Research Statement

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My research lies in the intersection of extremal graph theory, combinatorial number theory, and finite geometry. Broadly speaking, I work in extremal combinatorics, where the problems are phrased as maximizing or minimizing a combinatorial property subject to a given constraint. Thus, extremal combinatorics can be thought of as solving certain optimization problems, and as such has many real world applications. My specific research is related to several aspects of data transmission and communication. In addition to applications, the problems that I work on encourage beautiful mathematics that uses a variety of techniques. Interesting and open problems abound, and in this document I will describe a selection of problems that I have worked on and plan to work on in the future. Section 4 contains specific problems that are good projects to collaborate on with undergraduate students.

1 Introduction

The first area of focus of my research is on Turán numbers. For a graph F, the *Turán number of* F, denoted by $\mathrm{ex}(n, F)$, is the maximum number of edges in a simple n-vertex graph that does not contain F as a subgraph. Turán numbers for various graphs and graph families have a long history in combinatorics (see [13, 17, 29]), and most problems in extremal combinatorics can be phrased as a Turán problem for the appropriate family of forbidden graphs or hypergraphs.

The second area of study is that of avoiding solutions to equations. Given a fixed equation and algebraic structure R, the general problem is to maximize the size of a subset $A \subset R$ that has no solutions to the given equation. A famous example is that of maximizing a subset of integers that has no nontrivial solutions to the equation $x + y = 2z$ (c.f. [2, 23]). In this example, one is trying to find a set of integers that has no 3-term arithmetic progression. An example known even to the lay person is that of describing the solution set in the integers to $x^n + y^n = z^n$. Many other equations have been considered in the literature, and for most of them the extremal set A that avoids the equation is still not well-understood. One equation that plays prominently in my work is the *Sidon equation*, $x_1 + x_2 = x_3 + x_4$. A set avoiding nontrivial solutions to the Sidon equation is called a *Sidon set*. Sidon sets and their generalizations have been studied extensively since 1932 [28], though there are still many intriguing and difficult questions that remain unsolved.

The bulk of my research lies in these two well-studied areas. They are related to each other, and both are related to finite geometry. As these problems sit in the center of several areas of mathematics, many different techniques are applicable. My work utilizes tools from finite field and ring theory, finite geometry, linear algebra/spectral graph theory, and the probabilistic method. The benefits of working between many areas are numerous. First, there are several methods to attack a given problem. Second, these problems can bring together an eclectic group of mathematicians with different strengths, allowing for a unique collaboration. Third, making progress on the problems in my research can lead to new questions and techniques on the areas that it borders, allowing for progress in several areas of mathematics. Finally, many of the problems and techniques can be understood by a motivated undergraduate, and parts of my research are well-suited to be adapted to an undergraduate research project. At UC San Diego, I participated in the Graduate-Undergraduate Learning Program. I mentored 4 undergraduates as we studied the problem of finding the size of a maximal cap in $AG(n, 3)$ and its relationship to the card game SET.

In the remainder of this document, I will describe in more detail the problems that I have worked on and where I think my research will go from here. I will give several specific problems which I hope to solve in the next few years, and I will note which of these projects are most suitable for collaboration with undergraduates.

2 Avoiding solutions to equations

Questions asking to describe either a solution set or a maximal set avoiding a solution to some given equation have been asked for thousands of years (e.g. Diophantus lived in the 3rd century). But these questions are an active topic of research to the present day. Roth's Theorem [23] asserts that if $A \subset \{1, \dots, n\}$ has no nontrivial solutions to the equation $x + y = 2z$, then $A = o(n)$. A long series of work giving bounds on the maximum size of A followed (c.f. [2, 25]), but there is still no asymptotic formula. My work focuses on the Sidon equation and its generalizations. Hundreds of papers have been written on this topic, but there are still many open questions (c.f. [10, 22]).

Sum-Product Estimates: Sárközy [26, 27] studied the solvability of the equations $a+b+cd = \lambda$ and $ab + cd = \lambda$ in $\mathbb{F}_p$ where $\lambda \in \mathbb{F}_p$. This result was then generalized to $\mathbb{F}_q$ where q is a prime power [15], and to finite cyclic rings [35]. These questions are also related to sum-product estimates. Erdős and Szemerédi [9] showed that for a finite subset $A \subset \mathbb{Z}$, at least one of $|A+A|$ or $|A \cdot A|$ must be of size $\Omega(n^{1+\epsilon})$, the intuition being that a set of integers cannot look like both an arithmetic progression and a geometric progression at the same time. A corresponding theorem was proved set in $\mathbb{F}_p$ [5], set in $\mathbb{F}_q$ [16], and finally set in an arbitrary ring [33].

Thus, there is a general motif of asking which axioms determine the behavior of sets avoiding solutions to equations or when a non-trivial sum-product estimate holds. My work with Pham, Timmons, and Vinh shows that similar theorems hold in a finite quasifield, where in particular multiplication need not be associative. The proof technique uses finite geometry, for given a quasifield, one can use it to coordinatize a projective plane in such a way that point-line incidences correspond to either a solution to the equation $a+b+cd = \lambda$ or $ab+cd = \lambda$. One can then look at the bipartite incidence graph of this projective plane and its adjacency matrix A. Since projective planes are so well-behaved, one can compute the eigenvalues of A explicitly, and use quasirandomness to count point-line incidences quite precisely. From here we showed that a Szemerédi-Trotter type theorem holds in finite quasifields and use it to prove a sum-product estimate. The takeaway of our paper is that any algebraic structure which is rich enough to coordinatize a projective plane must satisfy a non-trivial sum-product estimate.

The main open question in this area is the conjecture of Erdős and Szemerédi, concerning a sum-product estimate in $\mathbb{Z}$ [9].

Problem 1. *Prove that for $A \subset \mathbb{Z}$,* $\max\{|A + A|, |A \cdot A|\} = n^{2-o(1)}$.

Invariant Equations Ruzsa [24] studied a generalization of Sidon sets where one is attempting to avoid solutions to

$$c_1x_1 + c_2x_2 = c_3x_3 + c_4x_4 \tag{1}$$

where $c_1 + c_2 = c_3 + c_4$. This equation is said to have *genus* 1 if $\{c_1, c_2\} \neq \{c_3, c_4\}$ and *genus* 2 if $\{c_1, c_2\} = \{c_3, c_4\}$. Let γ be the genus of (1). Ruzsa [24] posed the question of determining whether $A \subset \{1, \dots, n\}$ is a set of maximum size with no nontrivial solutions to (1) implies that $|A| = n^{1/\gamma - o(1)}$. This problem is open and is a main focus of my research. A possible method of attack is to build an auxiliary graph where solutions to (1) correspond to some forbidden subgraph. Analyzing this graph then involves tools such as exponential sums, spectral properties/expansion, and the size of certain sum or difference sets.

Problem 2. *If A is a set of maximum size with no nontrivial solutions to* (1)*, is it true that* $|A| = n^{1/\gamma - o(1)}$*?*

I believe that Ruzsa's guess is incorrect in general. In particular, it seems plausible that a maximum set avoiding the equation $2x_1 + 2x_2 = 3x_3 + x_4$, which has genus 1, must be of size $n^{1/2+o(1)}$.

Sidon Sets on Graphs: Another generalization of a Sidon set is to graphs. A Sidon set is a set where *all* sums are distinct. By defining a graph, one may choose which pairs of sums must be distinct. This question was investigated by Bollobás and Pikhurko [4], among others. Verstraëte and I investigated a multiplicative analog, where one avoids the equation $ab = cd$ on a graph. The methods used in our paper were heavily probabilistic.

For a graph G define a *sum-injective labeling* of G to be an injection $\phi : V(G) \to \mathbb{N}$ such that for every pair of edges uv and xy one has $\phi(u) + \phi(v) \neq \phi(x) + \phi(y)$. That is, the edge sums are distinct. Define $S(G)$ to be the minimum N such that G has a sum-injective labeling to $[N]$. Define $S(n, m)$ to be the maximum of $S(G)$ over all graphs G on n vertices and m edges. Bollobás and Pikhurko [4] asked the following.

Problem 3. *Determine the order of magnitude of* $S(n, m)$*.*

There is a logarithmic gap between the best known upper and lower bounds for $S(n, m)$. Work that we did while investigating the multiplicative analog indicates that this gap can be closed by labeling a graph using a semi-random method.

k-fold Sidon Sets: Yet another generalization is that of k-fold Sidon sets. A subset A of either $\{1, \ldots, n\}$ or $\mathbb{Z}/n\mathbb{Z}$ is called a k-fold Sidon set if it has only trivial solutions to (1) for all sets $\{c_1, c_2, c_3, c_4\}$ such that $c_1 + c_2 = c_3 + c_4$ and $|c_i| \leq k$. Lazebnik and Verstraëte [18] constructed 2-fold Sidon sets of size $\Omega\left(\sqrt{n}\right)$ and conjectured that k-fold Sidon sets of size $\Omega(\sqrt{n})$ exist for all k.

Problem 4. *For each k, construct a k-fold Sidon set of* $\{1, \ldots, n\}$ *of size* $\varepsilon_k \sqrt{n}$*.*

An affirmative answer to this question would determine the order of magnitude for the number of edges in the densest possible r-uniform hypergraph of girth 5. Thus, a main goal of my work is to construct k-fold Sidon sets in $\{1, \ldots, n\}$ of size $\Omega(\sqrt{n})$.

Related Work

T. Pham, M. Tait, C. Timmons, and L. A. Vinh, A Szemerédi-Trotter type theorem, sum-product estimates in finite quasifields, and related results, submitted.

M. Tait and J. Verstraëte, On sets of integers with restrictions on their products, *European Journal of Combinatorics* 51, (2016), 268–274.

3 Turán numbers

Turán [34] initiated the study of $\text{ex}(n, F)$ and it became a central problem in combinatorics (see [3] for a survey). When F is not bipartite, the Erdős-Stone Theorem [8] gives the asymptotic formula $\text{ex}(n, F) \sim (1 - \frac{1}{\chi(F)-1})\binom{n}{2}$. This formula breaks down when the forbidden subgraph F is bipartite, giving only $o(n^2)$, and determining the order of magnitude for a bipartite graph in general is notoriously difficult (c.f. [13]). The most well-studied bipartite Turán problem is that of determining the Turán number of an even cycle, $\text{ex}(n, C_{2k})$, where the order of magnitude is unknown when $k \notin \{2, 3, 5\}$. My research will focus on Turán numbers for even cycles, and particularly $\mathrm{ex}(n, C_4)$.

The Turán number for C_4: At the current time, all of the best lower bounds for $\mathrm{ex}(n, C_{2k})$ come from either finite incidence structures or algebraically, where edges correspond to solutions to some system of equations. A striking example of this is the work on $\mathrm{ex}(n, C_4)$. When n is the number of points in a projective plane of order q, work of Brown, Erdős-Rényi-Sós, and Füredi [6, 7, 12] give a rare exact formula:

$$\mathrm{ex}(q^2 + q + 1, C_4) = \frac{1}{2} q(q + 1)^2.$$

Further, *any* extremal graph is an orthogonal polarity graph of a projective plane [12]. When n is not the number of points of a projective plane, the problem gets very difficult. Timmons and I have used two methods to obtain the best-known lower bounds on $\mathrm{ex}(n, C_4)$ for certain values of n.

First, one can use a Sidon set to construct a C_4 free graph. Given a group Γ and a Sidon set $A \subset \Gamma$, the Cayley sum-graph generated by A is the graph with vertex set Γ and where $x \sim y$ if and only if $x + y \in A$. To see that this graph is C_4 free, consider a purported 4-cycle $wxyz$. Then $w + x = a$, $x + y = c$, $y + z = b$, $z + w = d$ for some $a, b, c, d \in A$. But $x + y + z + w$ is equal to both $a + b$ and $c + d$, and A a being a Sidon set yields a contradiction. Using Sidon sets, Timmons and I disproved a conjecture in [1] by giving the best-known lower bound for $\mathrm{ex}(q^2 - q - 2, C_4)$ for q an odd prime power. The techniques we used here were mostly algebraic, as the construction of many of the densest known Sidon sets come from finite fields.

Second, a technique for obtaining lower bounds on $\mathrm{ex}(n, C_4)$ when $n < q^2 + q + 1$ is to take an extremal graph on $q^2 + q + 1$ vertices and to remove vertices judiciously. Timmons and I used this method to improve our previous work, and to give the best-known lower bounds for $\mathrm{ex}(n, C_4)$ for many values of n. Since any extremal graph on $q^2 + q + 1$ vertices comes from a projective plane [12], these results heavily used finite geometry. Spectral graph theory and probabilistic techniques were also employed.

Problem 5. *Improve the lower bounds for* $\mathrm{ex}(n, C_4)$.

The best bounds that we were able to obtain are derived from a classical projective plane using the orthogonal polarity. It seems likely that a better understanding of non-desarguesian planes could improve our results. This is one of the topics of our current research.

The even cycle problem: Perhaps a more important problem is determining the order of magnitude for $\mathrm{ex}(n, C_{2k})$. Despite a long history of study, the correct exponent is not known for $k \notin \{2, 3, 5\}$ and determining the order of magnitude for any new value of k is tantalizing.

Problem 6. *Determine the order of magnitude of* $\mathrm{ex}(n, C_{2k})$ *for* $k \notin \{2, 3, 5\}$.

One possible approach is to consider B_k sets. A B_k set is a set with only trivial solutions to the equation $x_1 + \cdots x_k = y_1 + \cdots y_k$, and so a B_2 set is a Sidon set. Unfortunately, for $k > 2$, B_k sets do not yield C_{2k} free graphs as readily as Sidon sets yield C_4 free graphs. However, it is possible that one can construct a graph from a B_k set either by making a Cayley graph and altering it, or by some other more complicated construction. Improving the best-known bounds on $\mathrm{ex}(n, C_{2k})$ would be a good result, even if the order of magnitude proves too difficult to determine.

Related Work

M. Tait and C. Timmons, Sidon sets and graphs without 4-cycles, *Journal of Combinatorics* 5(2), (2014), 155–165.

M. Tait and C. Timmons, Small dense subgraphs of polarity graphs and the extremal number for the 4-cycle, *Australasian Journal of Combinatorics* 63(1), (2015), 107–114.

X. Peng, M. Tait, and C. Timmons, On the chromatic number of the Erdős-Rényi orthogonal polarity graph, *Electronic Journal of Combinatorics*, P2.21, (2015), 1–19.

4 REU suitable problems

I stated before that working on questions that border different areas of mathematics encourages collaboration with a large group of mathematicians. Doing mathematics with others is a joy, and I find it difficult not to share all of the problems that I am working on. For the past 2 years, I have attended the Graduate Student Workshop in Combinatorics, a two week problem-intensive workshop that focuses on building collaboration. Through these workshops, I have submitted 3 papers and gained 13 new coauthors. I hope to both continue to work with the GRWC and to continue to work with graduate students. Additionally, I hope to continue working with undergraduate students. The topics of my research that I feel are most accessible for this purpose are as follows.

REU Suitable Problem 1. *Improve the lower bounds for* $\mathrm{ex}(n, C_4)$.

Learning about non-desarguesian projective planes requires knowledge of abstract and linear algebra at the level of a talented undergraduate. Just a couple of years ago, Jason Williford led an REU that determined $\mathrm{ex}(q^2 + q, C_4)$ for q a power of 2 [11]. This represented one of the rare exact results in Turán theory and was published in top journal *Journal of Combinatorial Theory, Series B*, showing that undergraduates have the ability to prove extremely good results in this area.

REU Suitable Problem 2. *Determine the order of magnitude of the independence and chromatic numbers of various polarity graphs.*

An undergraduate student would need a strong background to work on this problem. However, one that has excelled in an undergraduate linear algebra, abstract algebra, and combinatorics course could research polarity graphs.

REU Suitable Problem 3. *Choose a particular family of projective planes and examine its subplane structure.*

This is a fundamental but poorly understood question in the field. It would be a good project for an undergraduate who excels at linear algebra and basic enumeration techniques to consider a specific family of projective planes and determine whether or not they contain subplanes of order 2, 3, or 4. Additionally, computer aided results are very helpful in this field, and a student with programming experience could be very valuable.

REU Suitable Problem 4. *Study the distance spectrum of various graphs.*

At the Graduate Research Workshop in Combinatorics this year, we worked on the distance spectrum of a graph. Given a graph, one can construct a matrix where the ij'th entry is the distance between vertex i and j in the graph. The eigenvalues of this matrix are its distance spectrum. This problem was studied in the 1970s at Bell Labs [14], but has largely remained dormant for the last 30 years or so. However, recently there is a renewed interest, and several papers have been written on this topic in the last few years. An undergraduate student who has excelled in a linear algebra course can get their hands dirty on this project right away, and it would be an excellent topic for an REU.

REU Suitable Problem 5. *Construct large k-fold Sidon sets.*

The best constructions of 1 and 2-fold Sidon sets are algebraic and at a level understandable by a student who has excelled in an undergraduate algebra course. Additionally, trying to find k-fold Sidon sets by computer is a good project for an undergraduate who has programming experience.

5 Future Work

Though I work on a variety of problems in several areas, there is a common theme to my research. The problems above are all related to each other and could have been moved between sections fluidly. Problems 1-6 represent the main questions that I hope to answer in the next few years. However, work on them yields many tangential problems and directions that are also fascinating. For example, work on the Turán number for C_4 led to questions about polarity graphs themselves, and this is now a part of my current work.

I am excited and passionate about the problems that I am working on. I hope that through my research I can foster collaboration between mathematicians with seemingly disjoint interests. I feel that the breadth of techniques I am able to employ to solve research problems is one of my greatest strengths. In the next several years, I hope to continue collaboration with undergraduates, graduate students, and senior faculty. I pursue these problems for the mathematics itself, however, graphs without small cycles are related to LDPC codes (c.f. [19]), and finite sets without solutions to the Sidon equation are equivalent to Golomb rulers. LDPC codes and Golomb rulers have several applications in error correction and data transmission, and thus it is likely that my work could yield practical and industrial applications.

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