References:
1) Section 7.3 of Nagle, Saff and Snider’s textbook
2) Section 4-2 of Paul’s notes

Let \( f : [0, \infty) \rightarrow \mathbb{R} \) be a function. We recall the definition of the Laplace transform of \( f \):
\[
\mathcal{L} \{ f \}(s) = \int_0^\infty e^{-st} f(t) dt = \lim_{N \to \infty} \int_0^N e^{-st} f(t) dt
\]
and this is defined for all \( s \) such that the limit above exists. We saw in the last lecture that if \( f \) is piecewise continuous and of exponential order \( \alpha \), then \( \mathcal{L} \{ f \}(s) \) exists for \( s > \alpha \). In this lecture, we’ll see three important properties of Laplace transforms that we will use to solve IVPs. Before that, let us see an example:

Example 0.1. Let \( b \) be a real number. Let us find \( \mathcal{L} \{ \sin bt \}(s) \). We have
\[
\mathcal{L} \{ \sin bt \}(s) = \lim_{N \to \infty} \int_0^N e^{-st} \sin b t dt
\]
This is an integral that can be calculated by using integration by parts twice. (Please complete it!) Also note that since \( |\sin bt| \leq 1 = e^{0t} \), \( \sin bt \) is of exponential order 0. We have
\[
\mathcal{L} \{ \sin bt \}(s) = \frac{b}{s^2 + b^2}, \quad s > 0
\]
We see the first property now. So let \( f \) be a function whose Laplace transform exists for \( s > \alpha \). Let us put \( F(s) = \mathcal{L} \{ f \}(s) \). Let \( a \) be a real number. Then we calculate the Laplace transform of \( e^{at} f(t) \):
\[
\mathcal{L} \{ e^{at} f(t) \}(s) = \int_0^\infty e^{-st} e^{at} f(t) dt = \int_0^\infty e^{-(s-a)t} f(t) dt = \mathcal{L} \{ f \}(s-a) = F(s-a)
\]
and this exists when \( s-a > \alpha \), i.e. \( s > a + \alpha \). Let us see an example:

Example 0.2. Find \( \mathcal{L} \{ e^t \sin 2t \}(s) \). We have, for \( s > 0 \),
\[
F(s) = \mathcal{L} \{ \sin 2t \}(s) = \frac{2}{s^2 + 4}
\]
Then using the property above with \( a = 1 \), we have, for \( s > 1 \),
\[
\mathcal{L} \{ e^t \sin 2t \}(s) = F(s-1) = \frac{2}{(s-1)^2 + 4}
\]

The second property is a generalization of something that we have already seen in the last lecture. Let \( f \) be a continuous function and \( f' \) be piecewise continuous with both \( f \) and \( f' \) of exponential order \( \alpha \). Then
\[
\mathcal{L} \{ f' \}(s) = s\mathcal{L} \{ f \}(s) - f(0)
\]
for all $s > \alpha$. The proof is using integration by parts, and we did it in the previous class. Let us see an application of this.

**Example 0.3.** We know that the Laplace transform of $\sin t$ is $\frac{1}{s^2 + 1}$. We can use this to calculate the Laplace transform of $\cos t$. If $f = \sin t$, then $f' = \cos t$ (both are of exponential order 0), and so, for $s > 0$,

$$\mathcal{L}\{\cos t\}(s) = s\mathcal{L}\{\sin t\}(s) = \frac{s}{s^2 + 1}$$

Now, the question is: What happens for higher derivatives of $f$? Let us first try $f''$. Using the above with $f'$, we have

$$\mathcal{L}\{f''\}(s) = s\mathcal{L}\{f'\}(s) - f'(0)$$

and again replacing $\mathcal{L}\{f'\}$ with the above expression, we get

$$\mathcal{L}\{f''\}(s) = s(s\mathcal{L}\{f\}(s) - f(0)) - f'(0) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$$

So similarly, we can extend it to higher derivatives (you can prove it by induction, try it out):

$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0)$$

**Example 0.4.** Let us see another example. Let $f(t) = \sin t$. Then $f' = \cos t$ and $f'' = -\sin t = -f$. So we have $-\mathcal{L}\{f\} = \mathcal{L}\{f''\}$. By the above property, we have

$$-\mathcal{L}\{f\}(s) = \mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0) = s^2\mathcal{L}\{f\}(s) - 1$$

Solving this, we get

$$\mathcal{L}\{f\}(s) = \frac{1}{s^2 + 1}$$

which we knew already. But note that this is a method to find $\mathcal{L}\{\sin t\}$ without any calculations involving integrals.

The third property that we see is the following: Let $f$ be a piecewise continuous function of exponential order $\alpha$. Let $F = \mathcal{L}\{f\}$. Let $n \geq 1$ be an integer. Then, for $s > \alpha$,

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n F}{ds^n}(s)$$

We will prove it for $n = 1$, i.e the statement

$$\mathcal{L}\{tf(t)\}(s) = -F'(s)$$

We have

$$F'(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t) dt$$

In general, it is not correct to interchange derivatives and integrals, but in this situation, with $f$ satisfying the properties mentioned above, it is known (we won’t prove it here, so keep it as a black box information) that we can actually interchange this derivative ans integral. So we have

$$F'(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t) dt = \int_0^\infty \frac{d}{ds} (e^{-st}) f(t) dt = \int_0^\infty (-t) e^{-st} f(t) dt = \mathcal{L}\{tf(t)\}(s)$$
This proves the statement when \( n = 1 \).

We summarize the 3 properties below:

- Let \( f \) be a function whose Laplace transform \( F(s) \) exists for \( s > \alpha \). Then
  \[
  \mathcal{L}\{e^{at}f(t)\}(s) = F(s - a)
  \]
  and this exists when \( s > a + \alpha \).

- Let \( f \) be a continuous function with \( f', \ldots, f^{(n-1)} \) continuous and \( f^{(n)} \) a piecewise continuous function and all of them of exponential order \( \alpha \), then
  \[
  \mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)
  \]
  for all \( s > \alpha \).

- Let \( f \) be piecewise continuous of exponential order \( \alpha \) with Laplace transform \( F(s) \). Then
  \[
  \mathcal{L}\{t^n f(t)\}(s) = (-1)^n\frac{d^n F}{ds^n}(s)
  \]
  for all \( s > \alpha \).

Let us see a quick example of how Laplace transforms are used to solve IVPs. Consider the following IVP:

\[
y'' + y = e^t, \quad y(0) = 1, y'(0) = 0
\]

We know how to solve it using the method of undetermined coefficients and other formulas, but let us try to use Laplace transforms here. We first apply the Laplace transform to the equation to get

\[
\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{e^t\}
\]

Denoting \( \mathcal{L}\{y\} \) by \( Y(s) \) and using the second property above, we have

\[
s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s - 1}
\]

(Check that the Laplace transform of \( e^t \) is \( \frac{1}{s - 1} \).) We then have, using \( y(0) = 1 \) and \( y'(0) = 0 \),

\[
s^2Y(s) - s + Y(s) = \frac{1}{s - 1}
\]

Then we have

\[
Y(s) = \frac{s^2 - s + 1}{(s - 1)(s^2 + 1)}
\]

So we know that

\[
\mathcal{L}\{y\}(s) = \frac{s^2 - s + 1}{(s - 1)(s^2 + 1)}
\]

Now, if we can obtain \( y \) from here, we would have solved the ODE. And this is what we will see in the next lecture: Suppose we know the Laplace transform of a function, then can we say what the function is? In certain cases, we can, and this will be called an Inverse Laplace Transform.