LECTURE 13

References:
1) Section 4.7 of Nagle, Saff and Snider’s textbook

We start with giving a quick overview of the types of second order ODEs that we have seen until now:

- We first started with understanding how to solve
  \[ ay'' + by' + cy = 0 \]
  where \( a, b \) and \( c \) are constants. This was done by looking at the characteristic equation, i.e. the equation \( ar^2 + br + c = 0 \) and then depending on the roots of this quadratic equation, we get the general solution.

- We then looked at equations of the form
  \[ ay'' + by' + cy = C t^m e^{rt} \text{ or } C t^m e^{at} (\cos \beta t / \sin \beta t) \]
  where \( m \) is a non-negative integer. This is solved using the method of undetermined coefficients.

- We then quickly talked about the superposition principle which states that we can obtain a solution for
  \[ ay'' + by' + cy = k_1 f_1(t) + k_2 f_2(t) \]
  by looking at solutions of \( ay'' + by' + cy = f_1(t) \) and \( ay'' + by' + cy = f_2(t) \) and then adding the two: If \( y_{p,1} \) is a solution to \( ay'' + by' + cy = f_1(t) \) and \( y_{p,2} \) is a solution to \( ay'' + by' + cy = f_2(t) \), then \( y_p = k_1 y_{p,1} + k_2 y_{p,2} \) is a solution to \( ay'' + by' + cy = k_1 f_1(t) + k_2 f_2(t) \).

- We then moved on to look at
  \[ ay'' + by' + cy = f(t) \]
  where \( f \) is any function of \( t \). For this, we used the method of variation of parameters: Suppose that \( y_1(t) \) and \( y_2(t) \) are two linearly independent solutions (we’ll talk a bit more about this in this lecture) of \( ay'' + by' + cy = 0 \), then we put \( y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) \) and we can then solve for \( v_1 \) and \( v_2 \). There are explicit equations for \( v_1 \) and \( v_2 \).

- We also saw that if you now replace \( a, b \) and \( c \) with functions of \( t \), the method of variation of parameters still works, i.e. for the ODE
  \[ a(t)y'' + b(t)y' + c(t)y = f(t), \]
  supposing that we know two linearly independent solutions \( y_1 \) and \( y_2 \) of \( a(t)y'' + b(t)y' + c(t)y = 0 \), we can use the same technique to obtain \( y_p \). But the problem here is that we don’t have a technique to find \( y_1 \) and \( y_2 \). So this implies that the problem of solving \( a(t)y'' + b(t)y' + c(t)y = f(t) \) is as easy or as difficult as solving the ODE \( a(t)y'' + b(t)y' + c(t)y = 0 \). In this lecture, we’ll see a special type of ODE (Cauchy-Euler equations) that falls in this scenario and for which we can actually find \( y_1 \) and \( y_2 \).
Before we move on to the above said specially type of ODEs, let us quickly discuss what it means for two solutions to be linearly independent: Two solutions $y_1$ and $y_2$ are said to be linearly independent if one is not a constant multiple of the other, or in other words, $\frac{y_2(t)}{y_1(t)} \neq$ a constant. For example, $y_1(t) = e^t$ and $y_2(t) = e^{2t}$ are linearly independent as one is not a constant multiple of the other, but $y_1(t) = e^t$ and $y_2(t) = 2e^t$ are not linearly independent as one is a constant multiple of the other.

For a second order linear ODE

$$a(t)y'' + b(t)y' + c(t)y = 0,$$

if you know two linearly independent solutions $y_1(t)$ and $y_2(t)$, then the general solution is $y(t) = c_1y_1(t) + c_2y_2(t)$. This is why finding two linearly independent solutions forms the crux of solving such an ODE.

Now we see how to solve Cauchy-Euler equations, i.e. equations of the form

$$at^2y'' + bt'y' + cy = 0$$

where $a$, $b$ and $c$ are real numbers. Note that the coefficient of $y$ is constant, but there is a $t$ term in the coefficient of $y'$ and a $t^2$ term in the coefficient of $y''$. So it seems that $y$ has to be something of the form $t^r$ because then $ty'$ would be of the form $r(t^{r-1}) = t^r$ and $t^2y''$ would be of the form $r(r-1)t^{r-2} = t^r$. So let us see if that works. Plugging in $y = t^r$, we get

$$at^2(r(r-1))t^{r-2} + btr^{r-1} + ct^r = 0$$

Taking $t^r$ common and removing it, we see that $r$ has to satisfy the quadratic equation

$$ar(r-1) + br + c = 0$$

Rewriting this, we get

$$ar^2 + (b-a)r + c = 0$$

So if $y = t^r$ is a solution, $r$ should satisfy the quadratic equation above.

Let us see an example:

**Example 0.1.** Solve the following ODE:

$$t^2y'' + 5ty' + 3y = 0$$

This is a Cauchy-Euler equation with $a = 1$, $b = 5$ and $c = 3$. Then $y = t^r$ is a solution if $r$ satisfies the quadratic equation

$$r^2 + 4r + 3 = 0$$

which gives us $r = -1$ or $r = -3$. So we get two linearly independent solutions $y_1(t) = t^{-1} = \frac{1}{t}$ and $y_2(t) = t^{-3} = \frac{1}{t^3}$. So we have the general solution

$$y(t) = c_1y_1 + c_2y_2 = \frac{c_1}{t} + \frac{c_2}{t^3}$$

So in this example, we got 2 real roots. And as expected, when the equation $ar^2 + (b-a)r + c = 0$ has 2 real roots $r_1$ and $r_2$, then the Cauchy-Euler equation has general solution

$$y(t) = c_1t^{r_1} + c_2t^{r_2}$$

But there are two more possibilities: when the quadratic equation has just one real root, and when it has 2 complex roots. Let us consider the case of just 1 real root $r$. 
Then we have $y_1(t) = t^r$ as a solution. Now how do we get another linearly independent solution $y_2(t)$? We’ll see a method of doing this. Note that this method works for any general second order homogeneous linear ODE. So let us see this method.

Consider the ODE

$$a(t)y'' + b(t)y' + c(t)y = 0$$

Dividing the equation by $a(t)$, we can rewrite this equation (with new notation) as

$$y'' + p(t)y' + q(t)y = 0$$

Suppose that we have one solution $y_1(t)$ to this ODE. The question we want to answer is: Can you use $y_1$ to find another solution $y_2$ such that $y_1$ and $y_2$ are linearly independent? We can, and the motivation for this method is quite similar to that for the method of variation of parameters. So we put $y_2(t) = v(t)y_1(t)$ and try to find what $v$ has to be. If $v$ is a non-constant function, then we are done, as that would be enough to ensure that $y_1$ and $y_2$ are linearly independent. We have

$$y_2' = v'y_1 + vy_1'$$

and

$$y_2'' = v''y_1 + 2v'y_1' + vy_1''$$

So for $y_2$ to be a solution, we should have

$$(v''y_1 + 2v'y_1' + vy_1'') + p(v'y_1 + vy_1') + qvy_1 = 0$$

Rearranging the terms above, we get

$$v(y''_1 + py_1' + qy_1) + (v''y_1 + 2v'y_1' + pv'y_1) = 0$$

Since $y_1$ is a solution to the ODE, we have $y''_1 + py_1' + qy_1 = 0$. So the above expression simplifies to

$$v''y_1 + v'(2y'_1 + py_1) = 0$$

Putting $w = v'$, we get

$$w'y_1 = -w(2y'_1 + py_1)$$

which gives us

$$\frac{dw}{dt} = -w\left(\frac{2y'_1}{y_1} + p\right)$$

We know that $\frac{2y'_1}{y_1} + p$ is a function of $t$, and so the equation above is a separable ODE. So we solve it:

$$\int \frac{dw}{w} = -\int \left(\frac{2y'_1}{y_1} + p\right) dt$$

Integrating, we get

$$\ln w = -2\int \frac{y'_1}{y_1} dt - \int p(t) dt$$

The first integral above is $\ln y_1$. So we have

$$\ln w = -2\ln y_1 - \int p(t) dt$$
Taking exponential, we get

\[ w = e^{-2\ln y_1}e^{-\int p(t)dt} = \frac{e^{-\int p(t)dt}}{y_1^2} \]

Since \( w = v' \), we can integrate \( w \) to get

\[ v(t) = \int \frac{e^{-\int p(t)dt}}{y_1(t)^2}dt \]

and this gives us

\[ y_2(t) = v(t)y_1(t) = y_1(t) \int \frac{e^{-\int p(t)dt}}{y_1(t)^2}dt \]

Let us see an example with a Cauchy-Euler equation:

**Example 0.2.** Solve the ODE

\[ t^2y'' - ty' + y = 0 \]

This is a Cauchy-Euler equation with \( a = 1 \), \( b = -1 \) and \( c = 1 \). Then the corresponding quadratic equation is \( r^2 - 2r + 1 = 0 \) which has one real root \( r = 1 \). So this tells us that \( y_1(t) = t \) is a solution. Now we use the method above to find \( y_2(t) \). First, we reduce our ODE to the form \( y'' + py' + qy = 0 \). For that, we divide by \( t^2 \) to get

\[ y'' + \left( -\frac{1}{t} \right)y' + \frac{1}{t^2}y = 0 \]

So we have \( p(t) = -\frac{1}{t} \) and

\[ e^{-\int p(t)dt} = e^{\int \frac{1}{t}dt} = e^{\ln t} = t \]

Putting this back in the equation for \( v \) above, we have

\[ v(t) = \int \frac{e^{-\int p(t)dt}}{y_1(t)^2}dt = \int \frac{t}{t^2}dt = \int \frac{1}{t}dt = \ln t \]

So we have

\[ y_2(t) = v(t)y_1(t) = t \ln t \]

This is true in general. If the equation \( ar^2 + (b - a)r + c = 0 \) has just 1 real root \( r \), then the ODE \( at^2y'' + bty' + cy = 0 \) has two linearly independent solutions \( y_1(t) = t^r \) and \( y_2(t) = t^r \ln t \) and general solution

\[ y(t) = c_1t^r + c_2t^r \ln t \]

We shall see the case of two complex roots in the next lecture.