Recall that last time we saw how to solve a special case of non-homogeneous linear ODEs: A particular solution to the 2nd order non-homogeneous linear ODE:

\[ ay'' + by' + cy = Cte^rt \]

where \( C \) and \( r \) are real numbers and \( m \) is a non-negative integer can be found by putting

\[ y_p(t) = t^s(A_m t^m + \ldots + A_0)e^{rt} \]

where

- \( s = 0 \) if \( r \) is not a root of \( ar^2 + br + c = 0 \)
- \( s = 1 \) if \( r \) is a simple root of \( ar^2 + br + c = 0 \)
- \( s = 2 \) if \( r \) is a double root of \( ar^2 + br + c = 0 \)

and then by trying it as a solution and finding out what \( A_0, \ldots, A_m \) has to be.

**Example 0.1.** Find the general solution to the ODE

\[ 4y'' + 11y' - 3y = -2te^{-3t} \]

We start with finding \( y_p \). We have \( m = 1 \) and \( r = -3 \). Then we take

\[ y_p(t) = t^s(A_1 t + A_0)e^{-3t} \]

Note that -3 is a simple root of the equation \( 4r^2 + 11r - 3 = 0 \), and so \( s = 1 \). So, we have

\[ y_p(t) = t(A_1 t + A_0)e^{-3t} = (A_1 t^2 + A_0 t)e^{-3t} \]

Now we can plug it back into the equation to find what \( A_0 \) and \( A_1 \) has to be. Let us do that. We first find \( y'_p \) and \( y''_p \).

\[ y'_p(t) = (2A_1 t + A_0)e^{-3t} - 3(A_1 t^2 + A_0 t)e^{-3t} = (-3A_1 t^2 + (2A_1 - 3A_0)t + A_0)e^{-3t} \]

and

\[ y''_p(t) = (-6A_1 t + 2A_1 - 3A_0)e^{-3t} - 3(-3A_1 t^2 + (2A_1 - 3A_0)t + A_0)e^{-3t} \]

\[ = (9A_1 t^2 + (9A_0 - 12A_1)t + 2A_1 - 6A_0)e^{-3t} \]

Plugging this back in, we get

\[ 4(9A_1 t^2 + (9A_0 - 12A_1)t + (2A_1 - 6A_0))e^{-3t} + 11(-3A_1 t^2 + (2A_1 - 3A_0)t + A_0)e^{-3t} - 3(A_1 t^2 + A_0 t)e^{-3t} = -2te^{-3t} \]

We see that the coefficient of \( t^2 e^{-3t} \) in the LHS is

\[ 4(9A_1) + 11(-3A_1) - 3A_1 = 0, \]

the coefficient of \( te^{-3t} \) in the LHS is

\[ 4(9A_0 - 12A_1) + 11(2A_1 - 3A_0) - 3A_0 = -26A_1 \]
and the coefficient of $e^{-3t}$ in the LHS is

$$4(2A_1 - 6A_0) + 11A_0 = 8A_1 - 13A_0$$

So we have

$$-26A_1te^{-3t} + (8A_1 - 13A_0)e^{-3t} = -2te^{-3t}$$

Comparing coefficients, we get

$$-26A_1 = -2 \quad \text{and} \quad 8A_1 - 13A_0 = 0$$

which gives us

$$A_1 = \frac{1}{13} \quad \text{and} \quad A_0 = \frac{8}{169}$$

So,

$$y_p(t) = \left( \frac{t^2}{13} + \frac{8t}{169} \right) e^{-3t}$$

Now that we have found $y_p$, we find the general solution. For that, we have to find the general solution of $4y'' + 11y' - 3y = 0$.

The characteristic equation is $4r^2 + 11r - 3 = 0$ which has as its roots $r_1 = \frac{1}{4}$ and $r_2 = -3$ and so the general solution of the homogeneous ODE is $c_1 e^{\frac{t}{4}} + c_2 e^{-3t}$. And therefore, the general solution of the non-homogeneous ODE is

$$y(t) = \left( \frac{t^2}{13} + \frac{8t}{169} \right) e^{-3t} + c_1 e^{\frac{t}{4}} + c_2 e^{-3t}$$

We now look at a slightly different case: A particular solution to the 2nd order non-homogeneous linear ODE:

$$ay'' + by' + cy = (Ct^m e^{at} \cos \beta t) \text{ or } (Ct^m e^{at} \sin \beta t)$$

where $C$, $a$ and $\beta$ are real numbers and $m$ is a non-negative integer, can be found by putting

$$y_p(t) = t^s(A_m t^m + \ldots + A_0)e^{at} \cos \beta t + t^s(B_m t^m + \ldots + B_0)e^{at} \sin \beta t$$

where

- $s = 0$ if $a + i\beta$ is not a root of $ar^2 + br + c = 0$
- $s = 1$ if $a + i\beta$ is a root of $ar^2 + br + c = 0$

and then by trying it as a solution and finding out what $A_0, \ldots, A_m, B_0, \ldots, B_m$ has to be.

**Example 0.2.** Find the general solution to the ODE

$$y'' + 4y = 8 \sin 2t$$

We start with finding $y_p$. We have $m = 0$, $\alpha = 0$ and $\beta = 2$. Then we take

$$y_p(t) = t^s(A_0)e^{at} \cos 2t + t^s(B_0)e^{at} \sin 2t$$

Note that $2i$ is a root of the equation $r^2 + 4 = 0$, and so $s = 1$. So, we have

$$y_p(t) = A_0 t \cos 2t + B_0 t \sin 2t$$

Now we can plug it back into the equation to find what $A_0$ and $B_0$ has to be. Let us do that. We first find $y'_p$ and $y''_p$.

$$y'_p(t) = A_0 \cos 2t - 2A_0 t \sin 2t + B_0 \sin 2t + 2B_0 t \cos 2t$$

$$y''_p(t) = -4A_0 \sin 2t - 4B_0 \cos 2t$$

$$y'' + 4y = 8 \sin 2t$$

So we have

$$-4A_0 \sin 2t - 4B_0 \cos 2t + 8 \sin 2t = 8 \sin 2t$$

Comparing coefficients, we get

$$A_0 = 2 \quad \text{and} \quad B_0 = 0$$

So, the particular solution is

$$y_p(t) = 2t \cos 2t$$

And the general solution of the non-homogeneous ODE is

$$y(t) = 2t \cos 2t + c_1 e^{\frac{t}{4}} + c_2 e^{-3t}$$
and
\[ y''(t) = -2A_0 \sin 2t - 2A_0 \sin 2t - 4A_0 t \cos 2t + 2B_0 \cos 2t + 2B_0 \cos 2t - 4B_0 t \sin 2t \]
\[ = -4A_0 \sin 2t - 4A_0 t \cos 2t + 4B_0 \cos 2t - 4B_0 t \sin 2t \]
Plugging this back in, we get
\[-4A_0 \sin 2t - 4A_0 t \cos 2t + 4B_0 \cos 2t - 4B_0 t \sin 2t + 4(A_0 t \cos 2t + B_0 t \sin 2t) = 8 \sin 2t \]
which gives us
\[-4A_0 \sin 2t + 4B_0 \cos 2t = 8 \sin 2t \]
Comparing coefficients, we get
\[-4A_0 = 8 \quad \text{and} \quad 4B_0 = 0 \]
which gives us
\[ A_0 = -2 \quad \text{and} \quad B_0 = 0 \]
So,
\[ y_p(t) = -2t \cos 2t \]
Now that we have found \( y_p \), we find the general solution. For that, we have to find the general solution of \( y'' + 4y = 0 \). The characteristic equation is \( r^2 + 4 = 0 \) which has as its roots \( r_1 = 2i \) and \( r_2 = -2i \) and so the general solution of the homogeneous ODE is \( c_1 \sin 2t + c_2 \cos 2t \). And therefore, the general solution of the non-homogeneous ODE is
\[ y(t) = -2t \cos 2t + c_1 \sin 2t + c_2 \cos 2t \]