Coupon colorings of regular graphs

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Definition of coupon coloring

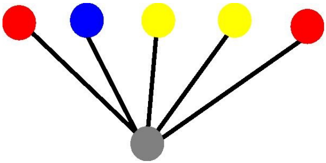
Let G be a graph with no isolated vertices.

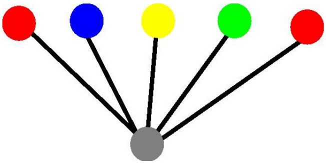
A *k-coupon coloring* is a coloring of the vertices from $[k]$ such that the neighborhood of every vertex of G contains all colors from $[k]$.

The maximum k for which a k-coupon coloring of G exists is called the *coupon coloring number of G* and will be denoted by $\chi_c(G)$.

Example of coupon coloring

Coloring with four colors.





Not coupon colored

Coupon coloring OK

$\chi_c(G)$ is well defined since we may color every vertex the same color.

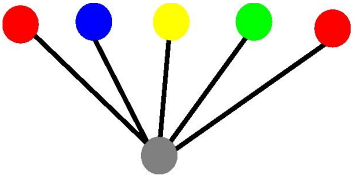
Definition of injective coloring

An *injective k-coloring* is a coloring of the vertices from $[k]$ such that the neighborhood of every vertex contains distinct colors. i.e. vertices with a path of length 2 between them receive different colors.

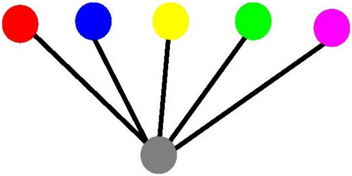
The minimum k for which an injective k-coloring exists is called the *injective coloring number of* G and will be denoted by $\chi_i(G)$

Example of injective coloring

Coloring with ≥ 5 colors.



Not injectively colored



Injective coloring OK

$\chi_i(G)$ is well defined since we may assign distinct colors to every vertex.

However, we can observe that if G has minimum degree δ and maximum degree Δ, then

$$\chi_c(G) \leq \delta \leq \Delta \leq \chi_i(G).$$

We will be interested in d-regular graphs with d large.

Previous Work

d-regular graphs that obtain $\chi_c(G) = d = \chi_i(G)$ are called *rainbow graphs*.

Figure: Lazebnik and Woldar

Previous Work

Coupon coloring has been studied in relation to large multi-robot networks.

Coupon coloring is related to panchromatic hypergraph coloring.

Many researchers have studied injective colorings, in particular on the Hamming graph in relation to scalability of optical networks

Coloring Q_n

The *Hypercube* Q_n is the graph with vertex set $\{0, 1\}^n$. Two vectors x and y are adjacent if they have Hamming distance 1.

$$(0, 1, 1, 0, 0) \sim (1, 1, 1, 0, 0)$$

$$(0, 1, 1, 0, 0) \not\sim (1, 1, 1, 0, 1)$$

Q_n is n-regular

Coloring Q_n

Theorem

Let $n = 2^t$. *Then*

$$\chi_c(Q_n) = \chi_i(Q_n) = n.$$

Proof: We will exhibit a coloring with n colors such that if $v \sim y$ and $v \sim z$, then y and z have distinct colors.

Identify $V(Q_n)$ with the power set of $\mathbb{F}_n$ in the natural way.

$$\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$$
$$v = (1, 0, 0, 1)$$
$$A_v = \{0, \quad \alpha^2\}$$

Identify colors with $\mathbb{F}_n$. Color A_v with

$$\sum_{x \in A_v} x$$

Coloring Q_n

Now assume $v \sim y$ and $v \sim z$. This means y and z each have Hamming distance 1 from v. Then there exists $\alpha, \beta \in \mathbb{F}_n$ such that

$$y \text{ colored with } (\pm)\alpha + \sum_{x \in A_v} x$$

$$z \text{ colored with } (\pm)\beta + \sum_{x \in A_v} x$$

$y \neq z$ implies $\alpha \neq \beta$. □

Theorem (Chen, Kim, MT, Verstraëte)

For every $\delta > 0$, there exists a $d_0(\delta)$ such that if $d \geq d_0(\delta)$, then every d-regular graph G has

$$\chi_c(G) \geq (1-\delta)\frac{d}{\log d}.$$

For every $\epsilon > 0$, there exists a $d_1(\epsilon)$ such that if $d \geq d_1(\epsilon)$, then as $n \to \infty$, almost every d-regular n-vertex graph has

$$\chi_c(G) \leq (1+\epsilon)\frac{d}{\log d}.$$

This gives $\chi_c(G) \sim \frac{d}{\log d}$ for almost all d-regular graphs.

Coupon Collector Problem

The expected time to collect n coupons drawing uniformly, independently, and with replacement is asymptotic to $n \log n$.

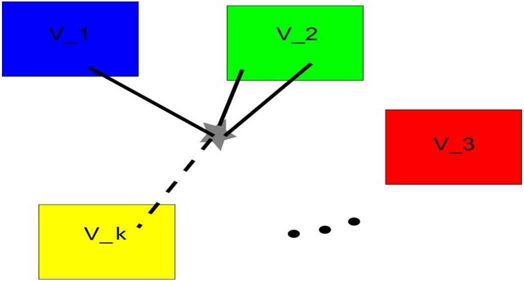
Theorem (Erdős and Rényi, 1961)

Let T_n be the time to collect n coupons. Then

$$\mathbb{P}(T_n < n \log n + cn) \to e^{-e^{-c}}$$

as $n \to \infty$.

Bounds for χ_c



d neighbors is the expected time to see $\frac{d}{\log d}$ colors if they were distributed randomly.

If there are $(1-\delta)\frac{d}{\log d}$ colors, coloring randomly gives each vertex a high chance of seeing all colors.

If there are $(1+\epsilon)\frac{d}{\log d}$ colors, it is very unlikely that a vertex sees every color when generating a random graph. $\square$

Open Problems

The *Hamming Graph* $H(n,q)$ is the graph with vertex set $[q]^n$ and two vectors adjacent if they have Hamming distance 1. $H(n,q)$ is $(q-1)n$ regular.
Östergard (2004) showed $\chi_i(H(n,q)) \sim (q-1)n$ for $q = 2, 3$.

Conjecture

Fix q, then as $n \to \infty$

$$\chi_i(H(n,q)) \sim \chi_c(H(n,q)) \sim (q-1)n$$

Open Problems

Can one find an explicit family of d-regular graphs with coupon coloring number $(1 + o(1))\frac{d}{\log d}$ as $d \to \infty$?

Paley graphs come within a factor of 4.

Thank You!