

**THE ALON-SAKS-SEYMOUR AND RANK-COLORING
CONJECTURES**

by
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We've taught you that the earth is round,
That red and white make pink,
And something else that matters more –
We've taught you how to think.

-Dr. Seuss

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ABSTRACT

This thesis studies the Alon-Saks-Seymour Conjecture and the Rank-Coloring Conjecture and their relationships to computer science. For a graph G , the chromatic number, $\chi(G)$, is the minimum number of colors needed to properly color the vertices of G . If $A(G)$ is the adjacency matrix of G , then $\text{rank}(A(G))$ denotes its rank. The biclique partition number, $\text{bp}(G)$, is the minimum number of complete bipartite subgraphs (bicliques) necessary to partition the edge set of G . The Rank-Coloring Conjecture states that for any graph G , $\chi(G) \leq \text{rank}(A(G))$ and the Alon-Saks-Seymour Conjecture states that for any graph G , $\chi(G) \leq \text{bp}(G) + 1$. Both of these conjectures have been previously disproven.

In this thesis we construct an infinite family of graphs that are counterexamples to both conjectures. This construction generalizes previous work of Razborov and Huang and Sudakov. We discuss the relationship between these conjectures and questions in theoretical computer science relating to communication complexity, which is the amount of information that two parties need to exchange in order to compute some objective boolean function. We also discuss a generalization of biclique partitions to hypergraphs, where we consider the minimum number of complete r -partite r -uniform hypergraphs necessary to partition the edge set of the complete r -uniform hypergraph on n vertices.

Chapter 1

INTRODUCTION

This thesis is about biclique partitions and their relationship to mathematics and theoretical computer science. Specifically we discuss two conjectures in graph theory, the Alon-Saks-Seymour Conjecture and the Rank-Coloring Conjecture, and their relationship to communication complexity. We will start with some basic definitions. Our notation is standard, and the reader should refer to West [37] for any missing definitions.

1.1 Definitions

A **graph** $G = (V, E)$ consists of a **vertex set** V (or $V(G)$), a set E (or $E(G)$) of **edges**, and a mapping associating each $e \in E(G)$ to an unordered pair of vertices x and y . We say x and y are endpoints of e . If x and y are endpoints of some edge e , we say x and y are **adjacent**, denoted $x \sim y$. We can denote the edge as $e = xy$ and we say the e is **incident** with both x and y .

We say two graphs G_1 and G_2 are **isomorphic** if there is a bijective mapping $f : V(G_1) \rightarrow V(G_2)$ such that $x \sim y$ in G_1 if and only if $f(x) \sim f(y)$ in G_2 .

The set of all vertices adjacent to a vertex $x \in V(G)$ is called the **neighborhood of x** and is denoted by $N(x)$. The number of edges incident with a vertex x is called the **degree** of x . If all the vertices of a graph have the same degree, we say the graph is **regular**.

If $x \neq y$ for each $xy \in E(G)$, we say that G is **simple**. All graphs discussed from this point in this thesis will be simple unless otherwise noted.

A set of vertices I is called an **independent set** if $x \not\sim y$ for all $x, y \in I$. A graph is called **bipartite** if the vertex set can be partitioned into two independent sets.

The **complete graph** on n vertices is the simple graph on n vertices that contains all $\binom{n}{2}$ possible edges, and is denoted by K_n . The **cycle** C_n on n vertices is the graph whose vertex set is $\{1, \dots, n\}$ where $i \sim i + 1$ for all $1 \leq i \leq n - 1$ and $i_1 \sim i_n$.

A **complete bipartite graph** (or **biclique**) $K_{a,b}$ is a bipartite graph with independent sets of size a and b and all ab possible edges between the independent sets. Figure 1.1 shows a $K_{4,3}$.

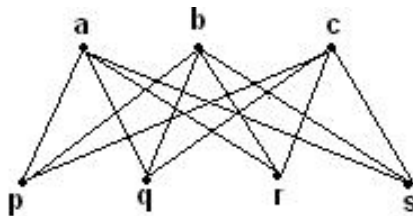


Figure 1.1: $K_{4,3}$

A **biclique partition** of a graph G is a set of bicliques such that the edges of the bicliques partition the edge set of G . The minimum number of bicliques necessary for a biclique partition of G is called the **biclique partition number** of G and is denoted $\text{bp}(G)$.

The **adjacency matrix** of G , denoted $A(G)$, has its rows and columns indexed after the vertices of G and its (u, v) -th entry equals 1 if the vertices u and v are adjacent in G and 0 otherwise. The rank of $A(G)$ will be denoted by $\text{rank}(A(G))$.

A **proper coloring** of a graph G is a function from the vertex set to a finite set of colors such that the endpoints of every edge have distinct colors. More formally, a proper coloring of a graph G is a function $f : V(G) \rightarrow \mathbb{N}$ such that $x \sim y$ in G implies $f(x) \neq f(y)$. A proper k -coloring of a graph is a proper coloring of the vertices using k colors. The **chromatic number** of a graph is the minimum natural number k such that a proper k -coloring exists, and is denoted by $\chi(G)$.

If A is an alphabet of symbols, we can define a word of length n to be an n -tuple of symbols from A . The **Hamming Distance** between two words of length n is the number of positions at which the corresponding symbols are different. It is the minimum number of letter substitutions necessary to change one word to the other.

1.2 Loop Switching

Consider the problem of communication among computers. At Bell Labs, J.R. Pierce proposed a scheme called “loop switching” [26] (see also [14]). Imagine computer terminals on one-way communication loops (see Figure 1.2). “Local” loops are connected by various switching points to one another as well as to other say “regional loops” which are connected to each other as well as say a “national” loop. If a message from one loop is sent to a terminal from another loop, it proceeds to a suitable switching point where it may choose to enter a different loop. This process continues until the message reaches its destination. The question that arises is how does the message know which sequence of loops to follow? It would be desirable for the message to have an “address” and for each switching point to be able to perform a simple test on this address to determine which loop to send it on.

A useful tool is to think of a loop system as a graph, where each loop corresponds to a vertex and two vertices are adjacent if and only if there is a switching

point from one's corresponding loop to the other's. Figure 1.2 shows a loop system and its equivalent graph.



Figure 1.2: A loop system and its graph

One idea is that a message is addressed by its destination, and each junction decides where the message should go based on this simple test: the message will go into the new loop if and only if it decreases the Hamming distance between the current location and the destination. For example, label the two-dimensional cube (isomorphic to C_4) with addresses 00, 01, 10, and 11 where $x \sim y$ if and only if the Hamming Distance between x and y is 1. Then if you want to send a message from loop 10 to 11, the message will not take the exit from 10 to 00 if it reaches that switching point first (because this would increase the Hamming distance). Rather, it would exit into 11 when it reaches it (this is the only junction that will decrease the Hamming distance).

This scheme seems to be the one we wish to follow. Each loop has an address that is n bits long and a message makes an exit from one loop to the next if and only if it decreases the Hamming distance between where it is and where it wants to go. Then the number of loops traversed should be equal exactly to the Hamming

distance between the sender and receiver since each loop switch decreases the Hamming distance by 1. Consider for example, addressing the three dimensional cube as below. To go from node 000 to node 011 would require exactly two loop switches because this is the Hamming distance between the two. For example, it could follow the path 000 to 001 to 011.

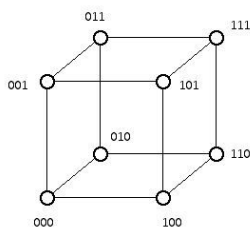


Figure 1.3: Addressing the cube

One immediate question is whether or not this scheme works for any loop system and associated graph. After some experimentation, it is clear that difficulties arise. For example, consider the loop structure that is represented by K_3 . In other words, how can we address three mutually adjacent loops? In this scheme, we cannot. In fact, using this scheme the only graphs that can be addressed properly are those which can be isometrically embedded into the hypercube (see [23]). The solution that Graham and Pollak [14] describe is to slightly generalize the binary address. Instead of 1's and 0's, we may make n -tuples as words from the alphabet $\{1, 0, *\}$ (to actually realize this in a binary setting, we can represent 0 as 00, 1 as 01 and * as either 10 or 11, but this is not as important to the mathematics of the discussion). The Hamming distance is then calculated by incrementing by 1 for every position which one address has a 1 and the other a 0. For example, the Hamming distance between $1 * 011$ and $10 * 00$ is 2 (coming from positions 4 and 5).

It is important to note here that we still call this Hamming distance only out of convention. The term Hamming *distance* makes sense when addressing with 1's and 0's because it is a metric. In other words, if x , y and z are words/ n -tuples, then the Hamming distance between x and y is positive unless $x = y$ in which case it is 0, the Hamming distance between x and y is the same as between y and x , and the Hamming distance between x and z is less than or equal to the sum of the Hamming distances between x and y and y and z . When we address with 0, 1, * instead and increment the "Hamming distance" only when one word has a 1 and one has a 0 in the corresponding spot, it is no longer a metric because it no longer satisfies the triangle inequality. For example, $d(111, 000) = 3$ while $d(111, 1*0) = 1$ and $d(1*0, 000) = 1$ implying that $d(111, 000) > d(111, 1*0) + d(1*0, 000)$. But, despite not being a metric, this notion of "distance" works in this situation. Now we can address a K_3 correctly, for example by 00, 10, and *1 (each pair of vertices has Hamming distance 1) and the scheme can be used again. Consider the example in Figure 1.2, which we can label as follows (figure and labeling from [14])

$A - 1111*$
 $B - 001**$
 $C - 11*0*$
 $D - 000*1$
 $E - 10**0$
 $F - 010**.$

In this example, the Hamming distance between any two addresses is exactly the distance between nodes. The question is, can all loop systems be addressed with

0's, 1's, and *'s in such a way that transferring between loops if and only if the Hamming distance is decreased will take every message to its correct destination? Further, if this is possible, how long must each address be? Graham and Pollak show in their 1971 paper [14] that any loop system on n vertices can be addressed in the way described, and that if s is the largest distance between two loops (i.e. the diameter of the graph), then at most $s(n - 1)$ coordinates must be used for each address (see also [4, 22, 23]). In fact, we will see that addressing a graph is equivalent to finding a biclique partition of the associated distance multigraph.

Graham and Pollak's algorithm is as follows. Let the vertices be labeled A_1, \dots, A_n . If B and C are disjoint sets of these vertices, we denote by $B \times C$ the following process of adding a 1, 0, or * to all n addresses. We add a 1 to A_i 's address if $A_i \in B$, we add a 0 if $A_i \in C$ and we add a * if A_i is in neither B nor C . This will increase the Hamming distance between A_i and A_j by exactly 1 if and only if $A_i \in B$ and $A_j \in C$ or $A_i \in C$ and $A_j \in B$. Now let $A_i(k)$ denote the set of all vertices A_j with $j > i$ such that the (graph) distance between A_i and A_j is at least k . Then if s is again diameter of the graph, addressing the loop system by

$$\begin{aligned}
&A_1 \times A_1(1) + A_1 \times A_1(2) + \dots + A_1 \times A_1(s) \\
&A_2 \times A_2(1) + A_2 \times A_2(2) + \dots + A_2 \times A_2(s) \\
&\dots \\
&A_{n-1} \times A_{n-1}(1) + A_{n-1} \times A_{n-1}(2) + \dots + A_{n-1} \times A_{n-1}(s)
\end{aligned}$$

gives an addressing that has at most $s(n - 1)$ coordinates and the Hamming distance between each is exactly the distance between the two vertices. Then the scheme described where a message will enter a loop if and only if it decreases the Hamming distance between its current position and its destination will work and will take exactly the distance between sender and receiver number of loop switches to arrive.

So any loop system can be addressed such that the scheme delivers correctly and at most $s(n - 1)$ coordinates are necessary. In fact, this scheme is equivalent to partitioning the edges of the distance multigraph into stars.

Graham and Pollak conjectured [14] that the maximum number of coordinates needed is actually $n - 1$ and not $s(n - 1)$. Winkler confirmed this conjecture in 1983 [38].

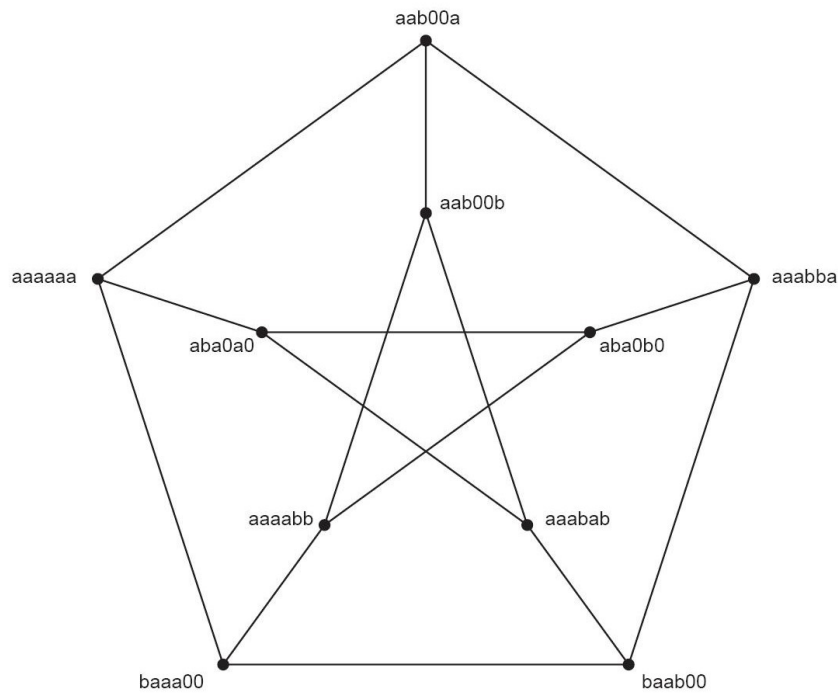


Figure 1.4: Addressing the Petersen Graph

An addressing of the Petersen graph is shown in Figure 1.4. Note that the authors use $\{a, b, 0\}$ for $\{1, 0, *\}$ respectively (figure from [11]). The paper [11] shows that 6 coordinates are necessary to address the Petersen Graph, and Figure 1.4 shows an optimal addressing.

It is a natural question to ask what is the minimum number of coordinates necessary. So let N be the number of coordinates necessary. Then the question is, what is the minimum N such that we can address any loop system on n vertices by

$$\sum_{\alpha=1}^N (A_{\alpha,i_1}, \dots, A_{\alpha,i_n}) \times (A_{\alpha,j_1}, \dots, A_{\alpha,j_n})?$$

If A_i and A_j are distance d_{ij} apart, they must appear on opposite sides of the products exactly d_{ij} times. Consider the distance multigraph G with vertex set $\{A_i\}_{i=1}^n$ obtained by putting d_{ij} edges between A_i and A_j and consider finding an optimal biclique partition of this graph. Let the bicliques be denoted by

$$B_1(U_1, V_1), \dots, B_{\text{bp}(G)}(U_{\text{bp}(G)}, V_{\text{bp}(G)}).$$

Then consider addressing the loop system as follows. In the k 'th position, address A_i with 1 if $A_i \in U_k$, with 0 if $A_i \in V_k$, and with $*$ otherwise. Then A_i and A_j will have exactly d_{ij} positions in their addresses where one has a 1 and the other a 0, and thus this addressing will produce the Hamming distance we require. Consider for any correct addressing of the loop system, creating a graph as follows. For each position k in the addresses, create a biclique with one partite set containing all vertices that have a 1 in the k 'th position of the address and one partite set containing all vertices that have a 0 there. Vertices with a $*$ in the k 'th position are not included in the k 'th biclique. Then it is clear that the union of these bicliques is a multigraph where the number of edges between A_i and A_j is equal to d_{ij} . Thus, there is an equivalence between addressing a loop system and finding a biclique partition of the distance multigraph of that loop system.

Consider the special case where the loop system is represented by the complete graph. Then we must find the minimum N such that for any i, j , A_i and A_j must appear on opposite sides of the product exactly one time. This is equivalent to finding

a biclique partition of the complete graph on n vertices. This case led Graham and Pollak to the following result.

Theorem 1 (Graham-Pollak [14]). *The edge-set of a complete graph on n vertices cannot be partitioned into fewer than $n - 1$ bicliques.*

This theorem is a consequence of the following result, attributed to Witsenhausen (see [6]).

Theorem 2. *For any graph G on n vertices with adjacency matrix A , let $n_+(A)$ and $n_-(A)$ denote the number of positive and negative eigenvalues of A respectively. Then*

$$\text{bp}(G) \geq \max\{n_+(A), n_-(A)\}.$$

Proof. Assume the edge set of a graph G is partitioned into $\text{bp}(G)$ bicliques. If S is a subset of the vertices of G , then the characteristic vector of S is the n -dimensional $(0, 1)$ column vector whose i -th position equals 1 if vertex i is in S and equals 0 otherwise. Denote by u_i and v_i the characteristic vectors of the partite sets of the i -th biclique of our decomposition. Define $D_i = u_i v_i^T + v_i u_i^T$. Then D_i is the adjacency matrix of the i -th biclique as a subgraph of G , and $A = \sum_{i=1}^{\text{bp}(G)} D_i$. Let

$$W = \text{Span}\{w \in \mathbb{R}^n \mid w^T u_i = 0, \forall 1 \leq i \leq \text{bp}(G)\}$$

$$P = \text{Span}\{\text{Eigenvectors of the positive eigenvalues of } A\}.$$

Since W is made up of n -dimensional vectors that are all orthogonal to $\text{bp}(G)$ vectors, we have that $\dim(W) \geq n - \text{bp}(G)$. On the other hand, since $p^T A p > 0$ for all nonzero $p \in P$, we have that $W \cap P = \{0\}$. Therefore

$$\dim(W) \leq n - \dim(P) = n - n_+(A).$$

It follows that $n - \text{bp}(G) \leq \dim(W) \leq n - n_+(A)$ which implies that $\text{bp}(G) \geq n_+(A)$. The argument for $n_-(A)$ follows similarly. Thus $\text{bp}(G) \geq \max\{n_+(A), n_-(A)\}$. \square

Since K_n has eigenvalue 1 with multiplicity $n - 1$, the Graham-Pollak Theorem follows. Graham and Pollak's addressing scheme can address a loop system corresponding to the complete graph with $n - 1$ coordinates, and because this is equivalent to a biclique partition of the complete graph, we see that $\text{bp}(K_n) \leq n - 1$. With the Graham-Pollak Theorem, this tells us that $\text{bp}(K_n) = n - 1$. We call a biclique a *star* when it is of the form $K_{1,a}$. It is clear that the edge set of K_n can be partitioned into $n - 1$ stars, but there are many decompositions of K_n into $n - 1$ bicliques. In fact, there are at least 2^{n-4} nonisomorphic decompositions of K_n into $n - 1$ bicliques ([4] Example 1.4.5).

1.3 Conjectures

The chromatic number of K_n is n . This means that Theorem 1 can be restated as $\chi(K_n) = \text{bp}(K_n) + 1$. Over the years, several proofs of the Graham-Pollak Theorem have been discovered (see [25, 34, 35]). Until recently, only algebraic proofs were known. However, recently the first counting proof has been discovered [36]. A natural generalization of the Graham-Pollak Theorem is to ask if any graph G can be properly colored with $\text{bp}(G) + 1$ colors. This question was first posed by Alon, Saks, and Seymour (cf. Kahn [18]).

Conjecture 3 (Alon-Saks-Seymour). *For any simple graph G , $\chi(G) \leq \text{bp}(G) + 1$.*

This conjecture was confirmed by Rho [30] for graphs G with n vertices and $\text{bp}(G) \in \{1, 2, 3, 4, n - 3, n - 2, n - 1\}$ and by Gao, McKay, Naserasr and Stevens [13] for graphs with $\text{bp}(G) \leq 9$. The Alon-Saks-Seymour Conjecture remained open for

twenty years until recently when Huang and Sudakov [17] constructed the first counterexamples. Huang and Sudakov's construction yields an infinite sequence of graphs G_n with arbitrarily large biclique partition number such that $\chi(G_n) \geq c(\text{bp}(G_n))^{6/5}$ for a fixed constant $c > 0$.

In 1976, van Nuffelen [24] (see also Fajtlowicz [12]) stated what became known as the Rank-Coloring Conjecture.

Conjecture 4 (Rank-Coloring). *For any simple graph G , $\chi(G) \leq \text{rank}(A(G))$.*

The Rank-Coloring Conjecture was disproved in 1989 by Alon and Seymour [3] when they constructed a graph with rank 29 and chromatic number 32. Razborov [28] found counterexamples with a superlinear gap between $\chi(G)$ and $\text{rank}(A(G))$ by constructing an infinite sequence of graphs G_n such that $\chi(G_n) \geq c(\text{rank}(A(G)))^{4/3}$ for some fixed constant $c > 0$. Other counterexamples were constructed from the Kasami graphs by Roy and Royle [31]. Nisan and Wigderson's construction from [21] yields the largest gap between the chromatic number and the rank at present time with a super polynomial gap between rank and chromatic number.

In the next chapter we construct an infinite family of counterexamples to both conjectures. The graphs presented (see also [8]) contain and generalize both Huang and Sudakov's counterexamples and Razborov's counterexamples. In Chapter 3, we will discuss the applications of these graphs to open problems in computer science. In Chapter 4 we will discuss a generalization of the Graham-Pollak Theorem. In chapter 5, we will discuss open problems.

Chapter 2

AN INFINITE FAMILY OF COUNTEREXAMPLES TO THE ALON-SAKS-SEYMOUR AND RANK-COLORING CONJECTURES

In this chapter, we construct infinitely many graphs that are counterexamples to both the Alon-Saks-Seymour Conjecture and the Rank-Coloring Conjecture (The results of this chapter have appeared in [8]). More precisely, we construct infinite families of graphs $G(n, k, r)$ with $n^{2k+2r+1}$ vertices for all integers $n \geq 2$, $k \geq 1$, $r \geq 1$ such that

$$\chi(G(n, k, r)) \geq \frac{n^{2k+2r}}{2r+1} \quad (2.1)$$

and for $k \geq 2$

$$2k(2r+1)(n-1)^{2k+2r-1} \leq \text{bp}(G(n, k, r)) < 2^{2k+2r-1}n^{2k+2r-1} \quad (2.2)$$

and

$$2k(2r+1)(n-1)^{2k+2r-1} \leq \text{rank}(A(G(n, k, r))) < 2k(2r+1)n^{2k+2r-1} + (n-1)^{2k}. \quad (2.3)$$

These inequalities imply that for fixed $k \geq 2$ and $r \geq 1$ and n large enough, the graphs $G(n, k, r)$ are counterexamples to both the Alon-Saks-Seymour Conjecture and the Rank-Coloring Conjecture. Our construction extends the constructions of Huang and Sudakov [17] and Razborov [28]. Taking $k = 2$ and $r = 1$, we get

Huang and Sudakov's graph sequence from [17]. When $k = 1$ and $r = 1$, we obtain Razborov's construction from [28].

In Section 2.1, we describe the construction of the graphs $G(n, k, r)$ and we prove inequality (2.1) and the upper bound on $\text{bp}(G(n, k, r))$ from (2.2). In Section 2.2, we obtain the bounds (2.3) on the rank of the adjacency matrix of $G(n, k, r)$ and deduce the lower bound on $\text{bp}(G(n, k, r))$ from (2.2).

2.1 The graphs $G(n, k, r)$

Let Q_n be the n -dimensional cube with vertex set $\{0, 1\}^n$ and two vertices x, y in Q_n adjacent if and only if they differ in exactly one coordinate. A k -dimensional subcube of Q_n is a subset of Q_n which can be written as

$$\{x = (x_1, \dots, x_n) \in Q_n \mid \forall i \in B_i \text{ with } x_i = b_i\} \quad (2.4)$$

where B is a set of $n - k$ fixed coordinates and each $b_i \in \{0, 1\}$. We represent the all ones and all zeros vectors as 1^n and 0^n respectively, and we define $Q_n^- = Q_n \setminus \{1^n, 0^n\}$. For any integer $n \geq 1$, we denote $\{1, \dots, n\}$ by $[n]$.

For given integers $n \geq 2, k \geq 1$, and $r \geq 1$, we define the graph $G(n, k, r)$ as follows. Its vertex set is

$$V(G(n, k, r)) = [n]^{2k+2r+1} = \{(x_1, \dots, x_{2k+2r+1}) \mid \forall i \in [2k+2r+1], x_i \in [n]\}.$$

For any two vertices $x = (x_1, \dots, x_{2k+2r+1}), y = (y_1, \dots, y_{2k+2r+1})$ let

$$\rho(x, y) = (\rho_1(x, y), \dots, \rho_{2k+2r+1}(x, y)) \in \{0, 1\}^{2k+2r+1} \quad (2.5)$$

where $\rho_i(x, y) = 1$ if $x_i \neq y_i$ and $\rho_i(x, y) = 0$ if $x_i = y_i$.

We define adjacency in $G(n, k, r)$ as follows: the vertices x and y are adjacent in $G(n, k, r)$ if and only if $\rho(x, y) \in S$ where

$$S = Q_{2k+2r+1} \setminus [(1^{2k} \times Q_{2r+1}^-) \cup \{0^{2k} \times 0^{2r+1}\} \cup \{0^{2k} \times 1^{2r+1}\}]. \quad (2.6)$$

We now prove the lower bound (2.1) for the chromatic number of $G(n, k, r)$. In the proofs in this chapter we will refer to $G(n, k, r)$ as G .

Theorem 5. For $n \geq 2$ and $k, r \geq 1$, $\chi(G(n, k, r)) \geq \frac{n^{2k+2r}}{2r+1}$.

Proof. For $x = (x_1, \dots, x_{2k}, x_{2k+1}, \dots, x_{2k+2r+1}) \in V(G)$, let $f(x) = (x_1, \dots, x_{2k})$ be the projection to the first $2k$ coordinates of x and $t(x) = (x_{2k+1}, \dots, x_{2k+2r+1})$ be the projection to the last $2r + 1$ coordinates of x .

Let I be an independent set in G . Any two vertices x and y of G which agree on one of the first $2k$ coordinates and satisfy $f(x) \neq f(y)$ are adjacent in G . This implies that any two distinct vectors in $f(I)$ differ in all of the first $2k$ of their coordinates and thus, $|f(I)| \leq n$.

If for every $u \in f(I)$, $|f^{-1}(u) \cap I| \leq 2r + 1$, then $|I| \leq (2r + 1)n$. Otherwise, there is a $\beta \in [n]^{2k}$ and distinct $x_1, x_2, \dots, x_{2r+2} \in I$ such that $f(x_i) = \beta$ for $1 \leq i \leq 2r + 2$. Then $\rho(t(x_i), t(x_j)) = 1^{2r+1}$ for any $1 \leq i \neq j \leq 2r + 2$. From the definition (2.6) of S , we know that any two vertices that differ in all $2k + 2r + 1$ coordinates are adjacent in G . If there exists a $z \in I$ such that $f(z)$ and β differ on every coordinate, then $t(z)$ and $t(x_i)$ are equal in at least one coordinate for each i . Thus at least two of $x_1, x_2, \dots, x_{2r+2}$ must agree in at least one coordinate of $t(z)$, contradicting that $t(x_i)$ must differ in every coordinate for distinct i . Thus, there must be only one element in $f(I)$. Again, the vertices in I must differ in all of the last $2r + 1$ coordinates, and thus $|I| = |f(I)| \leq n$.

Thus, we proved that the independence number of G satisfies the inequality $\alpha(G) \leq (2r + 1)n$. This fact and $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$ complete our proof. \square

To prove the upper bound (2.2) on the biclique partition number of $G(n, k, r)$, we need some auxiliary lemmas.

Lemma 6. *The set Q_{2k+1}^- can be partitioned into a disjoint union of 1-dimensional subcubes for $k \geq 1$.*

Proof. We prove the lemma by induction on k .

In the base case when $k = 1$, we can write

$$Q_3^- = \{(0, 0, 1), (0, 1, 1)\} \cup \{(0, 1, 0), (1, 1, 0)\} \cup \{(1, 0, 0), (1, 0, 1)\}. \quad (2.7)$$

This proves the base case.

Assume now that Q_{2k+1}^- can be partitioned into 1-dimensional subcubes. Then

$$\begin{aligned} Q_{2k+3} &= (Q_{2k+1} \times 1 \times 0) \cup (Q_{2k+1} \times 1 \times 1) \cup (Q_{2k+1} \times 0 \times 1) \cup (Q_{2k+1} \times 0 \times 0) \\ &= (Q_{2k+1} \times 1 \times 0) \\ &\cup (Q_{2k+1}^- \times 1 \times 1 \cup \{1^{2k+1} \times 1 \times 1\} \cup \{0^{2k+1} \times 1 \times 1\}) \\ &\cup (Q_{2k+1}^- \times 0 \times 1 \cup \{1^{2k+1} \times 0 \times 1\} \cup \{0^{2k+1} \times 0 \times 1\}) \\ &\cup (Q_{2k+1}^- \times 0 \times 0 \cup \{1^{2k+1} \times 0 \times 0\} \cup \{0^{2k+1} \times 0 \times 0\}). \end{aligned}$$

This implies

$$\begin{aligned} Q_{2k+3}^- &= (Q_{2k+1} \times 1 \times 0) \\ &\cup (Q_{2k+1}^- \times 1 \times 1 \cup \{0^{2k+1} \times 1 \times 1\}) \\ &\cup (Q_{2k+1}^- \times 0 \times 1 \cup \{1^{2k+1} \times 0 \times 1\} \cup \{0^{2k+1} \times 0 \times 1\}) \\ &\cup (Q_{2k+1}^- \times 0 \times 0 \cup \{1^{2k+1} \times 0 \times 0\}) \end{aligned}$$

which equals

$$\begin{aligned} &(Q_{2k+1} \times 1 \times 0) \cup (Q_{2k+1}^- \times 1 \times 1) \cup (Q_{2k+1}^- \times 0 \times 1) \cup (Q_{2k+1}^- \times 0 \times 0) \\ &\cup \{1^{2k+1} \times 0 \times 1, 1^{2k+1} \times 0 \times 0\} \cup \{0^{2k+1} \times 1 \times 1, 0^{2k+1} \times 0 \times 1\}. \end{aligned}$$

By induction hypothesis, it follows that Q_{2k+3}^- can be partitioned into 1-dimensional subcubes. \square

We use the previous lemma to prove that the set S defined in (2.6) can be partitioned into 2-dimensional subcubes.

Lemma 7. *For $k \geq 2$ and $r \geq 1$, the set*

$$S = Q_{2k+2r+1} \setminus [(1^{2k} \times Q_{2r+1}^-) \cup \{0^{2k} \times 0^{2r+1}\} \cup \{0^{2k} \times 1^{2r+1}\}]$$

can be partitioned into 2-dimensional subcubes.

Proof. We claim that the following three sets form a partition of S :

$$S' = (0^{2k-1} \times 0 \times Q_{2r+1}^-) \cup (0^{2k-1} \times 1 \times Q_{2r+1}^-) \cup (Q_{2k-1}^- \times 1 \times Q_{2r+1}^-) \quad (2.8)$$

$$S'' = (Q_{2k-1} \times 1 \times 0^{2r+1}) \cup (Q_{2k-1} \times 1 \times 1^{2r+1}) \quad (2.9)$$

and

$$S''' = (Q_{2k-1} \setminus \{0^{2k-1}\}) \times 0 \times Q_{2r+1}. \quad (2.10)$$

To show this is a partition, we first prove $S \subseteq S' \cup S'' \cup S'''$. To see this, consider the $2k$ -th coordinate of any vector $s = (s_1, \dots, s_{2k+2r+1})$ in S . As before, let $f(s) = (s_1, \dots, s_{2k})$ and $t(s) = (s_{2k+1}, \dots, s_{2k+2r+1})$. If $s_{2k} = 0$, and $f(s) \neq 0^{2k}$, then $s \in S'''$. If $f(s) = 0^{2k}$ then $s \in S'$. Now take $s \in S$ such that $s_{2k} = 1$. If $t(s) = 1^{2r+1}$ or $t(s) = 0^{2r+1}$, then $s \in S''$. Otherwise, $s \in S'$. Thus $S \subseteq S' \cup S'' \cup S'''$. Since S', S'', S''' are disjoint subsets of S , they must partition S .

The set Q_{2r+1} can be partitioned into 2-dimensional subcubes. It follows that for any $\beta \in Q_{2k}$, the set $\beta \times Q_{2r+1}$ can also be partitioned into 2-dimensional subcubes. For any x_1 adjacent to x_2 in Q_{2k} , y_1 adjacent to y_2 in Q_{2r+1} , the set $\{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)\}$ is a 2-dimensional subcube. By Lemma 6, Q_{2r+1}^-

can be decomposed into 1-dimensional subcubes. This implies that for any x_1 adjacent to x_2 in Q_{2k} , $(x_1 \times Q_{2r+1}^-) \cup (x_2 \times Q_{2r+1}^-)$ can be decomposed into 2-dimensional subcubes.

These remarks imply that S', S'', S''' and thus S can be partitioned into 2-dimensional subcubes. \square

Using the previous lemma, we are ready to prove the upper bound (2.2) for the biclique partition number of the graph $G(n, k, r)$.

Theorem 8. For $n \geq 2$, $k \geq 2$, $r \geq 1$, $\text{bp}(G(n, k, r)) < 2^{2k+2r-1} n^{2k+2r-1}$.

Proof. By Lemma 7, $S = \cup_{i=1}^t S_i$, where $t = \frac{2^{2k+2r+1} - 2^{2r+1}}{4} = 2^{2k+2r-1} - 2^{2r-1}$ and each S_i is a 2-dimensional subcube. For $1 \leq i \leq t$, let G_i be the subgraph of G such that $x, y \in V(G_i) = V(G) = [n]^{2k+2r+1}$ are adjacent if and only if $\rho(x, y) \in S_i$. Then the edge sets of the subgraphs G_1, G_2, \dots, G_t partition the edge set of the graph G . For each S_i there is a set $T_i = \{t_1, \dots, t_{2k+2r-1}\} \subset \{1, \dots, 2k+2r+1\}$ of fixed coordinates $a_1, \dots, a_{2k+2r-1} \in \{0, 1\}$ so that $S_i = \{(x_1, \dots, x_{2k+2r+1}) \mid j \in [2k+2r-1], x_{t_j} = a_j\}$.

Define G'_i with vertex set $[n]^{2k+2r-1}$ such that x' and y' adjacent in G'_i if and only if $\rho(x', y') = (a_1, \dots, a_{2k+2r-1})$. Then G_i is an n^2 -blowup of G'_i which means that G_i can be obtained from G'_i by replacing each vertex v of G'_i by an independent set I_v of n^2 vertices and by adding all edges between I_u and I_v in G_i whenever u and v are adjacent in G'_i . Note that a partition of G'_i into complete bipartite subgraphs becomes a partition into complete bipartite subgraphs in any blowup of G'_i . Thus $\text{bp}(G_i) \leq \text{bp}(G'_i) \leq |V(G'_i)| - 1 \leq n^{2k+2r-1} - 1$. Since the edge set of G is the disjoint union of the edge sets of G_1, \dots, G_t , we have that

$$\text{bp}(G) \leq \sum_{i=1}^t \text{bp}(G_i) \leq (2^{2k+2r-1} - 2^{2r-1})(n^{2k+2r-1} - 1) < 2^{2k+2r-1} n^{2k+2r-1}.$$

\square

2.2 The rank of $A(G(n, k, r))$

In this section, we obtain asymptotically tight bounds for the rank of the adjacency matrix of $G(n, k, r)$. We will use the following graph operation, called NEPS (Non-complete Extended P-Sum), introduced by Cvetković in his thesis [9] (see also [10], page 66).

Definition 1. For given $\mathcal{B} \subset \{0, 1\}^t \setminus \{0^t\}$ and graphs G_1, \dots, G_t , the NEPS with basis \mathcal{B} of the graphs G_1, \dots, G_t is the graph whose vertex set is the cartesian product of the sets of vertices of the graphs G_1, \dots, G_t and in which two vertices (x_1, \dots, x_t) and (y_1, \dots, y_t) are adjacent if and only if there is a t -tuple (b_1, \dots, b_t) in \mathcal{B} such that $x_i = y_i$ holds exactly when $b_i = 0$ and x_i is adjacent to y_i in G_i exactly when $b_i = 1$.

Note that when all the graphs G_1, \dots, G_t are isomorphic to the complete graph K_n , then the NEPS with basis \mathcal{B} of G_1, \dots, G_t will be the graph whose vertex set is $[n]^t$ with $(x_1, \dots, x_t) \sim (y_1, \dots, y_t)$ if and only if $\rho((x_1, \dots, x_t), (y_1, \dots, y_t)) = (b_1, \dots, b_t)$ for some $(b_1, \dots, b_t) \in \mathcal{B}$.

Hence, the graph $G(n, k, r)$ is the NEPS of $2k + 2r + 1$ copies of K_n with basis

$$S = Q_{2k+2r+1} \setminus [(1^{2k} \times Q_{2r+1}^-) \cup \{0^{2k} \times 0^{2r+1}\} \cup \{0^{2k} \times 1^{2r+1}\}].$$

Another important observation is given below.

Proposition 9. ([10], Theorem 2.21) *The adjacency matrix of the NEPS with basis \mathcal{B} of G_1, \dots, G_t equals*

$$\sum_{(b_1, \dots, b_t) \in \mathcal{B}} A(G_1)^{b_1} \otimes \dots \otimes A(G_t)^{b_t},$$

where $X \otimes Y$ denotes the Kronecker (tensor) product of two matrices X and Y .

Proof. In each of the graphs G_1, \dots, G_t , let the vertices be ordered arbitrarily. Lexicographically order the vertices of G (ordered n -tuples of the vertices of G_1, \dots, G_t) and form the adjacency matrix $A(G)$ according to this ordering. The entries of A are given by

$$A_{(x_1, \dots, x_t), (y_1, \dots, y_t)} = \sum_{(b_1, \dots, b_t) \in \mathcal{B}} (A(G_1)^{b_1})_{x_1, y_1} \cdots (A(G_t)^{b_t})_{x_t, y_t}.$$

By virtue of the lexicographic ordering, $A_{(x_1, \dots, x_t), (y_1, \dots, y_t)}$ equals 1 if and only if there exists a $(b_1, \dots, b_t) \in \mathcal{B}$ with $(A(G_i)^{b_i})_{x_i, y_i} = 1$ for $i = 1, \dots, t$

This means exactly that x_i and y_i are adjacent in G_i if $b_i = 1$ and equal if $b_i = 0$ and this completes the proof. \square

This allows us to prove a theorem about the spectrum of a NEPS of graphs.

Proposition 10. ([10], Theorem 2.23) *For $i = 1, 2, \dots, t$, let G_i be a graph with n_i vertices and let $\lambda_{i1}, \dots, \lambda_{in_i}$ be the spectrum of G_i . Then the spectrum of the NEPS with basis \mathcal{B} of G_1, \dots, G_t consists of all possible values of $\Lambda_{i_1 \dots i_n}$ where*

$$\Lambda_{i_1 \dots i_n} = \sum_{(b_1, \dots, b_t) \in \mathcal{B}} \lambda_{i_1}^{b_1} \cdots \lambda_{i_n}^{b_n}$$

Proof. Since A_i , the adjacency matrix of G_i , is symmetric, there exist vectors x_{ij} such that $A_i x_{ij} = \lambda_{ij} x_{ij}$ (i running from 1 to t , j running from 1 to n_i). Consider the vector

$$x = x_{1i_1} \otimes \cdots \otimes x_{ti_t}.$$

Then applying properties of tensor products to the equation given in Proposition 9 gives us

$$Ax = \Lambda_{i_1, \dots, i_t} x$$

which yields all $n_1 \cdot n_2 \cdots n_t$ possible eigenvalues

\square

These facts will enable us to compute the eigenvalues of $G(n, k, r)$ and to obtain the bounds from (2.3) on the rank of the adjacency matrix of $G(n, k, r)$.

Theorem 11. For $n \geq 2, k \geq 1, r \geq 1$,

$$2k(2r + 1)(n - 1)^{2k+2r-1} \leq \text{rank}(A(G(n, k, r))) < 2k(2r + 1)n^{2k+2r-1} + (n - 1)^{2k}.$$

Proof. By Proposition 10 the spectrum of the adjacency matrix of $G(n, k, r)$ has the following form:

$$\Lambda(G) = \{f(\lambda_1, \dots, \lambda_{2k+2r+1}) | \lambda_1, \dots, \lambda_{2k+2r+1} \text{ eigenvalues of } K_n\} \quad (2.11)$$

where

$$f(x_1, \dots, x_{2k+2r+1}) = \sum_{(s_1, \dots, s_{2k+2r+1}) \in S} \prod_{i=1}^{2k+2r+1} x_i^{s_i}. \quad (2.12)$$

Using the definition of S , we can simplify $f(x_1, \dots, x_{2k+2r+1})$ as follows

$$f(x_1, \dots, x_{2k+2r+1}) = \prod_{i=1}^{2k+2r+1} (1+x_i)^{-1} - \prod_{i=1}^{2k} x_i \left[\prod_{i=2k+1}^{2k+2r+1} (1+x_i) - 1 - \prod_{i=2k+1}^{2k+2r+1} x_i \right] - \prod_{i=2k+1}^{2k+2r+1} x_i. \quad (2.13)$$

Whenever the last $2r + 1$ positions are -1 , f evaluates as

$$f(x_1, \dots, x_{2k}, -1, \dots, -1) = -1 - \prod_{i=1}^{2k} x_i [-1 - (-1)^{2r+1}] - (-1)^{2r+1} = 0. \quad (2.14)$$

Whenever the first $2k$ positions are -1 and not all of the last $2r + 1$ positions are $n - 1$, f evaluates as

$$f(-1, \dots, -1, x_{2k+1}, \dots, x_{2k+2r}) = -1 - (-1)^{2k} \left[-1 - \prod_{i=2k+1}^{2k+2r+1} x_i \right] - \prod_{i=2k+1}^{2k+2r+1} x_i = 0. \quad (2.15)$$

Thus, we obtain 0 as an eigenvalue for $G(n, k, r)$ when all of the last $2r + 1$ positions are -1 or when the first $2k$ positions are -1 and at least one of the last

$2r + 1$ positions is -1 . The eigenvalues of K_n are $n - 1$ with multiplicity 1 and -1 with multiplicity $n - 1$. We will make use of the following simple inequality: $n^t - (n - 1)^t < tn^{t-1}$ for any integers $n, t > 1$.

These facts imply that $G(n, k, r)$ has eigenvalue 0 with multiplicity at least

$$\begin{aligned}
& n^{2k}(n - 1)^{2r+1} + (n - 1)^{2k}(n^{2r+1} - 1) - (n - 1)^{2k+2r+1} \\
&= n^{2k+2r+1} - (n^{2k} - (n - 1)^{2k})(n^{2r+1} - (n - 1)^{2r+1}) - (n - 1)^{2k} \\
&> n^{2k+2r+1} - 2kn^{2k-1} \cdot (2r + 1)n^{2r+1-1} - (n - 1)^{2k} \\
&= n^{2k+2r+1} - 2k(2r + 1)n^{2k+2r-1} - (n - 1)^{2k}
\end{aligned}$$

which shows that

$$\text{rank}(A(G(n, k, r))) < 2k(2r + 1)n^{2k+2r-1} + (n - 1)^{2k}. \quad (2.16)$$

To prove the lower bound, note that for fixed $u \in \{1, \dots, 2k\}$ and $v \in \{2k + 1, \dots, 2k + 2r + 1\}$, evaluating f when $x_i = -1$ for $i \neq u, v$ (by using (2.13)), we get

$$f(-1, \dots, x_u, \dots, x_v, \dots, -1) = -1 + x_u(-1 - x_v) - x_v = -(x_u + 1)(x_v + 1). \quad (2.17)$$

If $x_u = x_v = n - 1$, we obtain $f(-1, \dots, x_u, \dots, x_v, \dots, -1) = -n^2$. Since K_n has eigenvalue -1 with multiplicity $n - 1$, we deduce that $G(n, k, r)$ has the negative eigenvalue $-n^2$ with multiplicity at least $\binom{2k}{1} \binom{2r+1}{1} (n - 1)^{2k+2r-1}$. This shows $\text{rank}(A(G(n, k, r))) \geq 2k(2r + 1)(n - 1)^{2k+2r-1}$ and completes our proof. \square

Theorem 2 states that for any graph H

$$\text{bp}(H) \geq \max(n_+(A(H)), n_-(A(H))) \quad (2.18)$$

where $n_+(A(H))$ and $n_-(A(H))$ denote the number of positive and the number of negative eigenvalues of the adjacency matrix of H , respectively.

From the last part of the proof of Proposition 11, we deduce that $n_-(A(G(n, k, r))) \geq 2k(2r + 1)(n - 1)^{2k+2r-1}$. This result and inequality (2.18) imply

$$\text{bp}(G(n, k, r)) \geq 2k(2r + 1)(n - 1)^{2k+2r-1}.$$

As $\text{bp}(A(G(n, k, r))) \leq 2^{2k+2r-1}n^{2k+2r-1}$, these inequalities determine the order of $\text{bp}(G(n, k, r))$. More precisely, we say $f(n) = \Theta(g(n))$ if there exist constants $c_1, c_2 > 0$, and n_0 such that for any $n \geq n_0$, $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$. This shows that $\text{bp}(G(n, k, r)) = \Theta(n^{2k+2r-1})$ for fixed $k \geq 2$ and $r \geq 1$.

2.3 Concluding remarks of the chapter

In this chapter, we constructed families of counterexamples to the Alon-Saks-Seymour Conjecture and to the Rank-Coloring Conjecture. We computed the eigenvalues of the adjacency matrices of these graphs and obtained tight bounds for the rank of their adjacency matrices. We used these results to determine the asymptotic behavior of their biclique partition number. It would be interesting to determine other properties of these graphs.

At the present time, Huang and Sudakov's construction from [17] gives the biggest gap between biclique partition number and chromatic number and Nisan and Wigderson's construction from [21] gives the biggest gap between rank and chromatic number.

In the next chapter, we discuss applications of these graphs to questions in computer science.

Chapter 3

APPLICATIONS

An important application of graph theory thus far has been in computer science. As we saw with the addressing problem, questions about communicating between nodes (or computer terminals) can be reformulated into questions about graphs. The overlap of computer science and graph theory is seen prominently in the study of communication complexity (for example, see [28, 40]).

3.1 Communication Complexity

The deterministic model of communication complexity was considered by Yao [40] in 1979. The basic model is that there are two parties (traditionally named Alice and Bob), and two finite sets X and Y . The task is to evaluate a boolean function

$$f : X \times Y \rightarrow \{0, 1\}$$

where Alice is the only one who can see the input $x \in X$ and Bob is the only one who can see the input $y \in Y$. The objective is to find the protocol p in the set of all protocols \mathcal{P} which minimizes the amount of information exchanged between Alice and Bob to evaluate the function. We define the “cost” of evaluating the function given a protocol p , $\alpha_p(x, y)$, to be the number of bits that are exchanged between Alice and Bob before $f(x, y)$ can be determined, and the *two-way deterministic complexity* of

f , denoted $C(f)$, to be the cost given the most expensive inputs computed with the “best” protocol. More precisely

$$C(f) = \min_{p \in \mathcal{P}} \max_{x \in X, y \in Y} (\alpha_p(x, y)).$$

As noted by Razborov [29], a few things are left imprecise deliberately. For example, is the length of each message fixed or can it vary? Does communication end when one of the players knows the output or must both? These are left imprecise on purpose, as the details can be filled in according to the situation and the mathematical theory remains the same.

An obvious question is how can we bound $C(f)$. The first obvious answer is that

$$C(f) \leq \lceil \log_2 |X| \rceil + 1. \tag{3.1}$$

The protocol to see this upper bound is simple. Alice encodes her input x as a binary string of $\lceil \log_2 |X| \rceil$ bits using any injective map $f_1 : X \rightarrow \{0, 1\}^{\lceil \log_2 |X| \rceil}$ and sends it to Bob. Bob then computes $f(x, y)$ and sends $f(x, y)$ back to Alice. So we see that upper bounds on $C(f)$ are found by cooking up smart ways to exchange information. For example, if $g(x, y) \equiv x + y \pmod{2}$, then there is a smarter protocol than the one described above. Specifically, Alice can send 1 if her input is odd and 0 if it is even. Clearly, this is enough information for Bob to determine $g(x, y)$ and thus only 1 bit must be sent by Alice.

Now how do we determine lower bounds on $C(f)$? Here we see interesting and beautiful algebraic methods used.

Definition 2. *Let f be a boolean function on $X \times Y$. We call $S \times T$ (where $S \subset X$ and $T \subset Y$) a monochromatic rectangle if f is constant over $S \times T$. We define a 1-rectangle to be a monochromatic rectangle where f evaluates to 1 for every x, y in*

the rectangle, and a 0-rectangle to be a monochromatic rectangle where f evaluates to 0. A k -decomposition of f is a set

$$\{S_1 \times T_1, S_2 \times T_2, \dots, S_k \times T_k\}$$

of k disjoint monochromatic rectangles that partition $X \times Y$. We denote by $d(f)$ the minimum k such that a k -decomposition of f exists.

Theorem 12 (Yao [40]). $C(f) \geq \log_2 d(f)$.

Proof. We introduce a concept called a *history* or *transcript* (see [29]). By this we mean the whole sequence $(a_1, b_1, \dots, a_t, b_t)$ of messages exchanged by Alice and Bob given a particular input. Again denoting by $\alpha_p(x, y)$ the cost of the input (x, y) for an optimal protocol p , we denote $\alpha(p) = \max_{(x \in X, y \in Y)} \alpha(x, y)$. Then we observe that there are at most $2^{\alpha(p)}$ possible histories since there are only that many binary strings of length $\alpha(p)$. Given a fixed history h , we may form a set R_h consisting of all inputs (x, y) that lead to the history h .

First we see that every input (x, y) leads to exactly one history. Letting \mathcal{H} denote the set of all histories, this means that we can write

$$X \times Y = \bigcup_{h \in \mathcal{H}} R_h.$$

Notice that this is a disjoint union. Since each R_h ends in the output of the function, we see that each R_h is a monochromatic rectangle on f . So we have partitioned $X \times Y$ into at most $2^{\alpha(p)}$ monochromatic rectangles on f . Since $d(f)$ is the minimum number of monochromatic rectangles needed to partition $X \times Y$, we have

$$d(f) \leq 2^{\alpha(p)}$$

for any protocol p . Since $C(f)$ is the minimum $\alpha(p)$ over all protocols, we have proved the theorem. \square

A big breakthrough in the subject was the introduction of algebraic methods by Mehlhorn and Schmidt [33]. Given any boolean function $f : X \times Y \rightarrow \{0, 1\}$, we can arrange its values in the form of a matrix M_f . The rows of M_f are indexed by X and the columns are indexed by Y and $M_f(x, y) = f(x, y)$. The following theorem shows the relation between combinatorics and algebra in this problem.

Theorem 13 (Mehlhorn and Schmidt [33]). $d(f) \geq \text{rank}(M_f)$.

Proof. Define $d_0(f)$ to be the number of 0-rectangles necessary to partition the 0's of the matrix of some boolean function f and $d_1(f)$ to be the number of 1-rectangles necessary to partition the 1's. This means that $d(f) = d_0(f) + d_1(f)$. Let $R_1, \dots, R_{d_1(f)}$ be disjoint 1-rectangles covering all (x, y) with $f(x, y) = 1$. Let $f_i : X \times Y \rightarrow \{0, 1\}$ be the characteristic function of R_i . That is, $f_i(x, y) = 1$ if and only if $(x, y) \in R_i$, and let M_i be the matrix associated with f_i . Then $\text{rank}(M_i) = 1$ for all i , and

$$M_f = \sum_{i=1}^{d_1(f)} M_i$$

which implies $\text{rank}(M_f) \leq \sum_{i=1}^{d_1(f)} \text{rank}(M_i) = d_1(f) \leq d(f)$ □

This theorem along with Theorem 12 gives us that

$$C(f) \geq \log_2 \text{rank}(M_f).$$

3.2 The Log-Rank Conjecture

In this section we will talk about how the graphs presented in Chapter 2 are related to communication complexity. We saw in the previous section that

$$C(f) \geq \log_2 \text{rank}(M_f).$$

Lovász and Saks conjecture [19] that this bound is “almost” tight. This is given more precisely in Conjecture 14.

Conjecture 14. (*Log-Rank Conjecture*) *There exists a constant $k > 0$ such that for any function f*

$$C(f) \leq (\log_2 \text{rank}(M_f))^k.$$

Next we will explain the relation between the Log-Rank Conjecture and the Rank-Coloring conjecture described in Section 1.3. This relationship was first explained by Lovász and Saks [19].

Proposition 15. *If the Log-Rank Conjecture is true, then there exists a constant $l > 0$ such that for any graph G*

$$\log_2 \chi(G) \leq (\log_2 \text{rank}(A(G)))^l.$$

Proof. Assume the Log-Rank Conjecture is true. So there exists a fixed constant k such that for any boolean function f associated with matrix M_f we have

$$C(f) \leq (\log_2 \text{rank}(M_f))^k.$$

Let G be an arbitrary graph, and define $f : V(G) \times V(G) \rightarrow \{0, 1\}$ by the matrix $M_f = J - A(G)$. Then we notice that any 1-rectangle in f corresponds to an independent set in G . Let $d_1(f)$ and $d_0(f)$ be the number of 1-rectangles and 0-rectangles necessary to partition the 1's and 0's of M_f respectively. If we can partition the 1's of M_f with $d_1(f)$ 1-rectangles, we must be able to color G with $d_1(f) \leq d(f)$ colors, which shows $\chi(G) \leq d(f)$. Next we have that since $M_f = J - A(G)$, then $\text{rank}(M_f) \leq \text{rank}(J) + \text{rank}(A(G)) = 1 + \text{rank}(A(G))$. Combining these two inequalities and the Log-Rank Conjecture, we have

$$\log_2 \chi(G) \leq \log_2 d(f) \leq C(f) \leq (\log_2 \text{rank}(M_f))^k \leq (\log_2(1 + \text{rank}(A(G))))^k$$

which shows that there is a constant l such that for any graph G , $\log_2 \chi(G) \leq (\log_2 \text{rank}(A(G)))^l$. □

Next we show that the converse is also true.

Proposition 16 (Lovász and Saks). *If there exists a constant $l > 0$ such that for any graph G*

$$\log_2 \chi(G) \leq (\log_2 \text{rank}(A(G)))^l,$$

then the Log-Rank Conjecture is true.

Proof. Recall that $d(f)$ was defined to be the minimum number of monochromatic rectangles that partition some boolean function f . We define $t(f)$ to be the number of monochromatic rectangles necessary to cover a boolean function f . That is, the rectangles are allowed to overlap. This means that $d(f) \geq t(f)$ because any partition of a function f is also a covering. It is known that $C(f) \leq c(\log_2(t(f)))^2$ for some positive constant c [1, 28].

We want to show that there exists a constant k so that for any boolean function f with matrix M_f , $C(f) \leq (\log_2(\text{rank}(M_f)))^k$. We show that for any function f , we can construct a graph G such that

1. $\chi(G) = t(f)$
2. $\text{rank}(A(G)) \leq (\text{rank}(M_f) + 1)^2 + (\text{rank}(M_f))^2 + 1$

These two facts along with the assumption that there exists a constant $l > 0$ such that for any graph G $\log_2 \chi(G) \leq (\log_2 \text{rank}(A(G)))^l$ and the fact that $C(f) \leq c(\log_2(t(f)))^2$ for some constant c will imply that the Log-Rank Conjecture holds for any boolean function f .

Let f be a boolean function on $X \times Y$. Then we construct a graph G as follows. Its vertex set $V(G) = X \times Y$ and $(x, y) \sim (x_1, y_1)$ if and only if (x, y) , (x_1, y_1) , (x, y_1) , and (x_1, y) do *not* form a monochromatic rectangle in f . In other words, if $f(x, y) = f(x_1, y) = f(x, y_1) = f(x_1, y_1)$ then $(x, y) \not\sim (x_1, y_1)$.

First we discuss the chromatic number of G . Consider an independent set in G , say I . Then for any $(x, y), (x_1, y_1)$ in I we have that $(x, y), (x, y_1), (x_1, y)$, and (x_1, y_1) form a monochromatic rectangle in f which means that the vertices in an independent set of G correspond to a monochromatic rectangle in f . Since $\chi(G)$ is the minimum number of independent sets that cover the vertex set of G , this implies that $\chi(G) = t(f)$.

Next we discuss the rank of $A(G)$. The Hadamard product is the entrywise product of two matrices A and B , denoted $A \circ B$. That is

$$(A \circ B)_{ij} = A_{ij} \cdot B_{ij}.$$

The matrix $A \circ B$ is a submatrix of the tensor product of A and B and thus has $\text{rank}(A \circ B) \leq \text{rank}(A)\text{rank}(B) = \text{rank}(A \otimes B)$. We write $A(G)$ in terms of Hadamard products of matrices. Let C_1, D_1, E_1, F_1 be $|X||Y|$ by $|X||Y|$ matrices defined by

$$\begin{aligned} C_1[(x, y), (x_1, y_1)] &= 1 \text{ iff } M_f(x, y) = 1, \\ D_1[(x, y), (x_1, y_1)] &= 1 \text{ iff } M_f(x_1, y_1) = 1, \\ E_1[(x, y), (x_1, y_1)] &= 1 \text{ iff } M_f(x, y_1) = 1, \\ F_1[(x, y), (x_1, y_1)] &= 1 \text{ iff } M_f(x_1, y) = 1, \end{aligned}$$

and C_0, D_0, E_0, F_0 similarly by

$$\begin{aligned} C_0[(x, y), (x_1, y_1)] &= 1 \text{ iff } M_f(x, y) = 0, \\ D_0[(x, y), (x_1, y_1)] &= 1 \text{ iff } M_f(x_1, y_1) = 0, \\ E_0[(x, y), (x_1, y_1)] &= 1 \text{ iff } M_f(x, y_1) = 0, \\ F_0[(x, y), (x_1, y_1)] &= 1 \text{ iff } M_f(x_1, y) = 0. \end{aligned}$$

Then $C_1 \circ D_1 \circ E_1 \circ F_1 + C_0 \circ D_0 \circ E_0 \circ F_0$ is $J - A(G)$, which means that $\text{rank}(A(G)) \leq \text{rank}(C_1 \circ D_1 \circ E_1 \circ F_1 + C_0 \circ D_0 \circ E_0 \circ F_0) + 1$.

It is clear that C_1, D_1, C_0 and D_0 all have rank 1. For example, C_1 is the matrix whose rows are either all 0's (if $M_f(x, y) = 0$) or all 1's (if $M_f(x, y) = 1$).

If J_{YX} is the $|Y|$ by $|X|$ matrix of all 1's, then $E_1 = M_f \otimes J_{YX}$ and thus $\text{rank}(E_1) = \text{rank}(M_f)$. Similarly, if J_{XY} is the $|X|$ by $|Y|$ matrix of all 1's, then $F_1 = J_{XY} \otimes M_f^T$ and $\text{rank}(F_1) = \text{rank}(M_f)$. We also have $E_0 = (J - M_f) \otimes J_{YX}$ and $F_0 = J_{XY} \otimes (J - M_f)^T$ which means that $\text{rank}(E_0) = \text{rank}(F_0) = \text{rank}(J - M_f) \leq \text{rank}(M_f) + 1$. So we see that

$$\text{rank}(C_1 \circ D_1 \circ E_1 \circ F_1 + C_0 \circ D_0 \circ E_0 \circ F_0) \leq (\text{rank}(M_f))^2 + (\text{rank}(M_f) + 1)^2$$

which completes the proof. □

The next proposition shows that if one can construct a graph with a gap between rank and chromatic number, then one has a corresponding gap between rank and communication complexity of the associated function.

Proposition 17. *For any graph G such that $\text{rank}(A(G)) < \chi(G)$, there is a corresponding boolean function $f : V(G) \times V(G) \rightarrow \{0, 1\}$ such that*

$$\log_2(\text{rank}(M_f) - 1) < C(f).$$

Proof. Let G be a graph such that $\chi(G) > \text{rank}(A(G))$. Define a boolean function f on $V(G) \times V(G)$ by $M_f = J - A(G)$. Then, as before, any covering of f by $d_1(f)$ 1-rectangles corresponds to a coloring of G by $d_1(f) \leq d(f)$ colors. Because $M_f = J - A(G)$ we have $\text{rank}(M_f) \leq \text{rank}(A(G)) + 1$. This implies $\text{rank}(A(G)) < \chi(G)$ which yields

$$\log_2(\text{rank}(M_f) - 1) \leq \log_2(\text{rank}(A(G))) < \log_2(\chi(G)) \leq \log_2(d(f)) \leq C(f).$$

Thus, we have shown a separation between the known lower bound for $C(f)$ and the actual deterministic complexity. \square

In Chapter 2, we constructed graphs $G(n, k, r)$ such that

$$\chi(G(n, k, r)) \geq \frac{n^{2k+2r}}{2r+1}$$

and

$$\text{rank}A(G(n, k, r)) < 2k(2r+1)n^{2k+2r-1} + (n-1)^{2k}.$$

As above, for each $G(n, k, r)$ we create a boolean function defined by $M_f = J - A(G(n, k, r))$ such that

$$\log_2(\text{rank}(M_f) - 1) \leq \log_2(2k(2r+1)n^{2k+2r-1} + (n-1)^{2k}) < \log_2\left(\frac{n^{2k+2r}}{2r+1}\right) \leq C(f)$$

which means that we have constructed functions f that give

$$C(f) \geq \frac{2k+2r}{2k+2r-1} \log_2(\text{rank}(M_f)) - c$$

for a fixed constant $c > 0$. Thus, we have constructed examples where the deterministic communication complexity is a super-unitary constant term times the general lower bound of $\log_2(M_f)$.

3.3 Clique vs. Independent Set Problem

The question about deterministic communication complexity can be applied to many problems. One is called the clique vs. independent set problem, which was introduced by Yannakakis [39] and is denoted $CL - IS$ for short. In this problem, there is a publicly known graph G . Alice gets a complete subgraph C of G and Bob gets an independent set I of G . Let X be the set of all cliques in the graphs and Y the set of all independent sets. Then the objective function $f : X \times Y \rightarrow \{0, 1\}$

is given by $f(C, I) = |C \cap I|$. This is clearly either 1 or 0, and thus this defines a boolean function. If we denote the deterministic communication complexity of this problem on the graph G by $C(CL - IS_G)$, we see a lower bound for it right away. Let n be the number of vertices on the graph. If we consider that any single vertex can be a clique or an independent set, then there are n sets of vertices that may be given to either Bob or Alice (namely the n vertices). Then I_n is a submatrix of M_f . Indeed, for these n indices, if Bob and Alice receive vertices v_i and v_j respectively, then the corresponding entry of M_f will be 1 if $i = j$ and 0 otherwise. Since, I_n is a submatrix of M_f , we must have

$$\text{rank}(M_f) \geq \text{rank}(I_n) = n$$

which implies that $C(CL - IS_G) \geq \log_2 \text{rank}(M_f) \geq \log_2 n$. Surprisingly, this trivial lower bound is the best bound known at the present time. Next we discuss the connection between the clique vs. independent set problem and the Alon-Saks-Seymour Conjecture.

Proposition 18 (Alon-Haviv [17]). *For any graph G with $\chi(G) > \text{bp}(G) + 1$, there is a corresponding graph H with $C(CL - IS_H) > \log_2 |V(H)|$.*

Proof. During this proof, if f is a boolean function and M_f is its associated matrix, we will use $C(f)$ and $C(M_f)$ interchangeably. Similarly, we will use $d(f)$ and $d(M_f)$ interchangeably. Let G be a graph on vertex set $[n]$ and $\text{bp}(G) = m$. Let $\{B(U_i, W_i)\}_{i=1}^m$ be a partition of $E(G)$ into bicliques. Define the characteristic vector v_i of each biclique $v_i = (v_{i1}, \dots, v_{in})$, the same way as in Graham and Pollak's [14]

addressing scheme (see Section 1.2); that is

$$v_{ij} = \begin{cases} 0 & j \in U_i \\ 1 & j \in W_i \\ * & \text{otherwise} \end{cases}$$

Now we define a new graph H on the vertex set $[m]$. Two vertices i and i' are adjacent in H if there exists a $j \in [n]$ such that $v_{ij} = v_{i'j} = 1$ and nonadjacent if there exists $j' \in [n]$ such that $v_{ij'} = v_{i'j'} = 0$. In any other case, arbitrarily assign an edge or non-edge. To show that H is well defined, we must show that there cannot be a j and j' such that $v_{ij} = v_{i'j} = 1$ and $v_{ij'} = v_{i'j'} = 0$. If this were the case, then $j \in W_i \cap W_{i'}$ and $j' \in U_i \cap U_{i'}$ which is a contradiction because the edge jj' would be covered by two bicliques. So H is well defined.

Now we consider the $CL - IS$ problem on H . For $j \in [n]$, define $C_j = \{q \in [m] : v_{qj} = 1\}$ and $I_j = \{q \in [m] : v_{qj} = 0\}$. Then the $\{C_j\}$ are cliques and the $\{I_j\}$ are independent sets. Denote the matrix for $CL - IS_H$ by M and let M' denote the n by n submatrix which has its rows indexed by $\{C_j\}$ and columns indexed by $\{I_j\}$. Then by the inequalities discussed previously, we have

$$C(M) \geq C(M') \geq \log_2 d(M').$$

Now assume that R_1, \dots, R_t are 0-rectangles that cover the diagonal entries of M' . If (p, q) is covered by R_i , then $M'_{pq} = M'_{qp} = 0$ and $|C_p \cap I_q| = |C_q \cap I_p| = 0$. If pq were an edge in G , we would have an index i such that either $v_{ip} = 0, v_{iq} = 1$ or $v_{ip} = 1, v_{iq} = 0$, a contradiction. So pq is not an edge in G . This means that R_1, \dots, R_t corresponds to a covering of the vertices of G by independent sets and in particular $\chi(G) \leq t$. So we have

$$C(M) \geq C(M') \geq \log_2 d(M') \geq \log_2(t) \geq \log_2 \chi(G).$$

Then if $\text{bp}(G) < \chi(G)$, we have the existence of a graph H such that

$$C(CL - IS_H) \geq \log_2 \chi(G) > \log_2 \text{bp}(G) = \log_2 |V(H)|.$$

□

This result shows that a separation between chromatic number and biclique partition number yields a separation between the deterministic complexity of the clique vs. independent set problem for some graph and the logarithm of the size of that graph. It is not known if the converse is true. However, this tells us that a graph that is a counterexample to the Alon-Saks-Seymour Conjecture leads to a corresponding graph H that has a linear gap (super-unitary) between its deterministic complexity for the clique vs. independent set problem and $\log_2 |V(H)|$.

In Chapter 2, we constructed graphs $G(n, k, r)$ such that

$$\chi(G(n, k, r)) \geq \frac{n^{2k+2r}}{2r+1}$$

and

$$\text{bp}(G(n, k, r)) < 2^{2k+2r-1} n^{2k+2r-1}.$$

This means that these graphs correspond to graphs $H = H(n, k, r)$ such that

$$C(CL - IS_H) \geq \log_2 \left(\frac{n^{2k+2r}}{2r+1} \right) > \log_2 (2^{2k+2r-1} n^{2k+2r-1}) = \log_2 |V(H)|$$

which implies graphs $H = H(n, k, r)$ with a constant c such that

$$C(CL - IS_H) \geq \frac{2k+2r}{2k+2r-1} \log_2 |V(H)| - c.$$

Chapter 4

THE GRAHAM-POLLAK THEOREM FOR HYPERGRAPHS

4.1 Introduction

The question about partitioning graphs into bicliques can be generalized to hypergraphs. In this problem, one is attempting to partition the edge set of a hypergraph into complete multipartite subgraphs. If we consider the complete r -uniform hypergraph on n vertices, denoted $K_n^{(r)}$, we can ask how many complete r -partite r -uniform subgraphs are necessary to partition the edge set of $K_n^{(r)}$. This is a generalization of the Graham-Pollak Theorem. Indeed taking $r = 2$, the question asks how many bicliques are necessary to partition the edge set of K_n . The value for $K_n^{(r)}$ is unknown for $r \geq 4$. This question has applications to the complexity of computing bilinear forms and symmetric polynomials [15, 16, 27] and to the tensor rank computation of high dimensional arrays [27]. In this chapter we discuss the problem.

First we explain hypergraph notation (see [5]). Let $[n]$ denote the set $\{1, \dots, n\}$ and $[n]^{(r)}$ denote all r -subsets of $[n]$. If X_1, \dots, X_r are disjoint subsets of $[n]$, $\prod_{i=1}^r X_i$ denotes the set of subsets given by $\{(x_1, \dots, x_r) | x_i \in X_i\}$. The complete r -partite r -uniform hypergraph with parts X_1, \dots, X_r is the r -uniform hypergraph with edge set $\prod_{i=1}^r X_i$. Given an r -uniform hypergraph G , $f_r(H)$ denotes the minimum number

of complete r -partite r -uniform hypergraphs needed to partition the edge set of H . We denote $f_r(K_n^{(r)})$ by $f_r(n)$.

The best known bound is given by Cioabă, Kündgen, and Verstraëte [7] who improved a result of Alon [2] and showed

$$\frac{2\binom{n-1}{k}}{\binom{2k}{k}} \leq f_{2k}(n) \leq \binom{n-k}{k} \quad (4.1)$$

and

$$f_{2k}(n-1) \leq f_{2k+1}(n) \leq \binom{n-k-1}{k}. \quad (4.2)$$

4.2 The case $n = r + 2$

In this section, we determine $f_r(r+2)$ exactly. When $n = r+2$, each hyperedge of $K_n^{(r)}$ is an r -subset of $[n]$, and thus its complement has size 2. Thus, the complement of each hyperedge can be seen as an edge of K_{r+2} and we can consider K_{r+2} as a complement graph. If we decompose $K_{r+2}^{(r)}$ into complete r -partite r -uniform subgraphs, then the complements of each hyperedge will partition the complement graph K_{r+2} . Partitioning the complete r -uniform hypergraph is equivalent to partitioning K_{r+2} with certain graphs. Consider what an r -partite sub hypergraph of $K_{r+2}^{(r)}$ can look like. It can have r partite sets of size 1, which induces a K_2 in the complement graph. It can have $r-1$ partite sets of size 1 and 1 partite set of size 2, which induces a $K_{1,2}$ in the complement graph. It can have $r-1$ partite sets of size 1 and 1 set of size 3, which induces a K_3 in the complement graph. Finally, it can have $r-2$ partite sets of size 1 and 2 partite sets of size 2, which induces a $K_{2,2} = C_4$ in the complement graph. Since these are the only kinds of complete r -uniform r -partite subgraphs possible for the case $n = r+2$, partitioning $K_{r+2}^{(r)}$ into complete r -partite r -uniform subhypergraphs is equivalent to partitioning K_{r+2} into K_2 , K_3 , C_4 , or $K_{1,2}$.

Proposition 19. For any natural number r , $f_{8r-1}(8r+1) = r(8r+1)$.

Proof. From above, decomposing $K_{8r+1}^{(8r-1)}$ into complete $8r-1$ -partite $8r-1$ -uniform hypergraphs is equivalent to decomposing K_{8r+1} into K_2 , K_3 , C_4 , or $K_{1,2}$. K_{8r+1} has $4r(8r+1)$ edges. Because $4|4r(8r+1)$ and the number of vertices is odd, one can partition K_{8r+1} into $r(8r+1)$ copies of C_4 (see [32]). Thus $K_{8r+1}^{(8r-1)}$ can be partitioned into $r(8r+1)$ complete $8r-1$ -partite $8r-1$ -uniform hypergraphs with partite sets of size $\{2, 2, 2, 1, 1, \dots, 1\}$ and $f_{8r-1}(8r+1) \leq r(8r+1)$. The lower bound follows because each complement subgraph contains at most 4 edges. \square

Proposition 20. For $k \geq 2$, $\lceil \frac{(k+1)(2k+3)}{4} \rceil \leq f_{2k}(2k+2) \leq \frac{(k+1)(k+2)}{2}$.

Proof. The upper bound is given by (4.2). Partitioning the hyperedge set $K_{2k+2}^{(2k)}$ with complete $2k$ -partite $2k$ -uniform hypergraphs is equivalent to partitioning K_{2k+2} with K_2 , K_3 , C_4 , and $K_{1,2}$. Since each vertex of K_{2k+2} has odd degree, each vertex must be incident with at least one of K_2 or $K_{1,2}$. Thus at least $k+1$ graphs with at most 2 edges must be used. Then at least $\binom{2k+2}{2} - 2(k+1) = (k+1)(2k-1)$ edges remain and at least $\lceil \frac{(k+1)(2k-1)}{4} \rceil$ graphs must be used. So the total number of graphs used is at least $\lceil \frac{(k+1)(2k-1)}{4} \rceil + k+1 = \lceil \frac{(k+1)(2k+3)}{4} \rceil$. \square

Proposition 21. For any natural number r , $f_{8r}(8r+2) = 8r^2 + 5r + 1$.

Proof. Consider $K_{8r+2}^{(8r)}$ and let v be an arbitrary vertex. Then each hyperedge is an $8r$ -tuple that either contains v or does not. The hyperedges that contain v can be partitioned into $f_{8r-1}(8r+1)$ subgraphs. The hyperedges that do not contain v can be partitioned into $f_{8r}(8r+1)$ subgraphs. To partition $K_{8r+2}^{(8r)}$ into complete $8r$ -partite $8r$ -uniform subgraphs, we can take the following complete $8r$ -partite $8r$ -uniform hypergraphs with partite sets $G_1 = \{1, 2\}, \{3\}, \{4\}, \dots, \{8r+1\}$, $G_2 = \{1\}, \{2\}, \{3, 4\}, \dots, G_{4r} = \{1\}, \{2\}, \dots, \{8r-1, 8r\}, \{8r+1\}$, $G_{4r+1} =$

$\{1\}, \{2\}, \dots, \{8r\}$. Thus $f_{8r}(8r+1) \leq 4r+1$. By the Proposition 19, $f_{8r-1}(8r+1) = 8r^2 + r$. Because $f_{8r}(8r+2) \leq f_{8r}(8r+1) + f_{8r-1}(8r+1)$ we have $f_{8r}(8r+2) \leq 8r^2 + 5r + 1$. The lower bound is given by Proposition 20. \square

Theorem 22. For $k \geq 2$, $f_{2k}(2k+2) = \lceil \frac{2k^2+5k+3}{4} \rceil$

Proof. We prove by induction on k . For the base cases, it is known that $f_2(4) = 3$. Equation 4.2 gives $f_4(6) \leq 6$ and this is the lower bound given by Proposition 20. We also have that $f_6(8) \leq f_7(9) = 9$ by Proposition 19 and $f_6(8) \geq 9$ from Proposition 20. Now assume that $f_{2k}(2k+2) = \lceil \frac{2k^2+5k+3}{4} \rceil$. Then Proposition 20 gives us that

$$f_{2k+2}(2k+4) \geq \lceil \frac{2k^2 + 9k + 10}{4} \rceil.$$

Consider the following decomposition of $K_{2k+4}^{(2k+2)}$. We look at it in terms of the complement graph formed by the complements of the edges as before. So we want to find a partition of K_{2k+4} into C_4 , K_3 , K_2 , and $K_{1,2}$. Pick any two vertices of K_{2k+4} . Then consider an optimal decomposition into $\lceil \frac{2k^2+5k+3}{4} \rceil$ of the complete graph induced by the other $2k+2$ vertices. To complete the decomposition, we need to cover all edges between the $2k+2$ original vertices and the two remaining vertices, and we need to cover the one edge between the two remaining vertices. We use a K_2 to cover the edge between the two vertices. Then it is clear that we can use $k+1$ copies of C_4 , each having the two remaining vertices and one of the $k+1$ pairs of vertices as its nodes. Thus, we can decompose K_{2k+4} into C_4 , K_3 , K_2 , and $K_{1,2}$ by $\lceil \frac{2k^2+5k+3}{4} \rceil + 1 + (k+1) = \lceil \frac{2k^2+9k+11}{4} \rceil$ graphs. So we have that

$$\lceil \frac{2k^2 + 9k + 10}{4} \rceil \leq f_{2k+2}(2k+4) \leq \lceil \frac{2k^2 + 9k + 11}{4} \rceil.$$

These bounds are the same whenever $k \equiv 0, 1 \pmod{4}$. In the case that $k \equiv 3 \pmod{4}$, the result follows from Proposition 21. In the case that $k \equiv 2 \pmod{4}$, we

are attempting to find $f_{8r+6}(8r+8)$ for some r . Note, however, that $f_{8r+6}(8r+8) \leq f_{8r+7}(8r+9) = (r+1)(8r+9)$ by Proposition 19 and this completes our proof. \square

Theorem 23. For $k \geq 2$, $f_{2k+1}(2k+3) = \lceil \frac{2k^2+5k+3}{4} \rceil$.

Proof. We have $f_{2k+1}(2k+3) \geq f_{2k}(2k+2) = \lceil \frac{2k^2+5k+3}{4} \rceil$ by (4.2). To prove the upper bound we use a similar construction as in Theorem 22. For the induction hypothesis assume that $f_{2k-1}(2k+1) = \lceil \frac{2k^2+k}{4} \rceil$. Consider decomposing $K_{2k+3}^{(2k+1)}$. This is equivalent to decomposing K_{2k+3} into C_4 , K_3 , K_2 , and $K_{1,2}$. Pick any two vertices of K_{2k+3} . The other $2k+1$ vertices induce a K_{2k+1} which can be partitioned into $\lceil \frac{2k^2+k}{4} \rceil$ copies of C_4 , K_3 , K_2 , or $K_{1,2}$. The remaining edges needed to be covered are those between the two remaining vertices and the $2k+1$ vertices, and the single edge between the two remaining vertices. Partition the $2k+1$ vertices into k pairs and one single node. The edges between the two remaining vertices and the single node, and the single edge between the two remaining vertices can be covered with a K_3 . Then it is clear that the remaining edges can be covered by k copies of C_4 , each with the two remaining vertices and one of the k pairs of vertices as its nodes. This covers K_{2k+3} with $\lceil \frac{2k^2+k}{4} \rceil + 1 + k$ copies of C_4 , K_3 , K_2 , and $K_{1,2}$, equivalent to decomposing $K_{2k+3}^{(2k+1)}$ into $\lceil \frac{2k^2+5k+4}{4} \rceil$ hypergraphs. This gives us that

$$\lceil \frac{2k^2+5k+3}{4} \rceil \leq f_{2k+1}(2k+3) \leq \lceil \frac{2k^2+5k+4}{4} \rceil$$

These bounds are the same when $k \equiv 0, 1, 2 \pmod{4}$. The case where $k \equiv 3 \pmod{4}$ is handled by Proposition 19. \square

4.3 Improving the Upper Bound

Next we make an improvement on the upper bound in general. This is done by recursive construction. We can obtain an upper bound for $f_{2k}(n)$ by considering

the n vertices as a set of j vertices and a set of $n - j$ vertices for some j . Then we can use known upper bounds on the smaller sets to obtain an upper bound for $f_{2k}(n)$.

Lemma 24. *Fix j, k . Then for any $n \geq j$ we have*

$$\binom{n-k}{k} = \sum_{i=0}^{2k} \binom{j - \lceil \frac{i}{2} \rceil}{\lfloor \frac{i}{2} \rfloor} \binom{n-j - \lceil \frac{2k-i}{2} \rceil}{\lfloor \frac{2k-i}{2} \rfloor}.$$

Proof. We prove by induction on n . We define $\binom{m}{t}$ to be 0 if $m < 0$ and 1 if $m = t = 0$. Then in the base case $n = j$ the left hand side of the identity equals $\binom{j-k}{k}$ and the only nonzero term, $\binom{j-k}{k}$, of the right hand side is obtained when $i = 2k$. Now

$$\begin{aligned} & \sum_{i=0}^{2k} \binom{j - \lceil \frac{i}{2} \rceil}{\lfloor \frac{i}{2} \rfloor} \binom{n-j - \lceil \frac{2k-i}{2} \rceil}{\lfloor \frac{2k-i}{2} \rfloor} \\ &= \sum_{i=0}^{2k} \binom{j - \lceil \frac{i}{2} \rceil}{\lfloor \frac{i}{2} \rfloor} \left[\binom{n-j - \lceil \frac{2k-i}{2} \rceil - 1}{\lfloor \frac{2k-i}{2} \rfloor} + \binom{n-2-j - \lceil \frac{2k-i}{2} \rceil - 1}{\lfloor \frac{2k-i}{2} \rfloor - 1} \right] \end{aligned}$$

By the induction hypothesis we have that this equals $\binom{(n-1)-k}{k} + \binom{(n-2)-(k-1)}{k-1}$ which equals $\binom{n-k}{k}$. \square

Lemma 25. *For $n \geq 10$ we have*

$$f_8(n) < \binom{n-4}{4} - \frac{1}{20}n. \quad (4.3)$$

Proof. Let j be the largest integer such that $10 \cdot 2^j \leq n$. We prove the lemma by induction on j . For base case $j = 0$, we are attempting to find $f_8(n)$ where $10 \leq n < 20$. If we let $n = 10 + r$ where $0 \leq r < 10$, we break the n vertices into one part of size r and one part of size 10. Then a decomposition of $K_n^{(8)}$ into complete 8-partite 8-uniform hypergraphs can be obtained from a partition of $K_{10}^{(i)}$ into $f_i(10)$ complete i -partite i -uniform hypergraphs and a partition of $K_r^{(8-i)}$ into

$f_{8-i}(r)$ complete $(8-i)$ -partite $(8-i)$ -uniform hypergraphs, when i takes values between 0 and 8. Thus

$$f_8(n) \leq \sum_{i=0}^8 f_i(10) f_{8-i}(r)$$

We bound everything above by (4.2) except for the term $f_8(10)$, where we use Theorem 22. Using Lemma 24, this proves the base case.

If $j > 0$, we provide a recursive construction by breaking the n vertices into two sets of size $\frac{n}{2}$ and find

$$f_8(n) \leq \sum_{i=0}^8 f_i\left(\frac{n}{2}\right) f_{8-i}\left(\frac{n}{2}\right).$$

Using the upper bound given by (4.2), we have

$$f_8(n) \leq \sum_{i=1}^7 \binom{\frac{n}{2} - \lceil \frac{i}{2} \rceil}{\lfloor \frac{i}{2} \rfloor} \binom{\frac{n}{2} - \lceil \frac{8-i}{2} \rceil}{\lfloor \frac{8-i}{2} \rfloor} + 2f_8\left(\frac{n}{2}\right).$$

Using Lemma 24 and the induction hypothesis we have that

$$f_8(n) \leq \binom{n-4}{4} - 2 \cdot \frac{1}{20} \cdot \frac{n}{2}.$$

□

We find the improvement in general by constructing recursively and lowering the upper bound by the improvement on the $f_8(n)$ terms.

Theorem 26. $f_{2k}(n) \leq \binom{n-k}{k} - \frac{n}{20} \binom{\lfloor \frac{n}{2} \rfloor - k + 4}{k-4}$.

Proof. It suffices to show the case n even, so we assume this for convenience. Again, a decomposition of $K_n^{(2k)}$ into complete $2k$ -partite $2k$ -uniform hypergraphs can be obtained from a decomposition of $K_{\frac{n}{2}}^{(i)}$ into $f_i(\frac{n}{2})$ complete i -partite i -uniform subgraphs

and a partition of $K_{\frac{n}{2}}^{(2k-i)}$ into $f_{2k-i}(\frac{n}{2})$ complete $(2k-i)$ -partite $(2k-i)$ -uniform hypergraphs, when i takes values from 0 to $2k$. Thus

$$f_{2k}(n) \leq 2f_8\left(\frac{n}{2}\right) f_{2k-8}\left(\frac{n}{2}\right) + \sum_{i \neq 8, 2k-8} f_i\left(\frac{n}{2}\right) f_{2k-i}\left(\frac{n}{2}\right).$$

We bound each term of the previous sum from above by (4.2) for all $f_j(\frac{n}{2})$ except for when $j = 8$ where we use the bound given by Lemma 25. Then, using Lemma 24, we have

$$f_{2k}(n) \leq \binom{n-k}{k} - 2 \cdot \frac{1}{20} \cdot \frac{n}{2} \cdot \binom{\frac{n}{2}-k+4}{k-4}.$$

□

Chapter 5

FUTURE WORK

In this Chapter we talk about open problems and future work.

5.1 The Alon-Saks-Seymour and Rank-Coloring Conjectures

Both the Alon-Saks-Seymour and Rank-Coloring Conjectures have been disproven, but it remains an open question to see how large the gaps between parameters can be. As seen in Chapter 3, if it can be shown that the logarithm of the chromatic number of a graph cannot be bounded by a polynomial in the logarithm of the rank of its adjacency matrix, then the Log-Rank Conjecture is false.

Open Question 1. *Is the Log-Rank Conjecture true? Equivalently, does there exist a constant $k > 0$ such that for all graphs G*

$$\log_2 \chi(G) \leq (\log_2 \text{rank}(A(G)))^k?$$

In terms of the Alon-Saks-Seymour Conjecture, it remains an open problem to see how large the gap between the biclique partition number and the chromatic number of a graph can be in general. Huang and Sudakov conjecture in [17] that there exists a graph G with biclique partition number k and chromatic number at least $2^{c \log^2 k}$, for some constant $c > 0$. In communication complexity it is a long-standing open question to prove an $\Omega(\log^2 N)$ lower bound on the complexity of the

CL-IS problem for a graph on N vertices. The existence of the conjectured graph would resolve this problem.

Open Question 2. *Does there exist a graph G with biclique partition number k and chromatic number at least $2^{c \log^2 k}$, for some constant $c > 0$?*

5.2 The Graham-Pollak Theorem for Hypergraphs

Even with the improvement given in Chapter 4, the general bounds are still given by

$$\frac{2 \binom{n-1}{k}}{\binom{2k}{k}} \leq f_{2k}(n) \leq \binom{n-k}{k} - \frac{n}{20} \binom{\lfloor \frac{n}{2} \rfloor - k + 4}{k-4},$$

which are far apart.

Open Question 3. *What is the correct value of $f_{2k}(n)$ and $f_{2k+1}(n)$?*

This is an interesting question to investigate. We notice that the way we went about improving the bounds was by reducing the hypergraph question to a question on normal graphs. To close the gap between the bounds, perhaps a different approach is necessary.

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