Lecture 20

References:
1) Section 7.4 & 7.5 of Nagle, Saff and Snider's textbook
2) Section 4-3 & 4-5 of Paul’s notes

Let us start with an example:

Example 0.1. Calculate \( L^{-1} \left\{ \frac{25}{s^3(s^2 + 4s + 5)} \right\} \): Note that \( s^2 + 4s + 5 = (s + 2)^2 + 1 \). We want to find constants \( A_1, A_2, A_3, A_4 \) and \( A_5 \) such that
\[
\frac{25}{s^3(s^2 + 4s + 5)} = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{A_3}{s^3} + \frac{A_4(s + 2) + A_5}{s^2 + 4s + 5}
\]
Multiplying both sides by \( s^3(s^2 + 4s + 5) \) to get
\[
25 = A_1 s^2(s^2 + 4s + 5) + A_2 s(s^2 + 4s + 5) + A_3 (s^2 + 4s + 5) + A_4(s + 2)s^3 + A_5 s^3
\]
Plug in \( s = 0 \) to get \( 25 = 5A_3 \) to get \( A_3 = 5 \), Comparing coefficients of \( s^4 \), we have
\[
0 = A_1 + A_4
\]
Comparing coefficients of \( s^3 \) to get
\[
0 = 4A_1 + A_2 + 2A_4 + A_5
\]
Comparing coefficients of \( s^2 \) to get
\[
0 = 5A_1 + 4A_2 + A_3
\]
Comparing coefficients of \( s \) to get
\[
0 = 5A_2 + 4A_3
\]
From the last equation, we have \( A_2 = -4 \). We can solve the equations to get \( A_1 = \frac{11}{5} \), \( A_4 = -\frac{11}{5} \) and \( A_5 = -\frac{2}{5} \). So we have
\[
L^{-1} \left\{ \frac{25}{s^3(s^2 + 4s + 5)} \right\} = \frac{11}{5} L^{-1} \left\{ \frac{1}{s} \right\} - 4L^{-1} \left\{ \frac{1}{s^2} \right\} + 5L^{-1} \left\{ \frac{1}{s^3} \right\} - \frac{11}{5} L^{-1} \left\{ \frac{s + 2}{(s + 2)^2 + 1} \right\} - \frac{2}{5} L^{-1} \left\{ \frac{1}{(s + 2)^2 + 1} \right\}
\]
\[
= \frac{11}{5} - 4t + \frac{5}{2} t^2 - \frac{11}{5} e^{-2t} \cos t - \frac{2}{5} e^{-2t} \sin t
\]

In general, if there are quadratic factors in the denominator, say, something of the form \( ((s - \alpha)^2 + \beta^2)^m \), then the partial fraction decomposition takes the form
\[
\frac{A_1(s - \alpha) + B_1}{(s - \alpha)^2 + \beta^2} + \frac{A_2(s - \alpha) + B_2}{((s - \alpha)^2 + \beta^2)^2} + \cdots + \frac{A_m(s - \alpha) + B_m}{((s - \alpha)^2 + \beta^2)^m}
\]
We now see how we can use Laplace transforms to solve IVPs:
Example 0.2. Solve the following IVP using Laplace transforms:

\[ y'' - 2y' + 5y = -8e^{-t}, \quad y(0) = 2, \quad y'(0) = 12 \]

Let us denote \( \mathcal{L}\{y\}(s) \) by \( Y(s) \). Applying the Laplace transform on both sides of the equation above, we get

\[
\mathcal{L}\{y'' - 2y' + 5y\} = -8\mathcal{L}\{e^{-t}\} = -\frac{8}{s+1}
\]

Recall that \( \mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0) = sY(s) - 2 \) and \( \mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0) = s^2Y(s) - 2s - 12 \). So we have

\[
s^2Y(s) - 2s - 12 - 2(sY(s) - 2) + 5Y(s) = -\frac{8}{s+1}
\]

which gives us

\[
(s^2 - 2s + 5)Y(s) - 2s - 8 = -\frac{8}{s+1}
\]

and so

\[
(s^2 - 2s + 5)Y(s) = 2s + 8 - \frac{8}{s+1} = \frac{(2s+8)(s+1) - 8}{s+1} = \frac{2s^2 + 10s}{s+1}
\]

This gives us

\[
Y(s) = \frac{2s^2 + 10s}{(s+1)(s^2 - 2s + 5)}
\]

Since \( Y = \mathcal{L}\{y\} \), we have

\[
y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left\{ \frac{2s^2 + 10s}{(s+1)(s^2 - 2s + 5)} \right\}
\]

We find the partial fraction decomposition of this:

\[
\frac{2s^2 + 10s}{(s+1)(s^2 - 2s + 5)} = \frac{A_1}{s+1} + \frac{A_2(s-1) + A_3}{s^2 - 2s + 5}
\]

Multiplying both sides by \( (s+1)(s^2 - 2s + 5) \), we get

\[
2s^2 + 10s = A_1(s^2 - 2s + 5) + A_2(s-1)(s+1) + A_3(s+1)
\]

Plug in \( s = -1 \) to get \(-8 = 8A_1 \) and plug in \( s = 1 \) to get \( 12 = 4A_1 + 2A_3 \) which gives us \( A_1 = -1 \) and \( A_3 = 8 \). Let us compare the coefficient of \( s^2 \) in both sides of the equation above. In the LHS, the coefficient of \( s^2 \) is 2. In the RHS, the coefficient of \( s^2 \) is \( A_1 + A_2 \). So we should have \( A_1 + A_2 = 2 \), which gives us \( A_2 = 2 - A_1 = 3 \). So we have

\[
\mathcal{L}^{-1}\left\{ \frac{2s^2 + 10s}{(s+1)(s^2 - 2s + 5)} \right\} = -\mathcal{L}^{-1}\left\{ \frac{1}{s+1} \right\} + 3\mathcal{L}^{-1}\left\{ \frac{s-1}{(s-1)^2 + 4} \right\} + 4\mathcal{L}^{-1}\left\{ \frac{2}{(s-1)^2 + 4} \right\}
\]

\[
= -e^{-t} + 3e^t \cos 2t + 4e^t \sin 2t
\]

So we have

\[
y(t) = -e^{-t} + 3e^t \cos 2t + 4e^t \sin 2t
\]
This is how we use Laplace transforms to solve IVPs. Of course, we can solve the IVP above by other methods, i.e. first finding the general solution of \( y'' - 2y' + 5y = 0 \) and then finding a particular solution of \( y'' - 2y' + 5y = -8e^{-t} \) using the method of undetermined coefficients or the method of variation of parameters, and then plugging in the initial value conditions. Anyway, let us see another example, for which we cannot use any of the methods that we have seen until now.

**Example 0.3.** Solve the following IVP using Laplace transforms:

\[
y'' + 2ty' - 4y = 1, \quad y(0) = y'(0) = 0
\]

Let us denote \( \mathcal{L}\{y\}(s) \) by \( Y(s) \). Applying the Laplace transform on both sides of the equation above, we get

\[
\mathcal{L}\{y'' + 2ty' - 4y\} = \mathcal{L}\{1\} = \frac{1}{s}
\]

We have \( \mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0) = s^2Y(s) \). So we have

\[
s^2Y(s) + 2\mathcal{L}\{ty'\} - 4Y(s) = \frac{1}{s}
\]

Using the fact that \( \mathcal{L}\{tf\} = -\frac{d}{ds}(\mathcal{L}\{f\}) \) and \( \mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0) = sY(s) \), we have

\[
s^2Y(s) - 2Y(s) - 2sY'(s) - 4Y(s) = \frac{1}{s}
\]

and so

\[
(s^2 - 6)Y(s) - 2sY'(s) = \frac{1}{s}
\]

So we have

\[
2sY'(s) + (6 - s^2)Y(s) = -\frac{1}{s}
\]

Dividing by \( 2s \), we have

\[
Y'(s) + \left(\frac{3}{s} - \frac{s}{2}\right)Y(s) = -\frac{1}{2s^2}
\]

Note that this is a first order linear ODE with integrating factor

\[
\mu(s) = e^{\int \left(\frac{3}{s} - \frac{s}{2}\right) ds} = e^{3\ln s - \frac{s^2}{4}} = s^3e^{-\frac{s^2}{4}}
\]

So we have

\[
\frac{d}{ds}(\mu Y) = -\frac{1}{2s^2} s^3 e^{-\frac{s^2}{4}} = -\frac{s}{2} e^{-\frac{s^2}{4}}
\]

Integrating, we get

\[
\mu Y = -\int \frac{s}{2} e^{-\frac{s^2}{4}} ds = e^{-\frac{s^2}{4}} + C
\]

So we have

\[
Y(s) = \frac{e^{-\frac{s^2}{4}} + C}{s^3e^{-\frac{s^2}{4}}} = \frac{1}{s^3} + Ce^{\frac{s^2}{4}}
\]

Now, we use the following fact: If \( f \) is a piecewise continuous function of some exponential order, then

\[
\lim_{s \to \infty} \mathcal{L}\{f\}(s) = 0.
\]

(This is not difficult to prove. Try it!)

Anyway, as we have seen the theory of Laplace transforms mostly for functions of
some exponential order, and since we are looking at Laplace transforms of $y$, we assume that $y$ is also of some exponential order. So we have

$$\lim_{s \to \infty} Y(s) = 0$$

Since $Y(s) = \frac{1}{s^3} + C \frac{e^{\frac{t^2}{s}}}{s}$ and $\lim_{s \to \infty} \frac{1}{s^3} = 0$ and $\lim_{s \to \infty} \frac{e^{\frac{t^2}{s}}}{s^3} = \infty$, unless $C = 0$, we cannot have $\lim_{s \to \infty} Y(s) = 0$. So we have $C = 0$, and so

$$Y(s) = \frac{1}{s^3}$$

Hence

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s^3} \right\} = \frac{t^2}{2}$$

An important fact that we used above that needs to be highlighted again is that we assume that if $Y(s) = \mathcal{L}\{y\}$, then

$$\lim_{s \to \infty} Y(s) = 0$$