Let $f : [0, \infty) \to \mathbb{R}$ be a function. We recall the definition of the Laplace transform of $f$:

$$
\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t) dt = \lim_{N \to \infty} \int_0^N e^{-st} f(t) dt
$$

and this is defined for all $s$ such that the limit above exists. We also saw 3 main properties of Laplace transforms:

- Let $f$ be a function whose Laplace transform $F(s)$ exists for $s > \alpha$. Then
  $$
  \mathcal{L}\{e^{at}f(t)\}(s) = F(s - a)
  $$
  and this exists when $s > a + \alpha$.

- Let $f$ be a continuous function with $f', \ldots, f^{(n-1)}$ continuous and $f^{(n)}$ a piecewise continuous function and all of them of exponential order $\alpha$, then
  $$
  \mathcal{L}\{f^{(n)}\}(s) = s^n \mathcal{L}\{f\}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)
  $$
  for all $s > \alpha$.

- Let $f$ be piecewise continuous of exponential order $\alpha$ with Laplace transform $F(s)$. Then
  $$
  \mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n F}{ds^n}(s)
  $$
  for all $s > \alpha$.

In this lecture, we’ll define what the Inverse Laplace transform is, and see some examples. Before that, let us recall how Laplace transforms are used to solve IVPs and how Inverse Laplace transforms naturally come into the picture. Consider the following IVP:

$$
y'' + y = e^t, \quad y(0) = 0, y'(0) = 0
$$

We first apply the Laplace transform to the equation to get

$$
\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{e^t\}
$$

Denoting $\mathcal{L}\{y\}$ by $Y(s)$ and using the second property above, we have

$$
s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s - 1}
$$

We then have, using $y(0) = 0$ and $y'(0) = 0$,

$$
s^2 Y(s) + Y(s) = \frac{1}{s - 1}
$$

Then we have

$$
Y(s) = \frac{1}{(s - 1)(s^2 + 1)}
$$
So we know that
\[
Y(s) = \mathcal{L}\{y\}(s) = \frac{s^2 - s + 1}{(s - 1)(s^2 + 1)}
\]

Now, if we can obtain \(y\) from here, we would have solved the ODE. And this is where the Inverse Laplace transform comes into play: \(y\) is the Inverse Laplace transform of \(Y\).

Before we move on to the definition, let us copy down the Laplace transform table here:

<table>
<thead>
<tr>
<th>(f(t))</th>
<th>(\mathcal{L}{f}(s))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\frac{1}{s}, s &gt; 0)</td>
</tr>
<tr>
<td>(e^{at})</td>
<td>(\frac{1}{s-a}, s &gt; a)</td>
</tr>
<tr>
<td>(t^n, n = 1, 2, \ldots)</td>
<td>(\frac{n!}{s^{n+1}}, s &gt; 0)</td>
</tr>
<tr>
<td>(\sin bt)</td>
<td>(\frac{b}{s^2 + b^2}, s &gt; 0)</td>
</tr>
<tr>
<td>(\cos bt)</td>
<td>(\frac{s}{s^2 + b^2}, s &gt; 0)</td>
</tr>
<tr>
<td>(e^{at}t^n, n = 1, 2, \ldots)</td>
<td>(\frac{(s-a)^{n+1}}{n!}, s &gt; a)</td>
</tr>
<tr>
<td>(e^{at}\sin bt)</td>
<td>(\frac{b}{(s-a)^2 + b^2}, s &gt; a)</td>
</tr>
<tr>
<td>(e^{at}\cos bt)</td>
<td>(\frac{s-a}{(s-a)^2 + b^2}, s &gt; a)</td>
</tr>
</tbody>
</table>

Note that by knowing the Laplace transform of 1 and \(\sin bt\), you should be able to calculate the rest of the table using the 3 properties discussed in the last lecture.

**Definition 0.1.** Let \(f : [0, \infty) \to \mathbb{R}\) be a continuous function and let \(F\) be another function. If \(F = \mathcal{L}\{f\}\), then we say that \(f\) is the Inverse Laplace transform of \(F\) and we denote it by \(f = \mathcal{L}^{-1}\{F\}\).

Let us see a couple of simple examples:

**Example 0.2.** \(\mathcal{L}^{-1}\left\{\frac{3}{s^2 + 9}\right\} = \sin 3t\) and \(\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = e^t\). Both are easy to see from the table above.

Before we move on to more complicated examples, let us see a couple of easy properties of the Inverse Laplace transform:

- \(\mathcal{L}^{-1}\{F_1 + F_2\} = \mathcal{L}^{-1}\{F_1\} + \mathcal{L}^{-1}\{F_2\}\)

  This is quite easy to prove, but we will not spend time on a proof. Recall that an analogous statement is true for Laplace transforms, so it is natural to expect this.

- \(\mathcal{L}^{-1}\{cF\} = c\mathcal{L}^{-1}\{F\}\)

  where \(c\) is a real number. Again, this is very easy to prove and recall that an analogous statement is true for Laplace transforms.

Let us see an example:

**Example 0.3.** Calculate \(\mathcal{L}^{-1}\left\{\frac{5}{(s-1)^3}\right\}\) : From the table, we see that this resembles the Laplace transform of \(e^{at}t^n\), and we should expect \(a = 1\) and \(n = 3\). We have

\[
\mathcal{L}^{-1}\left\{\frac{3!}{(s-1)^3}\right\} = e^t t^3
\]

So, using the second property above, we can take out the \(3! = 6\) outside to get

\[
6\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\} = e^t t^3
\]
Multiplying both sides by \( \frac{5}{6} \), we get
\[
5\mathcal{L}^{-1}\left\{ \frac{1}{(s-1)^2} \right\} = \frac{5}{6} e^t t^3
\]
Now we can take the 5 inside (using the second property again), to get
\[
\mathcal{L}^{-1}\left\{ \frac{5}{(s-1)^2} \right\} = \frac{5}{6} e^t t^3
\]

Let us see a slightly more complicated example:

**Example 0.4.** Calculate \( \mathcal{L}^{-1}\left\{ \frac{-3s^2}{s^2 + 2s + 10} \right\} \) : This doesn’t immediately resemble anything from the table, but note that we can write the denominator as \( s^2 + 2s + 10 = (s + 1)^2 + 9 = (s + 1)^2 + 3^2 \). So we have
\[
\frac{3s + 2}{s^2 + 2s + 10} = \frac{3s + 2}{(s + 1)^2 + 3^2}
\]
This closely resembles the Laplace transform of \( e^{-t} \cos 3t \), but not quite. We need \( s + 1 \) in the numerator for it to actually be the Laplace transform of \( e^{-t} \cos 3t \). So let us write \( 3s + 2 = 3(s + 1) - 1 \) to get
\[
\frac{3s + 2}{s^2 + 2s + 10} = \frac{3(s + 1) - 1}{(s + 1)^2 + 3^2} = \frac{3}{(s + 1)^2 + 3^2} - \frac{1}{(s + 1)^2 + 3^2}
\]
Using the properties of the Inverse Laplace transform above, we have
\[
\mathcal{L}^{-1}\left\{ \frac{3s + 2}{s^2 + 2s + 10} \right\} = 3\mathcal{L}^{-1}\left\{ \frac{s + 1}{(s + 1)^2 + 3^2} \right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{ \frac{3}{(s + 1)^2 + 3^2} \right\}
\]
From the table, we then get
\[
\mathcal{L}^{-1}\left\{ \frac{3s + 2}{s^2 + 2s + 10} \right\} = 3e^{-t} \cos 3t - \frac{1}{3} e^{-t} \sin 3t
\]

Let us see one more example:

**Example 0.5.** Calculate \( \mathcal{L}^{-1}\left\{ \frac{3}{s^2 - 49} \right\} \) : Note that here we have \( s^2 - 49 = s^2 - 7^2 \), which doesn’t resemble anything from the table. (It would have, has it been \( s^2 + 49 \).) But we can factorize it to \( s^2 - 49 = (s - 7)(s + 7) \) and this suggests that we could use the theory of partial fractions to try to solve this. So let us recall what it means to write this as a partial fraction decomposition. We want to find constants \( A \) and \( B \) such that
\[
\frac{1}{(s - 7)(s + 7)} = \frac{A}{s - 7} + \frac{B}{s + 7}
\]
Multiplying both sides by \( (s - 7)(s + 7) \), we get
\[
1 = A(s + 7) + B(s - 7)
\]
We can solve for \( A \) and \( B \) in two ways. We can either expand the RHS above to get
\[
1 = As + 7A + Bs - 7B = (A + B)s + 7(A - B)
\]
and since \( s \) is a variable, we have \( A + B = 0 \) and \( 7(A - B) = 1 \), which gives us \( A = \frac{1}{14} \) and \( B = -\frac{1}{14} \). Else, we can look at \( 1 = A(s + 7) + B(s - 7) \) and plug in \( s = 7 \) to get \( 1 = 14A \) and plug in \( s = -7 \) to get \( 1 = -14B \) which gives us \( A = \frac{1}{14} \) and \( B = -\frac{1}{14} \). The second method is usually better. Anyway, we now have
\[
\frac{1}{(s - 7)(s + 7)} = \frac{1}{14(s - 7)} - \frac{1}{14(s + 7)}
\]
So we have
\[
\mathcal{L}^{-1}\left\{ \frac{3}{s^2 - 49} \right\} = 3\mathcal{L}^{-1}\left\{ \frac{1}{s^2 - 49} \right\} = 3\mathcal{L}^{-1}\left\{ \frac{1}{14 \cdot s - 7} - \frac{1}{14 \cdot s + 7} \right\}
\]
which gives us
\[
\mathcal{L}^{-1}\left\{ \frac{3}{s^2 - 49} \right\} = \frac{3}{14} \mathcal{L}^{-1}\left\{ \frac{1}{s - 7} \right\} - \frac{3}{14} \mathcal{L}^{-1}\left\{ \frac{1}{s + 7} \right\}
\]
Looking at the table, we have
\[
\mathcal{L}^{-1}\left\{ \frac{3}{s^2 - 49} \right\} = \frac{3}{14} e^{7t} - \frac{3}{14} e^{-7t}
\]

The theory of partial fractions is what we will be using to find Inverse Laplace transforms. We will only be dealing with functions of the form \(\frac{\text{polynomial}}{\text{polynomial}}\) as in all the examples that we have seen above. Also, the degree of the polynomial in the numerator would be always less than the degree of the polynomial in the denominator. There are three cases to consider. In this lecture, we’ll see the first one: the denominator can be written as a product of distinct linear factors, i.e. we want to find
\[
\mathcal{L}^{-1}\left\{ \frac{f(s)}{(s - a_1)\ldots(s - a_r)} \right\}
\]
where \(\deg(f) < r\) and all the \(a_i\)’s are distinct. For this, we write it as
\[
f(s) = \frac{A_1}{s - a_1} + \ldots + \frac{A_r}{s - a_r}
\]
and then we can multiply the whole thing by \((s - a_1)\ldots(s - a_r)\) to get
\[
f(s) = A_1(s - a_2)\ldots(s - a_r) + A_2(s - a_1)(s - a_3)\ldots(s - a_r) + \ldots + A_r(s - a_1)\ldots(s - a_{r-1})
\]
We can plug in \(s = a_1\) to obtain \(A_1\), \(s - a_2\) to obtain \(A_2\) etc. Now we can use this new form to find the Inverse Laplace transform:
\[
\mathcal{L}^{-1}\left\{ \frac{f(s)}{(s - a_1)\ldots(s - a_r)} \right\} = \mathcal{L}^{-1}\left\{ \frac{A_1}{s - a_1} \right\} + \ldots + \mathcal{L}^{-1}\left\{ \frac{A_r}{s - a_r} \right\}
\]
which, from the table above, gives us
\[
\mathcal{L}^{-1}\left\{ \frac{f(s)}{(s - a_1)\ldots(s - a_r)} \right\} = A_1 e^{a_1 t} + \ldots + A_r e^{a_r t}
\]
We end this lecture with an example:

**Example 0.6.** Calculate \(\mathcal{L}^{-1}\left\{ \frac{s+7}{s^2-3s-10} \right\} : \) Note that here we have \(s^2 - 3s - 10 = (s - 5)(s + 2)\). We want to find constants \(A_1\) and \(A_2\) such that
\[
\frac{s + 7}{(s - 5)(s + 2)} = \frac{A_1}{s + 2} + \frac{A_2}{s - 5}
\]
Multiplying both sides by \((s - 5)(s + 2)\), we get
\[
s + 7 = A_1(s - 5) + A_2(s + 2)
\]
Plug in \(s = 5\) to get \(12 = 7A_2\) and plug in \(s = -2\) to get \(5 = -7A_1\) which gives us \(A_1 = -\frac{5}{7}\) and \(A_2 = \frac{12}{7}\). So we have
\[
\frac{s + 7}{(s - 5)(s + 2)} = \frac{12}{7} \frac{1}{s - 5} - \frac{5}{7} \frac{1}{s + 2}
\]
So we have
\[
\mathcal{L}^{-1}\left\{ \frac{s + 7}{s^2 - 3s - 10} \right\} = \frac{12}{7}\mathcal{L}^{-1}\left\{ \frac{1}{s - 5} \right\} - \frac{5}{7}\mathcal{L}^{-1}\left\{ \frac{1}{s + 2} \right\}
\]

Looking at the table, we have
\[
\mathcal{L}^{-1}\left\{ \frac{s + 7}{s^2 - 3s - 10} \right\} = \frac{12}{7}e^{5t} - \frac{5}{7}e^{-2t}
\]