Interpolation and Approximation: Hermite Interpolation

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Background

Hermite Interpolation

Background

Suppose we have pairwise distinct nodal points $x_0, x_1, \ldots, x_m \in \mathbb{R}$.

Lemma

The Newton polynomials $p_k(x)$ *with*

$$p_k(x) = \prod_{i=0}^{k-1} (x - x_i), \quad 0 \leq k \leq m,$$

are a basis of $\mathcal{P}_m$.

Proof.

Let $p(x) \in \mathcal{P}_m$ be a degree m polynomial with

$$p(x) = a_0 p_0(x) + a_1 p_1(x) + \cdots + a_m p_m(x).$$

Assume that $p(x) = 0$. Then obviously

$$p(x_0) = p(x_1) = \cdots = p(x_m) = 0.$$

We show step by step that all coefficients $a_0, a_1, \ldots, a_m$ are equal zero.

First, since by the definition of the Newton polynomials

$$p_0(x_0) = 1, \quad p_1(x_0) = p_2(x_0) = \cdots = p_m(x_0) = 0,$$

so we get that $0 = p(x_0) = a_0 p_0(x_0) = a_0$, which implies $a_0 = 0$.

Proof.

Next, since by the definition of the Newton polynomials

$$p_1(x_1) = x_1 - x_0, \quad p_2(x_1) = p_3(x_1) = \cdots = p_m(x_1) = 0,$$

so we get that $0 = p(x_1) = a_1 p_1(x_1) = a_1(x_1 - x_0)$, which implies $a_1 = 0$.

Next, since by the definition of the Newton polynomials

$$p_2(x_2) = (x_2 - x_1)(x_2 - x_0), \quad p_3(x_2) = \cdots = p_m(x_2) = 0,$$

so we get that $0 = p(x_2) = a_2 p_2(x_2) = a_2(x_2 - x_1)(x_2 - x_0)$, which implies $a_2 = 0$.

Repeating this shows $a_0, a_1, \ldots, a_m$ are all equal zero, and so the polynomials are linearly independent. □

Lemma

The Lagrange polynomials $L_k(x)$ *with*

$$L_k(x) = \prod_{\substack{0\leq j\leq m\\ k\neq j}} \frac{x - x_j}{x_k - x_j}, \quad 0 \leq k \leq m,$$

are a basis of $\mathcal{P}_m$.

Proof.

Let $p(x) \in \mathcal{P}_m$ be a degree m polynomial with

$$p(x) = a_0 L_0(x) + a_1 L_1(x) + \cdots + a_m L_m(x).$$

Assume that $p(x) = 0$. Then obviously

$$p(x_0) = p(x_1) = \cdots = p(x_m) = 0.$$

For any $0 \leq k \leq m$ we have

$$\begin{aligned} 0 = p(x_k) &= a_0 L_0(x_k) + a_1 L_1(x_k) + \cdots + a_m L_m(x_k) \\ &= a_k L_k(x_k) = a_k. \end{aligned}$$

So $a_k = 0$. Since all the coefficients vanish, the polynomials are linearly independent, and thus they are a basis. $\square$

Background

Suppose we have nodal points $x_0, x_1, \ldots, x_m \in \mathbb{R}$, not necessarily pairwise distinct.

Having nodal points with duplicates corresponds to the case of interpolation that takes into account higher order derivatives. That is called **Hermite interpolation**.

Background

Hermite Interpolation

Hermite Interpolation

Suppose we have pairwise distinct nodal points $x_0, x_1, \ldots, x_n \in \mathbb{R}$ and non-negative integers $k_0, k_1, \ldots, k_n \geq 0$.

The Hermite interpolation problem seeks a polynomial $p \in \mathcal{P}_m$ of degree

$$m = n + k_0 + k_1 + \cdots + k_n.$$

with the property that for some fixed function f we have

$$\begin{aligned}
&p(x_0) = f(x_0), \quad p'(x_0) = f'(x_0), \quad \ldots \quad p^{(k_0)}(x_0) = f^{(k_0)}(x_0),\\
&p(x_1) = f(x_1), \quad p'(x_1) = f'(x_1), \quad \ldots \quad p^{(k_1)}(x_1) = f^{(k_1)}(x_1),\\
&\qquad\vdots\\
&p(x_n) = f(x_n), \quad p'(x_n) = f'(x_n), \quad \ldots \quad p^{(k_n)}(x_n) = f^{(k_n)}(x_n).
\end{aligned}$$

In the discussion of that problem, it will be helpful to interpret the k_i as multiplicities of the nodal points x_i.

Hermite Interpolation

Suppose we have pairwise distinct nodal points $x_0, x_1, \dots, x_n \in \mathbb{R}$ and non-negative integers $k_0, k_1, \dots, k_n \geq 0$.

The Hermite interpolation problem seeks a polynomial $p \in \mathcal{P}_m$ of degree

$$m = n + k_0 + k_1 + \cdots + k_n.$$

with the property

$$\begin{aligned}
p(x_0) &= y_0, \quad p'(x_0) = y_0', \quad \dots \quad p^{(k_0)}(x_0) = y_0^{(k_0)}, \\
p(x_1) &= y_1, \quad p'(x_1) = y_1', \quad \dots \quad p^{(k_1)}(x_1) = y_1^{(k_1)}, \\
&\vdots \\
p(x_n) &= y_n, \quad p'(x_n) = y_n', \quad \dots \quad p^{(k_n)}(x_n) = y_n^{(k_n)}.
\end{aligned}$$

In the discussion of that problem, it will be helpful to interpret the k_i as multiplicities of the nodal points x_i.

Hermite Interpolation

The Hermite interpolation problem seeks a polynomial $p \in \mathcal{P}_m$ of degree

$$m = n + k_0 + k_1 + \cdots + k_n.$$

with the property

$$p^{(k)}(x_i) = f^{(k)}(x_i), \quad 0 \le k \le k_i, \quad 0 \le i \le n.$$

For later usage, we introduce nodal points with multiplicities:

$$\underbrace{z_0, z_1, z_2, \ldots, z_{k_0}}_{k_0+1 \text{ copies of } x_0}, \underbrace{z_{k_0+1}, z_{k_0+2}, \ldots, z_{k_0+k_1}}_{k_1+1 \text{ copies of } x_1}, \ldots,$$

Example (Lagrange interpolation)

We search for a polynomial $p(x)$ of degree m such that

$$p(x_0) = f(x_0), \quad p(x_1) = f(x_1), \quad \dots \quad p(x_m) = f(x_m)$$

where $x_0, x_1, \dots, x_m \in \mathbb{R}$ are $m+1$ pairwise distinct points. Here,

$$n = m, \quad k_0 = k_1 = \dots = k_n = 0,$$

Example (Taylor interpolation)

We search for a polynomial p of degree m such that

$$p(x_0) = f(x_0), \quad p'(x_0) = f'(x_0), \quad \dots \quad p^{(m)}(x_0) = f^{(m)}(x_0).$$

for some point $x_0 \in \mathbb{R}$. Here,

$$n = 0, \quad k_0 = m.$$

Example

We search for a polynomial p of degree 3 such that

$$p(1) = 1, \quad p'(1) = 2, \quad p(2) = 2, \quad p'(2) = 3.$$

We express these four constraints as a linear system of equations with invertible system matrix. We define

$$N_0(x) = 1, \quad N_1(x) = (x-1),$$
$$N_2(x) = (x-1)^2, \quad N_3(x) = (x-2)(x-1)^2.$$

Observe

$$N_0'(x) = 0, \quad N_1'(x) = 1,$$
$$N_2'(x) = 2(x-1), \quad N_3'(x) = (x-1)^2 + 2(x-2)(x-1).$$

We will express the solution as

$$p(x) = a_0 N_0(x) + a_1 N_1(x) + a_2 N_2(x) + a_3 N_3(x).$$

Example

The linear system in that form reads

$$\begin{aligned}
p(1) &= a_0 N_0(1) + a_1 N_1(1) + a_2 N_2(1) + a_3 N_3(1) = 1. \\
p'(1) &= a_0 N_0'(1) + a_1 N_1'(1) + a_2 N_2'(1) + a_3 N_3'(1) = 2. \\
p(2) &= a_0 N_0(2) + a_1 N_1(2) + a_2 N_2(2) + a_3 N_3(2) = 2. \\
p'(2) &= a_0 N_0'(2) + a_1 N_1'(2) + a_2 N_2'(2) + a_3 N_3'(2) = 3.
\end{aligned}$$

Evaluating the polynomials and their derivatives, and setting this matrix form reveals that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix}.$$

We easily see that

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = -1, \quad a_3 = 3.$$

We express the Hermite interpolation as a linear system of equations.

Lemma

The Hermite interpolation problem has got a unique solution.

Proof.

The idea is the following: we use a modification of the Newton basis for Lagrange interpolation.

That will provide a basis of $\mathcal{P}_m$ with respect to which the Hermite interpolation problem can be expressed as an invertible triangular system.

Proof.

Consider the system

$$
\begin{aligned}
&p(x_0) = y_0, \quad p'(x_0) = y_0', \quad \ldots \quad p^{(k_0)}(x_0) = y_0^{(k_0)}, \\
&p(x_1) = y_1, \quad p'(x_1) = y_1', \quad \ldots \quad p^{(k_1)}(x_1) = y_1^{(k_1)}, \\
&\quad \vdots \\
&p(x_n) = y_n, \quad p'(x_n) = y_n', \quad \ldots \quad p^{(k_n)}(x_n) = y_n^{(k_n)}.
\end{aligned}
$$

We will construct a basis for $\mathcal{P}_m$, with respect to which the system has a triangular form with non-zero diagonal.

Proof.

We define

$$N_p(x) = \prod_{j=0}^{p-1} (z - z_j), \quad 0 \le p \le m.$$

Notice that these are linearly independent. From here, with $k < k_i$ we see

$$N_{k_0+k_1+\cdots+k_{i-1}+k}(x) = (x - x_i)^k \prod_{j=0}^{i-1} (x - x_j)^{k_j+1}.$$

We already see that $N_{k_0+k_1+\cdots+k_{i-1}+k}$ vanishes at $x_0, x_1, \dots, x_{i-1}$. We generalize this observation to higher derivatives.

We recall the generalized product rule:

$$(f_0 f_1 \cdots f_n)^{(l)} = \sum_{l_0+l_1+\cdots+l_n=l} \frac{n!}{l_0! l_1! \cdots l_n!} f_0^{(l_0)} f_1^{(l_1)} \cdots f_n^{(l_n)}.$$

Furthermore, we know that

$$\Big((x - x_i)^k\Big)^{(l)} = k(k-1)\cdots(k-l)(x - x_i)^{k-l}, \quad l \le k,$$
$$\Big((x - x_i)^k\Big)^{(l)} = 0, \quad l > k.$$

Proof.

Obviously,

$$N^{(l)}_{k_0+k_1+\cdots+k_{i-1}+k}(x) = \sum_{\substack{l_0+l_1+\cdots+l_i=l \\ l_0\le k_0, l_1\le k_1,\ldots,l_i\le k_i \\ l_i \le k}} C(l_0,l_1,\ldots,l_i)(x-x_i)^{k-l_i} \prod_{0\le j<i} (x-x_j)^{k_j+1-l_j}.$$

where $C(l_0,l_1,\ldots,l_n)$ is a positive integer.

- If $l \le k_j$ for some $0 \le j < i$, then in each summand $l_j < k_j + 1$. We conclude that
$$N^{(l)}_{k_0+k_1+\cdots+k_{i-1}+k}(x_j) = 0, \quad 0 \le l \le k_j, \quad 0 \le j < i.$$
- If $l < k$, then in each summand $l_i < k$. We conclude that
$$N_{k_0+k_1+\cdots+k_{i-1}+k}(x_i)^{(l)} = 0, \quad 0 \le l \le k-1.$$
- Finally, if $l = k$, then all summands vanish except the one where $l_i = k$. Thus
$$N^{(k)}_{k_0+k_1+\cdots+k_{i-1}+k}(x_i) = C(0,0,\ldots,0,l_i) \prod_{0\le j<i} (x-x_j)^{k_j+1} \ne 0.$$

Proof.

Abbreviating

$$N(x) = N_{k_0+k_1+\cdots+k_{i-1}+k}(x),$$

we have for $0 \leq i \leq n$ and $0 \leq k \leq k_i$ that

$$\begin{aligned}
N(x_0) = 0, \quad N'(x_0) = 0, \quad \ldots \quad N^{(k_0)}(x_0) = 0, \\
\vdots \\
N(x_{i-1}) = 0, \quad N'(x_{i-1}) = 0, \quad \ldots \quad N^{(k_{i-1})}(x_{i-1}) = 0, \\
N(x_i) = 0, \quad N'(x_i) = 0, \quad \ldots \quad N^{(k-1)}(x_i) = 0, \quad N^{(k)}(x_i) \neq 0.
\end{aligned}$$

The linear problem of Hermite interpolation has got a triangular matrix with non-zero diagonal entries when expressed via Newton polynomials.

The system has got a unique solution. □

Summary

- Lagrange and Taylor interpolation are special cases of Hermite interpolation.
- Possible expression in terms of Newton basis, other bases are possible.
- Error analysis more complicated, not part of this lecture.