Characterizing graphs of maximum principal ratio

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Abstract

The principal ratio of a connected graph, denoted $\gamma(G)$, is the ratio of the maximum and minimum entries of its first eigenvector. Cioabă and Gregory conjectured that the graph on n vertices maximizing $\gamma(G)$ is a kite graph: a complete graph with a pendant path. In this paper we prove their conjecture.

1 Introduction

Several measures of graph irregularity have been proposed to evaluate how far a graph is from being regular. In this paper we determine the extremal graphs with respect to one such irregularity measure, answering a conjecture of Cioabă and Gregory [5].

All graphs in this paper will be simple and undirected, and all eigenvalues are of the adjacency matrix of the graph. For a connected graph G, the eigenvector corresponding to its largest eigenvalue, the *principal eigenvector*, can be taken to have all positive entries. If $\mathbf{x}$ is this eigenvector, let $x_{\min}$ and $x_{\max}$ be the smallest and largest eigenvector entries respectively. Then define the *principal ratio*, $\gamma(G)$ to be

$$\gamma(G) = \frac{x_{\max}}{x_{\min}}.$$

Note that $\gamma(G) \geq 1$ with equality exactly when G is regular, and it therefore can be considered as a measure of graph irregularity.

Let $P_r \cdot K_s$ be the graph attained by identifying an end vertex of a path on r vertices to any vertex of a complete graph on s vertices. This has been called a *kite graph* or a *lollipop graph*. Cioabă and Gregory [5] conjectured that the connected graph on n vertices maximizing γ is a kite graph. Our main theorem proves this conjecture for n large enough.

Theorem 1. *For sufficiently large n, the connected graph G on n vertices with largest principal ratio is a kite graph.*

We note that Brightwell and Winkler [4] showed that a kite graph maximizes the expected hitting time of a random walk. Other irregularity measures for graphs have been well–studied. Bell [3] studied the irregularity measure $\epsilon(G) := \lambda_1(G) - \bar{d}(G)$, the difference between the spectral radius and the average degree of G. He determined the extremal graph over all (not necessarily connected) graphs on n vertices and e edges. It is not known what the extremal connected graph is, and Aouchiche et al [2] conjectured that this extremal graph is a 'pineapple':

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a complete graph with pendant vertices added to a single vertex. Bell also studied the *variance* of a graph,

$$var(G) = \frac{1}{n} \sum_{v \in V(G)} \left| d_v - \bar{d} \right|^2 .$$

Albertson [1] defined a measure of irregularity by

$$\sum_{uv \in E(G)} |d(u) - d(v)|$$

and the extremal graphs were characterized by Hansen and Mélot [6].

Nikiforov [9] proved several inequalities comparing $var(G)$, $\epsilon(G)$ and $s(G) := \sum_v |d(u) - \bar{d}|$. Bell showed that $\epsilon(G)$ and $var(G)$ are incomparable in general [3]. Finally, bounds on $\gamma(G)$ have been given in [5, 10, 8, 7, 11].

2 Preliminaries

Throughout this paper G will be a connected simple graph on n vertices. The eigenvectors and eigenvalues of G are those of the adjacency matrix A of G. The vector v will be the eigenvector corresponding to the largest eigenvalue λ_1, and we take v to be scaled so that its largest entry is 1. Let x_1 and x_k be the vertices with smallest and largest eigenvector entries respectively, and if several such vertices exist then we pick any of them arbitrarily. Let $x_1, x_2, \cdots, x_k$ be a shortest path between x_1 and x_k. Let $\gamma(G)$ be the principal ratio of G. We will abuse notation so that for any vertex x, the symbol x will refer also to $v(x)$, the value of the eigenvector entry of x. For example, with this notation the eigenvector equation becomes

$$\lambda v = \sum_{w \sim v} w.$$

We will make use of the Rayleigh quotient characterization of the largest eigenvalue of a graph,

$$\lambda_1(G) = \max_{0 \neq v} \frac{v^T A(G) v}{v^t v} \tag{1}$$

Recall that the vertices $v_1, v_2, \cdots, v_m$ are a *pendant path* if the induced graph on these vertices is a path and furthermore if, in G, v_1 has degree 1 and the vertices $v_2, \cdots, v_{m-1}$ have degree 2 (note there is no requirement on the degree of v_m).

Lemma 2. *If $\lambda_1 \geq 2$ and $\sigma = (\lambda_1 + \sqrt{\lambda_1^2 - 4})/2$, then for $1 \leq j \leq k$,*

$$\gamma(G) \leq \frac{\sigma^j - \sigma^{-j}}{\sigma - \sigma^{-1}} x_j^{-1}.$$

Moreover we have equality if the vertices $x_1, x_2, \cdots, x_j$ are a pendant path.

Proof. We have the following system of inequalities

$$\begin{aligned}
\lambda_1 x_1 &\geq x_2 \\
\lambda_1 x_2 &\geq x_1 + x_3 \\
\lambda_1 x_3 &\geq x_2 + x_4 \\
&\vdots \\
\lambda_1 x_{j-1} &\geq x_j + x_{j-2}
\end{aligned}$$

The first inequality implies that

$$x_1 \geq \frac{1}{\lambda_1} x_2$$

Plugging this into the second equation and rearranging gives

$$x_2 \geq \frac{\lambda_1}{\lambda_1^2 - 1} x_3$$

Now assume that

$$x_i \geq \frac{u_{i-1}}{u_i} x_{i+1}.$$

with u_j positive for all $j < i$. Then

$$\lambda_1 x_{i+1} \geq x_i + x_{i+2}$$

implies that

$$x_{i+1} \geq \frac{u_i}{\lambda_1 u_i - u_{i-1}} x_{i+2}.$$

where $\lambda_1 u_i - u_{i-1}$ must be positive because x_j is positive for all j. Therefore the coefficients u_i satisfy the recurrence

$$u_{i+1} = \lambda_1 u_i - u_{i-1}$$

Solving this and using the initial conditions $u_0 = 1$, $u_1 = \lambda$ we get

$$u_i = \frac{\sigma^{i+1} - \sigma^{-i-1}}{\sigma - \sigma^{-1}}$$

In particular, u_i is always positive, a fact implicitly used above. Finally this gives,

$$x_1 \geq \frac{u_0}{u_1} x_2 \geq \frac{u_0}{u_1} \cdot \frac{u_1}{u_2} x_3 \geq \cdots \geq \frac{x_j}{u_{j-1}}$$

Hence

$$\gamma(G) = \frac{x_k}{x_1} = \frac{1}{x_1} \leq \frac{\sigma^j - \sigma^{-j}}{\sigma - \sigma^{-1}} x_j^{-1}$$

If these vertices are a pendant path, then we have equality throughout. □

We will also use the following lemma which comes from the paper of Cioabă and Gregory [5].

Lemma 3. *For $r \geq 2$ and $s \geq 3$,*

$$s - 1 + \frac{1}{s(s-1)} < \lambda_1(P_r \cdot K_s) < s - 1 + \frac{1}{(s-1)^2}.$$

In the remainder of the paper we prove Theorem 1. We now give a sketch of the proof that is contained in Section 3.

1. We show that the vertices $x_1, x_2, \cdots, x_{k-2}$ are a pendant path and that x_k is connected to all of the vertices in G that are not on this path (lemma 5).
2. Next we prove that the length of the path is approximately $n - n/\log(n)$ (lemma 6).
3. We show that x_{k-2} has degree exactly 2 (lemma 9), which extends our pendant path to $x_1, x_2, \cdots, x_{k-1}$. To do this, we find conditions under which adding or deleting edges increases the principal ratio (lemma 7).
4. Next we show that x_{k-1} also has degree exactly 2 (lemma 11). At this point we can deduce that our extremal graph is either a kite graph or a graph obtained from a kite graph by removing some edges from the clique. We show that adding in any missing edges will increase the principal ratio, and hence the extremal graph is exactly a kite graph.

3 Proof of Theorem 1

Let G be the graph with maximal principal ratio among all connected graphs on n vertices, and let k be the number of vertices in a shortest path between the vertices with smallest and largest eigenvalue entries. As above, let $x_1, \cdots, x_k$ be the vertices of the shortest path, where $\gamma(G) = x_k/x_1$. Let C be the set of vertices not on this shortest path, so $|C| = n - k$. Note that there is no graph with $n - k = 1$, as the endpoints of a path have the same principal eigenvector entry. Also $\lambda_1(G) \geq 2$, otherwise $P_{n-2} \cdot K_3$ would have larger principal ratio. Finally note that k is strictly larger than 1, otherwise $x_k = x_1$ and G would be regular.

Lemma 4. $\lambda_1(G) > n - k$.

Proof. Let H be the graph $P_k \cdot K_{n-k+1}$. It is straightforward to see that in H, the smallest entry of the principal eigenvector is the vertex of degree 1 and the largest is the vertex of degree $n-k+1$. Also note that in H, the vertices on the path P_k form a pendant path. By maximality we know that $\gamma(G) \geq \gamma(H)$. Combining this with lemma 2, we get

$$\frac{\sigma^k - \sigma^{-k}}{\sigma - \sigma^{-1}} \geq \gamma(G) \geq \gamma(H) = \frac{\sigma_H^k - \sigma_H^{-k}}{\sigma_H - \sigma_H^{-1}}$$

where $\sigma_H = \left(\lambda_1(H) + \sqrt{\lambda_1(H)^2 - 4}\right)/2$.
Now the function

$$f(x) = \frac{x^k - x^{-k}}{x - x^{-1}}$$

is increasing when $x \geq 1$. Hence we have $\sigma \geq \sigma_H$, and so $\lambda_1(G) \geq \lambda_1(H) > n - k$. □

Lemma 5. *$x_1, x_2, \cdots, x_{k-2}$ are a pendant path in G, and x_k is connected to every vertex in G that is not on this path.*

Proof. By our choice of scaling, $x_k = 1$. From lemma 4

$$n - k < \lambda_1(G) = \sum_{y \sim x_k} y \leq |N(x_k)|.$$

Now $|N(x_k)|$ is an integer, so we have $|N(x_k)| \geq n-k+1$. Moreover because $x_1, x_2, \cdots, x_k$ is an induced path, we must have that $|N(x_k)| = n-k+1$ exactly, and hence the $N(x_k) = C \cup \{x_{k-1}\}$. It follows that $x_1, x_2, \cdots, x_{k-3}$ have no neighbors off the path, as otherwise there would be a shorter path between x_1 and x_k. □

Lemma 6. *For the extremal graph G, we have $n - k = (1 + o(1))\frac{n}{\log n}$.*

Proof. Let H be the graph $P_j \cdot K_{n-j+1}$ where $j = \left\lfloor n - \frac{n}{\log n} \right\rfloor$, and let G be the connected graph on n vertices with maximum principal ratio. Let $x_1, \cdots, x_k$ be a shortest path from x_1 to x_k where $\gamma(G) = \frac{x_k}{x_1}$. By lemma 5, we have

$$\lambda_1(G) \leq \Delta(G) \leq n - k + 1.$$

By the eigenvector equation, this gives that

$$\gamma(G) \leq (n - k + 1)^k \tag{2}$$

Now, lemma 2 gives that

$$\gamma(H) = \frac{\sigma_H^j - \sigma_H^{-j}}{\sigma_H - \sigma_H^{-1}},$$

where

$$\sigma(H) = \frac{\lambda_1(H) + \sqrt{\lambda_1(H)^2 - 4}}{2}.$$

Now, $s - 1 + \frac{1}{s(s-1)} < \lambda_1(P_r \cdot K_s) < s - 1 + \frac{1}{(s-1)^2}$, so we may choose n large enough that $\frac{n}{\log n} + 1 > \sigma_H - \sigma_H^{-1} > \frac{n}{\log n}$. By maximality of $\gamma(G)$, we have

$$(n-k+1)^k \geq \gamma(G) \geq \gamma(H) \geq \left(\frac{n}{\log n}\right)^{n - \frac{n}{\log n} - 2}.$$

Thus, $n - k = (1 + o(1))\frac{n}{\log n}$. □

For the remainder of this paper we will explore the structure of G by showing that if certain edges are missing, adding them would increase the principal ratio, and so by maximality these edges must already be present in G. We have established that the vertices $x_1, x_2, \cdots, x_{k-2}$ are a pendant path, and so we have

$$\gamma(G) = \frac{\sigma^{k-2} - \sigma^{-k+2}}{\sigma - \sigma^{-1}} \frac{1}{x_{k-2}} \tag{3}$$

We will not add any edges that affect this path, and so the above equality will remain true. The change in γ is then completely determined by the change in λ_1 and the change in x_{k-2}. The next lemma gives conditions on these two parameters under which γ will increase or decrease.

Lemma 7. *Let $x_1, x_2, \cdots, x_{m-1}$ form a pendant path in G, where $n - m = (1 + o(1))n/\log(n)$. Let G_+ be a graph obtained from G by adding some edges from x_{m-1} to $V(G) \setminus \{x_1, \cdots, x_{m-1}\}$, where the addition of these edges does not affect which vertex has largest principal eigenvector entry. Let λ_1^+ be the largest eigenvalue of G_+ with leading eigenvector entry for vertex x denoted x^+, also normalized to have maximum entry one. Define δ_1 and δ_2 such that $\lambda_1^+ = (1 + \delta_1)\lambda_1$ and $x_{m-1}^+ = (1 + \delta_2)x_{m-1}$. Then*

- *$\gamma(G_+) > \gamma(G)$ whenever $\delta_1 > 4\delta_2/n$*
- *$\gamma(G_+) < \gamma(G)$ whenever $\delta_1 \exp(2\delta_1\lambda_1 \log n) < \delta_2/3n$.*

Proof. We have

$$\sigma = \lambda_1 - \lambda_1^{-1} - \lambda_1^{-3} - 2\lambda_1^{-5} - \cdots - \frac{2}{2n-3}\binom{2n-2}{n}\lambda_1^{-(2n-1)} - \cdots$$

So

$$\lambda_1^+ - \lambda_1 < \sigma_+ - \sigma < \lambda_1^+ - \lambda_1 - 2((\lambda_1^+)^{-1} - \lambda_1^{-1})$$

when λ_1 is sufficiently large, which is guaranteed by lemma 6. Plugging in $\lambda_1^+ = (1 + \delta_1)\lambda_1$, we get

$$\delta_1\lambda_1 < \sigma_+ - \sigma < \delta_1\lambda_1 + 2\lambda_1^{-1}(1 - (1 + \delta_1)^{-1}) < \delta_1\lambda_1 + \delta_1$$

In particular

$$(1 + \delta_1/2)\sigma < \sigma_+ < (1 + 2\delta_1)\sigma$$

To prove part (i), we wish to find a lower bound in the change in the first factor of equation 3. Let

$$f(x) = \frac{x^{m-1} - x^{-m+1}}{x - x^{-1}}.$$

Then $2mx^{m-3} > f'(x) > (m-2)x^{m-3} - mx^{m-5}$, and using that $n - m \sim n/\log(n)$ and $\sigma \sim \lambda_1$ which goes to infinity with n, we get $f'(x) \gtrsim (m-2)x^{m-3}$. By linearization and because $f(\sigma) \sim \sigma^{m-2}$, it follows that

$$\frac{\sigma_+^{m-1} - \sigma_+^{-m+1}}{\sigma_+ - \sigma_+^{-1}} \geq \left(1 + \frac{\delta_1(m-3)}{2}\right) \frac{\sigma^{m-1} - \sigma^{-m+1}}{\sigma - \sigma^{-1}}$$

Hence, if

$$\frac{\delta_1(m-3)}{2} > \delta_2$$

then $\gamma(G_+) > \gamma(G)$. In particular it is sufficient that $\delta_1 > 4\delta_2/n$.

To prove part (ii), recall from above that $f'(x) < 2mx^{m-3}$. Then, when $x = (1 + o(1))(n/\log(n))$

$$\begin{aligned} f'(x+\varepsilon) &< 2m(x+\varepsilon)^{m-3} \\ &= 2mx^{m-3}\left(1 + \frac{\varepsilon}{x}\right)^{m-3} \\ &\leq 2mx^{m-3}\exp\left(\frac{m\varepsilon}{x}\right) \\ &\leq 2nx^{m-3}\exp(2\log(n)\varepsilon) \end{aligned}$$

So for $0 < \varepsilon < \delta_1\lambda_1$, we have

$$f'(x+\varepsilon) < 2nx^{m-3}\exp(2\log(n)\delta_1\lambda_1)$$

Hence

$$\big(1 + 3n\exp(2\delta_1\lambda_1\log n)\delta_1\big)\frac{\sigma^{m-1} - \sigma^{-m+1}}{\sigma - \sigma^{-1}} > \frac{\sigma_+^{m-1} - \sigma_+^{-m+1}}{\sigma_+ - \sigma_+^{-1}}$$

□

Lemma 8. *For every subset of U of $N(x_k)$, we have*

$$|U| - 1 < \sum_{y \in U} y \leq |U|.$$

An immediate consequence is that there is at most one vertex in the neighborhood of x_k with eigenvector entry smaller than $1/2$.

Proof. The upper bound follows from $y \leq 1$, and the lower bound from the inequalities

$$\sum_{y \in N(x_k)\setminus U} y \leq |N(x_k)| - |U|$$

and

$$\sum_{y \in N(x_k)} y = \lambda_1(G) > |N(x_k)| - 1.$$

□

Lemma 9. *The vertex x_{k-2} has degree exactly 2 in G.*

Proof. Assume to the contrary. Let $U = N(x_{k-2}) \cap N(x_k)$. Then $|U| \geq 2$, so by lemma 8 we have

$$\sum_{y\in U} y > |U| - 1 \geq 1.$$

Now, by the same argument as the in the proof of lemma 2, we have that

$$\gamma(G) = \frac{\sigma^{k-1} - \sigma^{-k+1}}{\sigma - \sigma^{-1}} \left(\sum_{y\in U} y \right)^{-1}$$

Let $H = P_{k-1} \cdot K_{n-k+2}$. Then by maximality of $\gamma(G)$ we have

$$\frac{\sigma^{k-1} - \sigma^{-k+1}}{\sigma - \sigma^{-1}} > \gamma(G) \geq \gamma(H) = \frac{\sigma_H^{k-1} - \sigma_H^{-k+1}}{\sigma_H - \sigma_H^{-1}}$$

So $\sigma > \sigma_H$, which means $\lambda_1(G) > \lambda_1(H) > n - k + 1$. This means that $\Delta(G) > n - k + 1$, but we have established that $\Delta(G) = n - k + 1$. □

We now know that $x_1, x_2, \cdots, x_{k-1}$ is a pendant path in G, and so equation 3 becomes

$$\gamma(G) = \frac{\sigma^{k-1} - \sigma^{-k+1}}{\sigma - \sigma^{-1}} \frac{1}{x_{k-1}} \tag{4}$$

Lemma 10. *The vertex x_{k-1} has degree less than $11|C|/\sqrt{\log n}$.*

Proof. Assume to the contrary, so throughout this proof we assume that the degree of x_{k-1} is at least $11|C|/\sqrt{\log n}$. Let G_+ the graph obtained form G with an additional edge from x_{k-1} to a vertex $z \in C$ with $z \geq 1/2$. Let $\lambda_1^+ = \lambda_1(G_+)$ and let x^+ be the principal eigenvector entry of vertex x in G_+, where this eigenvector is normalized to have $x_k^+ = 1$.
Change in λ_1: By equation 1, we have $\lambda_1^+ - \lambda_1 \geq 2\frac{x_{k-1}z}{||v||_2^2}$. A crude upper bound on $||v||_2^2$ is

$$||v||_2^2 \leq 1 + \sum_{y\sim x_k} y + \frac{2}{\lambda_1} + \frac{4}{\lambda_1^2} + \cdots < 2\lambda_1$$

We also have that $z \geq 1/2$ so

$$\lambda_1^+ \geq \left(1 + \frac{x_{k-1}}{2\lambda_1^2}\right)\lambda_1.$$

Change in x_{k-1}: Let $U = N(x_{k-1} \cap C)$. By the eigenvector equation we have

$$\begin{aligned} x_{k-1} &= \frac{1}{\lambda_1}\left(x_{k-2} + x_k + \sum_{y\in U} y\right) \\ x_{k-1}^+ &= \frac{1}{\lambda_1^+}\left(x_{k-2}^+ + x_k^+ + z^+ + \sum_{y\in U} y^+\right) \end{aligned}$$

Subtracting these, and using that $\lambda_1 < \lambda_1^+$ and $x_k = x_k^+ = 1$, we get

$$x_{k-1}^+ - x_{k-1} \leq \frac{1}{\lambda_1}\left(x_{k-2}^+ - x_{k-2} + z^+ + \sum_{y\in U} y^+ - y\right).$$

By lemma 8, we have $\sum_{y \in U} y^+ - y \leq 1$. We also have $x_{k-2}^+ - x_{k-2} < 1$ and $z^+ \leq 1$. Hence $x_{k-1}^+ - x_{k-1} \leq 3/\lambda_1$, or

$$x_{k-1}^+ \geq \left(1 + \frac{3}{\lambda_1 x_{k-1}}\right) x_{k-1}$$

We can only apply lemma 7 if x_k^+ is the largest eigenvector entry in G_+. So we must consider two cases.

Case 1: If in G^+ the largest eigenvector entry is still attained by vertex x_k, then we can apply lemma 7, and see that $\gamma(G^+) > \gamma(G)$ if

$$\frac{x_{k-1}}{2\lambda_1^2} \geq \frac{12}{\lambda_1 x_{k-1} n}$$

or equivalently

$$x_{k-1}^2 \geq \frac{24\lambda_1}{n}.$$

We have that $\lambda_1 = (1 + o(1))(n - n/\log(n))$, so it suffices for

$$x_{k-1} \geq \frac{5}{\sqrt{\log n}}. \tag{5}$$

We know that

$$x_{k-1} > \frac{|U| - 1}{2\lambda_1}.$$

By assumption

$$|U| + 2 = N(x_{k-1}) \geq 11|C|/\sqrt{\log n}$$

Equation 5 follows from this, so $\gamma(G^+) > \gamma(G)$.

Case 2: Say the largest eigenvector entry of G^+ is no longer attained by vertex x_k. It is easy to see that the largest eigenvector entry is not attained by a vertex with degree less than or equal to 2, and comparing the neighborhood of any vertex in C with the neighborhood of x_k we can see that $x_k \geq y$ for all $y \in C$. So the largest eigenvector entry must be attained by x_{k-1}. Then equation 4 no longer holds, instead we have

$$\gamma(G_+) = \frac{\sigma_+^{k-1} - \sigma_+^{-k+1}}{\sigma_+ - \sigma_+^{-1}}. \tag{6}$$

Recall that in lemma 7 we determined the change from $\gamma(G_+)$ to $\gamma(G)$ by considering $\lambda_1^+ - \lambda_1$ and $x_{k-1}^+ - x_{k-1}$. In this case, by (6), we must consider $\lambda_1^+ - \lambda_1$ and $1 - x_{k-1}$. Now if $x_{k-1}^+ > x_k^+$, then vertex x_{k-1} in G is connected to all of C except perhaps a single vertex. Hence in G, the vertex x_{k-1} is connected to all of C except at most two vertices. This gives the bound

$$1 - x_{k-1} \leq 3/\lambda_1$$

and so as in the previous case, $\gamma(G_+) > \gamma(G)$.

So in all cases, x_{k-1} is connected to all vertices in C that have eigenvector entry larger than 1/2. If all vertices in C have eigenvector entry larger than 1/2, then x_{k-1} is connected to all of C, and this implies that $x_{k-1} > x_k$, which is a contradiction. At most one vertex in C is smaller than 1/2, and so there is a single vertex $z \in C$ with $z < 1/2$. We will quickly check that adding the edge $\{x_{k-1}, z\}$ increases the principal ratio. As before let G_+ be the graph obtained by adding this edge. The largest eigenvector entry in G_+ is attained by x_{k-1}, as its neighborhood strictly contains the neighborhood of x_k. As above, adding the edge $\{z, x_k\}$ increases the spectral radius at least

$$\lambda_1^+ > \left(1 + \frac{z}{2\lambda_1^2}\right) \lambda_1$$

and we have $1 - x_{k-1} < 1 - z/\lambda_1$. Applying lemma 7 we see that $\gamma(G_+) > \gamma(G)$, which is a contradiction. Finally we conclude that the degree of x_{k-1} must be smaller than $11|C|/\sqrt{\log n}$. □

We note that this lemma gives that $x_{k-1} < 1/2$ which implies that any vertex in C has eigenvector entry larger than $1/2$.

Lemma 11. *The vertex x_{k-1} has degree exactly 2 in G. It follows that $x_{k-1} < 2/\lambda_1$.*

Proof. Let $U = N(x_{k-1}) \cap C$, $c = |U|$. If $c = 0$ then we are done. Otherwise let G_- be the graph obtained from G by deleting these C edges. We will show that $\gamma(G_-) > \gamma(G)$.
(1) Change in λ_1: We have by equation 1,

$$\lambda_1 - \lambda_1^- \leq 2c\frac{x_{k-1}}{||v||_2^2}$$

By Cauchy–Schwarz,

$$||v||_2^2 > \sum_{x \in N(x_k)} x^2 \geq \frac{\left(\sum_{x \in N(x_k)} x\right)^2}{|C|+1} \geq \frac{(n-k)^2}{n-k+1}$$

We also have

$$x_{k-1} \leq \frac{c+2}{\lambda_1}$$

Combining these we get

$$\lambda_1 - \lambda_1^- < \frac{9c^2}{\lambda_1(n-k+1)} \Rightarrow \lambda_1 < \left(1 + \frac{9c^2}{\lambda_1\lambda_1^-(n-k+1)}\right)\lambda_1^-$$

We have $\lambda_1\lambda_1^- > (n-k)^2$, so

$$\lambda_1 < \left(1 + \frac{10c^2}{(n-k)^3}\right)\lambda_1^-$$

(2) Change in x_{k-1}: At this point, we know that in G_- the vertices $x_1, \cdots, x_k$ form a pendant path, and so by the proof of lemma 2, we have $x_{k-1}^- = (1+o(1))/\lambda_1$. By the eigenvector equation and using that the vertices in C have eigenvector entry at least $1/2$, we have $x_{k-1} > (1+c/2)/\lambda_1$. So

$$x_{k-1} - x_{k-1}^- > \frac{1}{\lambda_1}\left(\frac{c}{2} + o(1)\right)$$

In particular,

$$x_{k-1} > \left(1 + \frac{c}{3x_{k-1}^-\lambda_1}\right)x_{k-1}^-$$

Applying lemma 7, it suffices now to show that

$$\frac{10c^2}{(n-k)^3}\exp\left(2\frac{10c^2}{(n-k)^3}\lambda_1^- \log n\right) < \frac{c}{9x_{k-1}^-\lambda_1 n}. \tag{7}$$

Now

$$\frac{10c^2}{(n-k)^3} < 10\frac{11^2}{\log(n)}\frac{|C|^2}{(n-k)^3} < \frac{11^3}{\log n}\frac{\log n}{n} = \frac{11^3}{n}.$$

Similarly $2\frac{10c^2}{(n-k)^3}\lambda_1^- \log n < 2 \cdot 11^3$, so the lefthand side of equation 7 is smaller than C_0/n, where C_0 is an absolute constant. For the righthand side, recall that $x_{k-1}^-\lambda_1 = 1 + o(1)$, and also that

$$c > \frac{11}{\sqrt{\log n}}\left(\frac{n}{\log n} + o(1)\right) > \frac{10n}{\log^{3/2} n}.$$

So the righthand side is larger than $1/\log^{3/2} n$. Hence for large enough n, the righthand side is larger than the lefthand side.

□

We are now ready to prove the main theorem.

Theorem 1. *For sufficiently large n, the connected graph G on n vertices with largest principal ratio is a kite graph.*

Proof. It remains to show that C induces a clique. Assume it does not, and let H be the graph $P_k \cdot K_{n-k+1}$. We will show that $\gamma(H) > \gamma(G)$, and this contradiction tells us that C is a clique. As before, lemma 2 gives that

$$\gamma(H) = \frac{\sigma_H^k - \sigma_H^{-k}}{\sigma_H - \sigma_H^{-1}},$$

where

$$\sigma(H) = \frac{\lambda_1(H) - \sqrt{\lambda_1(H)^2 - 4}}{2}.$$

Since $x_1, \cdots x_k$ form a pendant path we also know that

$$\gamma(G) = \frac{\sigma^k - \sigma^{-k}}{\sigma - \sigma^{-1}}.$$

Now, $\lambda_1(H) > \lambda_1(G)$ because $E(G) \subsetneq E(H)$. Since the functions $g(x) = x + \sqrt{x^2 - 4}$ and $f(x) = (x^k - x^{-k})/(x - x^{-1})$ are increasing when $x \geq 1$, we have $\gamma(H) > \gamma(G)$.

□

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References

[1] Michael O Albertson. The irregularity of a graph. *Ars Combinatoria*, 46:219–225, 1997.

[2] Mustapha Aouchiche, Francis K Bell, D Cvetković, Pierre Hansen, Peter Rowlinson, SK Simić, and D Stevanović. Variable neighborhood search for extremal graphs. 16. some conjectures related to the largest eigenvalue of a graph. *European Journal of Operational Research*, 191(3):661–676, 2008.

[3] Francis K Bell. A note on the irregularity of graphs. *Linear Algebra and its Applications*, 161:45–54, 1992.

[4] Graham Brightwell and Peter Winkler. Maximum hitting time for random walks on graphs. *Random Structures & Algorithms*, 1(3):263–276, 1990.

[5] Sebastian M Cioabă and David A Gregory. Principal eigenvectors of irregular graphs. *Electron. J. Linear Algebra*, 16:366–379, 2007.

[6] Pierre Hansen, Hadrien Mélot, and Québec) Groupe d'études et de recherche en analyse des décisions (Montréal. *Variable neighborhood search for extremal graphs 9: Bounding the irregularity of a graph*. Montréal: Groupe d'études et de recherche en analyse des décisions, 2002.

[7] Geoff A Latham. A remark on minc's maximal eigenvector bound for positive matrices. *SIAM Journal on Matrix Analysis and Applications*, 16(1):307–311, 1995.

[8] Henryk Minc. On the maximal eigenvector of a positive matrix. *SIAM Journal on Numerical Analysis*, 7(3):424–427, 1970.

[9] Vladimir Nikiforov. Eigenvalues and degree deviation in graphs. *Linear Algebra and its Applications*, 414(1):347–360, 2006.

[10] Britta Papendieck and Peter Recht. On maximal entries in the principal eigenvector of graphs. *Linear Algebra and its Applications*, 310(1):129–138, 2000.

[11] Xiao-Dong Zhang. Eigenvectors and eigenvalues of non-regular graphs. *Linear Algebra and its Applications*, 409:79–86, 2005.