LECTURE 17

References:
1) Section 7.3 of Nagle, Saff and Snider's textbook
2) Section 4-2 of Paul's notes

In the last lecture, we saw two examples, one where $\mathscr{L}\{f\}$ is defined for $s > 0$ and one where $\mathscr{L}\{f\}$ is defined for $s > 4$. Now, the following question arises: Let $\alpha > 0$. Can we give some condition on f to ensure that its Laplace transform will be defined for all $s > \alpha$? We can, and we will see that today. This lecture will be a bit more theoretical in nature in the beginning.

Definition 0.1. A function $f : [0, \infty) \to \mathbb{R}$ is said to be of exponential order α if there exists positive constants M, T such that

$$|f(x)| \leq Me^{\alpha x}$$

for all $x \geq T$.

In more informal terms, f is said to be of exponential order α if for x large enough, f is bounded above by a constant multiple of $e^{\alpha x}$.

Example 0.2. The function

$$f(x) = \begin{cases} 2 & \text{if } 0 \leq x < 5 \\ 0 & \text{if } 5 \leq x < 10 \\ e^{4x} & \text{if } 10 \leq x \end{cases}$$

from last lecture clearly satisfies $|f(x)| = e^{4x} \leq e^{4x}$ for $x > 10$ and so with $M = 1$ and $T = 10$, we can see that f is of exponential order 4, and we saw that the Laplace transform exists for $s > 4$.

Example 0.3. The function

$$f(x) = \begin{cases} e^{2x} & \text{if } 0 \leq x < 15 \\ e^{6x} & \text{if } 5 \leq x < 100 \\ e^{8x} & \text{if } 100 \leq x \end{cases}$$

is of exponential order 8, and the function

$$g(x) = \begin{cases} e^{2x} & \text{if } 0 \leq x < 15 \\ e^{8x} & \text{if } 5 \leq x < 100 \\ 100e^{6x} & \text{if } 100 \leq x \end{cases}$$

is of exponential order 6.

In general, we have the following result:

Theorem 0.4. *If $f : [0, \infty) \to \mathbb{R}$ is a piecewise continuous function of exponential order α, then $\mathscr{L}\{f\}(s)$ is defined for all $s > \alpha$.*

Anyway, we shouldn't get lost in all this technicality, and redo a calculation from the last lecture: Suppose that f is a piecewise continuous function of exponential order α. Then, using integration by parts, we get

$$\int_0^N e^{-sx} f'(x)dx = e^{-sx} f(x)\Big|_{x=0}^{N} - \int_0^N (-s)e^{-sx} f(x)dx$$
$$= e^{-sN} f(N) - f(0) + s\int_0^N e^{-sx} y(x)dx$$

Since $s > \alpha$ and $|f(N)| \leq Me^{\alpha N}$ for N large enough, we have $|e^{-sN} f(N)| \leq Me^{-(s-\alpha)N}$ and since $e^{-(s-\alpha)N} \to 0$ as $N \to \infty$, this shows that

$$\lim_{N\to\infty} e^{-sN} f(N) = 0$$

So we get that $\mathscr{L}\{f'\}$ also exists for $s > \alpha$ and

$$\mathscr{L}\{f'\}(s) = s\mathscr{L}\{f\}(s) - f(0)$$

In the previous lecture, we just gave a vague explanation as to why we can ignore the $e^{-sN} f(N)$ but now we have proof that it becomes 0 when f satisfies some nice conditions (exponential order α).
To conclude, we should just keep in mind that all the f that we see in this course will be nice enough to have a Laplace transform for $s > \alpha$ for some α, and not get bogged down by all these technical details.

Now, the question is: What happens for higher derivatives of f? Let us first try f''. Using the above with f', we have

$$\mathscr{L}\{f''\}(s) = s\mathscr{L}\{f'\}(s) - f'(0)$$

and again replacing $\mathscr{L}\{f'\}$ with the above expression, we get

$$\mathscr{L}\{f''\}(s) = s(s\mathscr{L}\{f\}(s) - f(0)) - f'(0) = s^2\mathscr{L}\{f\}(s) - sf(0) - f'(0)$$

Example 0.5. Let us see another example. Let $f(x) = \sin bx$, where b is any real number. Then $f' = b\cos bx$ and $f'' = -b^2 \sin bx = -b^2 f$. So we have $-b^2\mathscr{L}\{f\} = \mathscr{L}\{f''\}$. By the above property, we have

$$-b^2\mathscr{L}\{f\} = \mathscr{L}\{f''\} = s^2\mathscr{L}\{f\} - sf(0) - f'(0) = s^2\mathscr{L}\{f\} - b$$

Solving this, we get

$$\mathscr{L}\{f\} = \frac{b}{s^2 + b^2}$$

This method to find $\mathscr{L}\{\sin bx\}$ does not involve calculations involving integrals.

Example 0.6. We know that the Laplace transform of $\sin bx$ is $\frac{b}{s^2+b^2}$. We can use this to calculate the Laplace transform of $\cos bx$. If $f = \sin bx$, then $f' = b\cos bx$ (both are of exponential order 0), and so, for $s > 0$,

$$b\mathscr{L}\{\cos bx\}(s) = s\mathscr{L}\{\sin bx\}(s) - \sin 0 = \frac{sb}{s^2 + b^2}$$

Dividing by b, we get

$$b\mathscr{L}\{\cos bx\}(s) = \frac{s}{s^2 + b^2}$$

Similarly, we can extend this result to higher derivatives (try to prove it yourself):

$$\mathscr{L}\{f^{(n)}\}(s) = s^n \mathscr{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0)$$

We now see another property. Let f be a function whose Laplace transform exists for $s > \alpha$. Let us put $F(s) = \mathscr{L}\{f\}(s)$. Let a be a real number. Then we calculate the Laplace transform of $e^{ax}f(x)$:

$$\mathscr{L}\{e^{ax}f(x)\}(s) = \int_0^\infty e^{-sx}e^{ax}f(x)dx = \int_0^\infty e^{-(s-a)x}f(x)dx = \mathscr{L}\{f\}(s-a) = F(s-a)$$

and this exists when $s - a > \alpha$, i.e. $s > a + \alpha$. Let us see an example:

Example 0.7. Find $\mathscr{L}\{e^{2x}\sin 3x\}(s)$. We have, for $s > 0$,

$$F(s) = \mathscr{L}\{\sin 3x\}(s) = \frac{3}{s^2+9}$$

Then using the property above with $a = 2$, we have, for $s > 2$,

$$\mathscr{L}\{e^{2x}\sin 3x\}(s) = F(s-2) = \frac{3}{(s-2)^2+9}$$