - 15}{(s+2)^2(2s^2 + 3s - 2)} = \frac{-4s^2 - 16s - 15}{(s+2)^3(2s-1)} \]

So we have \( y(t) = \mathcal{L}^{-1}\{Y\} \), i.e.

\[ y(t) = \mathcal{L}^{-1}\left\{ \frac{-4s^2 - 16s - 15}{(s+2)^3(2s-1)} \right\} \]

Partial fraction expansion is

\[ F(s) = \frac{A_1}{s+2} + \frac{A_2}{(s+2)^2} + \frac{A_3}{(s+2)^3} + \frac{A_4}{2s-1} \]

So we have

\[ -4s^2 - 16s - 15 = A_1(s+2)^2(2s-1) + A_2(s+2)(2s-1) + A_3(2s-1) + A_4(s+2)^3 \]

Some computation gives us \( A_1 = \frac{96}{125}, A_2 = -\frac{2}{25}, A_3 = -\frac{1}{5}, A_4 = -\frac{192}{125} \). So the Inverse Laplace transform is

\[ y(t) = \frac{96}{125}e^{-2t} - \frac{2}{25}te^{-2t} - \frac{1}{10}t^2e^{-2t} - \frac{96}{125}e^{\frac{t}{2}} \]
**Problem 6.** Solve the following IVP:

\[ ty'' - ty' + y = 2, \quad y(0) = 2, \quad y'(0) = -1 \]

Note: The last part of the solution here is different from something that you have seen before, so this question is not a part of the final (or anything similar).

**Answer.** We apply Laplace transform to the ODE to get

\[
\mathcal{L}\{ty''\} - \mathcal{L}\{ty'\} + \mathcal{L}\{y\} = \mathcal{L}\{2\} = \frac{2}{s}
\]

Denoting \( \mathcal{L}\{y\} \) by \( Y(s) \), and recalling that \( \mathcal{L}\{ty'\} = -\frac{d}{ds}(sY(s) - y(0)) = -(Y(s) + sY'(s)) \)
and \( \frac{d}{ds}(\mathcal{L}\{ty''\}) = -\frac{d}{ds}(s^2Y(s) - sy(0) - y'(0)) = -(2sY(s) + s^2Y'(s) - y(0)) = -2sY(s) - s^2Y'(s) + 2 \), we get

\[
-2sY(s) - s^2Y'(s) + 2 + Y(s) + sY''(s) + Y(s) = \frac{2}{s}
\]

Rewriting this, we have

\[
(s^2 - s)Y''(s) + (2s - 2)Y(s) = 2 - \frac{2}{s}
\]

So, we have

\[
Y'(s) + \frac{2}{s}Y(s) = \frac{2s - 2}{s(s^2 - s)} = \frac{2}{s^2}
\]

This is a linear ODE with integrating factor \( s^2 \). So we have

\[
s^2Y'(s) + 2sY(s) = 2
\]

This becomes

\[
(s^2Y(s))' = 2
\]

which gives us

\[
s^2Y(s) = 2s + c
\]

and so

\[
Y(s) = \frac{2}{s} + \frac{c}{s^2}
\]

So the Inverse Laplace transform is

\[
y(t) = 2 + ct
\]

Since \( y'(0) = -1 \), then \( c = -1 \), so \( y(t) = 2 - t \).
**Problem 7.** Solve the following IVP:

\[ y'' + y = t^2 + 2, \quad y(0) = 1, \quad y'(0) = -1 \]

**Answer.** We apply Laplace transform to the ODE to get

\[
\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{t^2 + 2\} = \frac{2}{s^3} + \frac{2}{s}
\]

Denoting \( \mathcal{L}\{y\} \) by \( Y(s) \), and recalling that \( \mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 1 \), we get

\[
s^2Y(s) - s + 1 + Y(s) = \frac{2}{s^3} + \frac{2}{s} = \frac{2(s^2 + 1)}{s^3}
\]

Rewriting this, we have

\[
Y(s) = \frac{2}{s^3} + \frac{s - 1}{s^2 + 1} = \frac{2}{s^3} + \frac{s}{s^2 + 1} - \frac{1}{s^2 + 1}
\]

So the Inverse Laplace transform is

\[ y(t) = t^2 + \cos t - \sin t \]