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A comment on a conjecture of N. Wiener

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ABSTRACT

N. Wiener conjectured that a necessary and sufficient condition for a stationary process to be representable as a one-sided function of a sequence of independent, identically distributed random variables and its shifts is that its backward tail field be trivial. Here it is shown that the condition is not sufficient for such a representation.

1. Introduction

Let $\{X_n, -\infty < n < \infty\}$ be a stationary process with

$$\mathcal{B}_n = \mathcal{B}\{X_j, j \leq n\}$$

the σ-field generated by the random variables $X_j, j \leq n$. Let $\{\xi_n, -\infty < n < \infty\}$ be a sequence of independent, identically distributed random variables. In Wiener (1958) the question of under what circumstances a stationary process $\{X_n\}$ could have a one-sided representation

$$X_n = f(\xi_n, \xi_{n-1}, \ldots) \tag{1}$$

in terms of iid random variables was discussed. It was conjectured there that a necessary and sufficient condition for such a representation was that the backward tail field

$$\mathcal{B}_{-\infty} = \bigcap_n \mathcal{B}_n = \{\emptyset, \Omega\} \tag{2}$$

be trivial. This was shown to be true for stationary countable state Markov chains in Rosenblatt (1960). A partial extension of these results to continuous state Markov sequences was given by Hanson (1963). In this note it will be shown that there are stationary sequences $\{X_n\}$ with trivial tail field that cannot have such a one-sided representation in terms of independent, identically distributed random variables.

2. A factor

Let $x = (x_n, n = \ldots, -1, 0, 1, \ldots)$ with the x_n's real, $\mathfrak{M}$ the product σ-algebra of the 1-dimensional Borel sets and μ a Bernoulli measure on $\mathfrak{M}$. T, the shift operator acting on x ($(Tx)_n = x_{n+1}$) is a Bernoulli or B-automorphism of $(M, \mathfrak{M}, \mu)$ where M is the space of sequences x. Let $y_0 = f(x_0, x_{-1}, \ldots)$ be a Borel measurable function with

$$y_n = f(T^n x)$$

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and $y = (y_n, n = \dots, -1, 0, 1, \dots)$. Consider T_1 the shift operator on y sequences. M_1 is the space of y sequences, $\mathfrak{M}_1$ the σ-algebra on y sequences and μ_1 the measure on $\mathfrak{M}_1$ induced by $(M, \mathfrak{M}, \mu)$. Let

$$\phi(x) = \{y_n(x), n = \dots, -1, 0, 1, \dots\}.$$

Then

$$\begin{aligned}\phi(Tx) &= \{y_{n+1}(x), n = \dots, -1, 0, 1, \dots\} \\ &= T_1\phi(x)\end{aligned} \tag{3}$$

so that $\phi : M \to M_1$ is a homomorphism and T_1 is a factor automorphism of the B-automorphism T (see Cornfield and Sinai (1989)). But it is known that a factor-automorphism of a B-automorphism is also a B-automorphism (see Ornstein (1974)). So *the shift T_1 acting on a process (1) with a one-sided representation is a B-automorphism.*

If for any measurable set $A \in \mathfrak{M}$,

$$\lim_{n\to\infty} P(T^n x \in A | x_j, j \le 0) = \lim_{n\to\infty} P(A|x_j, j \le -n) = P(A) \tag{4}$$

($\mathcal{B}_{-\infty}$ is trivial), any automorphism with this property is called a K-automorphism.

In Kalikow (1982) a transformation referred to as "T, T^{-1}" leads to a process that is shown to be a K-automorphism, but not Bernoulli. Set $Q = (1, -1)$ and the random variables $\{w_i\}_{i\in\mathbb{Z}}$ independent, identically distributed random variables (iid) with

$$w_i = \begin{cases} 1 & \text{with probability } \dfrac{1}{2} \\ -1 & \text{with probability } \dfrac{1}{2}. \end{cases}$$

Let T be the shift $(T(w))_i = w_{i+1}$ for each $w = \{w_i\}_{i\in\mathbb{Z}}$ in $\Omega = Q^Z$. The transformation S on $\Omega_1 \times \Omega_2$ is set up so that

$$S((_1w, {}_2w)) = \begin{cases} (T(_1w), T(_2w)) & \text{if } {}_2w_0 = 1 \\ (T^{-1}(_1w), T(_2w)) & \text{if } {}_2w_0 = -1 \end{cases}$$

and $(_1w', {}_2w')_n = (S^n(_1w, {}_2w))_0$. Let

$$X(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ \displaystyle\sum_{j=0}^{i-1} w_j & \text{if } i > 0 \\ \displaystyle -\sum_{j=-1}^{-i} w_j & \text{if } i < 0. \end{cases}$$

One can show that

$${}_2w'_i = {}_2w_i, \quad {}_1w'_i = {}_1w_{X(i, {}_2w)}.$$

The T, T^{-1} transformation is a K-transformation that Kalikow has shown is not a Bernoulli transformation. By the discussion given earlier it is clear we have correspondingly a stationary process $(_1w', {}_2w')_n$ with trivial backward tail field that cannot have a representation of the form (1).

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