SMOOTHED PROJECTIONS OVER WEAKLY LIPSCHITZ DOMAINS

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Abstract. We develop finite element exterior calculus over weakly Lipschitz domains. Specifically, we construct commuting projections from L^p de Rham complexes over weakly Lipschitz domains onto finite element de Rham complexes. These projections satisfy uniform bounds for finite element spaces with bounded polynomial degree over shape-regular families of triangulations. Thus we extend the theory of finite element differential forms to polyhedral domains that are weakly Lipschitz but not strongly Lipschitz. As new mathematical tools, we use the collar theorem in the Lipschitz category, and we show that the degrees of freedom in finite element exterior calculus are flat chains in the sense of geometric measure theory.

1. Introduction

The aim of this article is to contribute to the understanding of finite element methods for partial differential equations over domains of low regularity. For partial differential equations associated to a differential complex, projections that commute with the relevant differential operators are central to the analysis of mixed finite element methods. In particular, *smoothed projections* from Sobolev de Rham complexes to finite element de Rham complexes are used in *finite element exterior calculus* (FEEC) [1, 3]. This was researched for the case that the underlying domain is a Lipschitz domain. In this article, we study finite element exterior calculus more generally when the underlying domain is merely a *weakly Lipschitz domain*. Specifically, we construct and analyze smoothed projections. Thus we enable the abstract Galerkin theory of finite element exterior calculus within that generalized geometric setting.

It is easy to motivate the class of weakly Lipschitz domains in the context of finite element methods. A domain is called weakly Lipschitz if its boundary can be flattened locally by a Lipschitz coordinate transformation. This generalizes the classical notion of (strongly) Lipschitz domains, whose boundaries, by definition, can be written locally as Lipschitz graphs. Although Lipschitz domains are a common choice for the geometric ambient in the theoretical and numerical analysis of partial differential equations, they exclude several practically relevant domains. It is easy to find three-dimensional polyhedral domains that are not Lipschitz domains, such as the "crossed bricks domain" [24, p.39, Figure 3.1]. But as we show

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in this article, every three-dimensional polyhedral domain is still a weakly Lipschitz domain (see Theorem 4.1 for the precise statement).

Moreover, weakly Lipschitz domains have attracted interest in the theory of partial differential equations because basic results in vector calculus, well-known for strongly Lipschitz domains, are still available in this geometric setting [19, 20, 14, 6, 5]. For example, one can show that the differential complex

$$H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \tag{1.1}$$

over a bounded three-dimensional weakly Lipschitz domain Ω satisfies Poincaré-Friedrichs inequalities, and realizes the Betti numbers of the domain on cohomology. Furthermore, a vector field version of a Rellich-type compact embedding theorem is valid, and the scalar and vector Laplacians over Ω have a discrete spectrum. Recasting this in the calculus of differential forms, one can more generally establish the analogous properties for the L^2 de Rham complex

$$H\Lambda^0(\Omega) \xrightarrow{\mathsf{d}} H\Lambda^1(\Omega) \xrightarrow{\mathsf{d}} \cdots \xrightarrow{\mathsf{d}} H\Lambda^n(\Omega) \tag{1.2}$$

over a bounded weakly Lipschitz domain $\Omega \subset \mathbb{R}^n$.

It is therefore of interest to develop finite element analysis over weakly Lipschitz domains. Since the analytical theory is formulated within the calculus of differential forms, we wish to adopt this calculus on the discrete level. Specifically, we use the framework of finite element exterior calculus; our agenda is to extend that framework to numerical analysis on weakly Lipschitz domains. The foundational idea is to mimic the L^2 de Rham complex by a finite element de Rham complex

$$\mathcal{P}\Lambda^0(\mathcal{T}) \xrightarrow{\mathsf{d}} \mathcal{P}\Lambda^1(\mathcal{T}) \xrightarrow{\mathsf{d}} \cdots \xrightarrow{\mathsf{d}} \mathcal{P}\Lambda^n(\mathcal{T}). \tag{1.3}$$

Here, each $\mathcal{P}\Lambda^k(\mathcal{T})$ is a subspace of $H\Lambda^k(\Omega)$ whose members are piecewise polynomial with respect to a triangulation $\mathcal{T}$ of the domain. Arnold, Falk, and Winther [1] have determined specific finite element de Rham complexes that aid the construction and analysis of stable mixed finite element methods.

A central component of finite element exterior calculus are uniformly bounded *smoothed projections*. These are an instance of commuting finite element projection operators, of which various examples are known in the literature [7, 26, 17, 10]. Our main contribution in this article is to devise such a projection when the domain is merely weakly Lipschitz (see Theorem 7.11). As an immediate consequence, the a priori error estimates of finite element exterior calculus are applicable over weakly Lipschitz domains. The following theorem is a condensed version of the main result.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded weakly Lipschitz domain, and let $\mathcal{T}$ be a simplicial triangulation of Ω. Let* (1.3) *be a differential complex of finite element spaces of differential forms as in finite element exterior calculus* [1]. *Then there exist bounded linear projections $\pi^k : L^2\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T}) \subseteq L^2\Lambda^k(\Omega)$ such that*

$$\begin{array}{ccccccc}
H\Lambda^0(\Omega) & \xrightarrow{\mathsf{d}} & H\Lambda^1(\Omega) & \xrightarrow{\mathsf{d}} & \cdots & \xrightarrow{\mathsf{d}} & H\Lambda^n(\Omega) \\
\downarrow{\scriptstyle \pi^0} & & \downarrow{\scriptstyle \pi^1} & & & & \downarrow{\scriptstyle \pi^n} \\
\mathcal{P}\Lambda^0(\mathcal{T}) & \xrightarrow{\mathsf{d}} & \mathcal{P}\Lambda^1(\mathcal{T}) & \xrightarrow{\mathsf{d}} & \cdots & \xrightarrow{\mathsf{d}} & \mathcal{P}\Lambda^n(\mathcal{T})
\end{array} \tag{1.4}$$

is a commuting diagram. Moreover, $\pi^k\omega = \omega$ for $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$. The operator norm of each π^k is uniformly bounded in terms of the maximum polynomial degree of (1.3), *the shape measure of the triangulation, and geometric properties of Ω.*

Let us outline the construction of the smoothed projection and the new tools which we employ in this article. We largely follow ideas in the published literature [1, 9] but introduce significant technical modifications. Given a differential form over the domain, the smoothed projection is composed in several steps.

We first extend the differential form beyond the original domain by reflection along the boundary, using a parametrized tubular neighborhood of the boundary. For strongly Lipschitz domains, such a parametrization can be constructed using the flow along a smooth vector field transversal to the boundary [1, 9], but for weakly Lipschitz domains such a transversal vector field does not necessarily exist. Instead we obtain the desired parametrized tubular neighborhood via a variant of the collaring theorem in Lipschitz topology [22].

Next, a regularizaiton operator smooths the extended differential form. Local control of the smoothing radius by a smoothed mesh size function guarantees uniform bounds for shape-regular families of meshes. This is similar to [9], but we elaborate the details of the construction and make a minor correction; see also Remark 7.12. The smoothed differential form has well-defined degrees of freedom.

We then apply the canonical finite element interpolant to the smoothed differential form. The resulting *smoothed interpolant* commutes with the exterior derivative and satisfies uniform bounds but is not idempotent generally. We can, however, control the interpolation error over the finite element space. If the smoothed interpolant is sufficiently close to the identity over the finite element space, then a commuting and uniformly bounded discrete inverse exists. Following an idea of Schöberl [25], the composition of this discrete inverse with the smoothed interpolant yields the desired smoothed projection.

In order to derive the aforementioned interpolation error estimate over the finite element space, we call on geometric measure theory [12, 27]. The principal motivation in utilizing geometric measure theory is the low regularity of the boundary, which requires new techniques in finite element theory. A key observation, which we believe to be of independent interest, is the identification of the degrees of freedom as *flat chains* in the sense of geometric measure theory. The desired estimate of the interpolation error over the finite element space is proven eventually with distortion estimates on flat chains. Moreover, we identify a non-trivial gap in the corresponding proofs of previous works [1, 9]; see also Remark 7.10. This gives further motivation for our recourse to geometric measure theory.

Most of the literature on commuting projections focuses on the L^2 theory (but see also [8, 11]). We consider differential forms with coefficients in general L^p spaces, following [15]. This article moreover prepares future research on smoothed projections which preserve partial boundary conditions.

The remainder of this work is structured as follows. In Section 2, we introduce weakly Lipschitz domains and a collar theorem. We recapitulate the calculus of differential forms in Section 3. We briefly review triangulations in Section 4. The relevant background in geometric measure theory is given in Section 5. Then we introduce finite element spaces, degrees of freedom, and interpolation operators in Section 6. In Section 7, we finally construct the smoothed projection.

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2. Geometric Setting

We begin by establishing the geometric background. We review the notion of *weakly Lipschitz domains* and prove the existence of a *closed two-sided Lipschitz collar* along the boundaries of such domains. We refer to [22] for further background in the area of Lipschitz topology.

Throughout this article, and unless stated otherwise, we let finite-dimensional real vector spaces $\mathbb{R}^n$ and their subsets be equipped with the canonical Euclidean norms, which we write as $\|\cdot\|$. We let $B_r(U)$ be the closed Euclidean r-neighborhood, $r > 0$, of any set $U \subseteq \mathbb{R}^n$, and we write $B_r(x) := B_r(\{x\})$.

We introduce some basic notions of Lipschitz analysis. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, and let $f : X \to Y$ be a mapping. For a subset $U \subseteq X$, we let the *Lipschitz constant* $\operatorname{Lip}(f, U) \in [0, \infty]$ of f over U be the minimal $L \in [0, \infty]$ that satisfies

$$\forall x, x' \in U : \|f(x) - f(x')\| \leq L\|x - x'\|.$$

We simply write $\operatorname{Lip}(f) := \operatorname{Lip}(f, U)$ if U is understood. We call f *Lipschitz* if $\operatorname{Lip}(f, X) < \infty$. We call f *locally Lipschitz* or *LIP* if for each $x \in X$ there exists a relatively open neighborhood $U \subseteq X$ of x such that $f_{|U} : U \to Y$ is Lipschitz. If f is invertible, then we call f *bi-Lipschitz* if both f and f^{-1} are Lipschitz, and we call f a *lipeomorphism* if both f and f^{-1} are locally Lipschitz. If $f : X \to Y$ is locally Lipschitz and injective such that $f : X \to f(X)$ is a lipeomorphism, then we call f a *LIP embedding*. The composition of Lipschitz mappings is again Lipschitz, and the composition of locally Lipschitz mappings is again locally Lipschitz. If X is compact, then every locally Lipschitz mapping is also Lipschitz.

Let $\Omega \subseteq \mathbb{R}^n$ be open. We call Ω a *weakly Lipschitz domain* if for all $x \in \partial\Omega$ there exist a closed neighborhood U_x of x in $\mathbb{R}^n$ and a bi-Lipschitz mapping $\varphi_x : U_x \to [-1, 1]^n$ such that $\varphi_x(x) = 0$ and

$$\varphi_x(\Omega \cap U_x) = [-1, 1]^{n-1} \times [-1, 0), \tag{2.1a}$$

$$\varphi_x(\partial\Omega \cap U_x) = [-1, 1]^{n-1} \times \{0\}, \tag{2.1b}$$

$$\varphi_x\left(\overline{\Omega}^c \cap U_x\right) = [-1, 1]^{n-1} \times (0, 1]. \tag{2.1c}$$

Note that Ω is a weakly Lipschitz domain if and only if $\overline{\Omega}^c$ is a weakly Lipschitz domain. The closed sets $\{\partial\Omega \cap U_x \mid x \in \partial\Omega\}$ constitute a covering of $\partial\Omega$ and the mappings $\varphi_{x|\partial\Omega\cap U_x} : \partial\Omega \cap U_x \to [-1, 1]^{n-1}$ are bi-Lipschitz.

Remark 2.1. In other words, a weakly Lipschitz domain is a domain whose boundary can be flattened locally by a bi-Lipschitz coordinate transformation. The notion of *weakly Lipschitz domain* contrasts with the classical notion of *Lipschitz domain*, then also called *strongly Lipschitz domain*. A strongly Lipschitz domain is an open subset Ω of $\mathbb{R}^n$ whose boundary $\partial\Omega$ can be written locally as the graph of a Lipschitz function in some orthogonal coordinate system. Strongly Lipschitz domains are weakly Lipschitz domains, but the converse is generally false.

We also note that a different access towards the idea originates from differential topology: a weakly Lipschitz domain is a *locally flat Lipschitz submanifold* of $\mathbb{R}^n$ in the sense of [22]. Weakly Lipschitz domains inside Lipschitz manifolds are defined similarly [14].

Example 2.2. Every bounded domain $\Omega \subset \mathbb{R}^3$ with a finite triangulation is a weakly Lipschitz domain. We will specify and prove this statement in Section 4 after having formally defined triangulations. At this point, we review a concrete and well-known example, namely the *crossed bricks domain* Ω_{CB}, which has already been mentioned in the introduction. Let

$$\begin{aligned}\Omega_{CB} := (-1,1)\times(0,1)\times(0,-1) \cup (0,1)\times(0,-1)\times(-1,1)\\ \cup\,(0,1)\times\{0\}\times(0,-1);\end{aligned} \tag{2.2}$$

see also the left part of Figure 1. The domain Ω_{CB} is not a Lipschitz domain because at the origin it is not possible to write $\partial\Omega_{CB}$ as the graph of a Lipschitz function in any orthogonal coordinate system. Indeed, the contrary would imply the existence of a vector that has positive angle both with the second coordinate vector $(0,1,0)$ and its negative, corresponding to $\partial\Omega_{CB}$ being partly the lower side of the upper brick and the upper side of the lower brick near the origin.

But Ω_{CB} is a weakly Lipschitz domain. This follows from Theorem 4.1 later in this chapter, but it is easy to verify in the particular example of Ω_{CB}. We first observe that near every non-zero $x \in \partial\Omega_{CB}$ we can write $\partial\Omega_{CB}$ as a Lipschitz graph, from which we can easily construct a suitable Lipschitz coordinate chart around x. But this approach does not work at the origin.

It is possible, however, to deform Ω_{CB} into a strongly Lipschitz domain $\widetilde{\Omega}_{CB}$ by a bi-Lipschitz mapping; see the right part of Figure 1. By the definition of strongly Lipschitz domains, we can write $\widetilde{\Omega}_{CB}$ as the hypograph of a Lipschitz function near the origin in an orthogonal coordinate system. From this observation, the existence of a U_0 and $\varphi_0 : U_0 \to [-1,1]$ with $\varphi_0(0) = 0$ and (2.1) is easily deduced.

For technical completeness, we describe the relevant mappings explicitly. We first we define $\varphi_{CB} : \mathbb{R}^3 \to \mathbb{R}^3$ by setting $\varphi(x_1,x_2,x_3) := (x_1,x_2',x_3)$, where

$$x_2' = \begin{cases} x_2 - {}^3\!/\!_4(1-x_2)\min(1,|x_3|) & \text{if } x_2 \in [0,1],\\ x_2 - {}^3\!/\!_4(1+x_2)\min(1,|x_3|) & \text{if } x_2 \in [-1,0],\\ x_2 & \text{if } x_2 \notin [-1,1].\end{cases}$$

This mapping is obviously Lipschitz. Its inverse φ_{CB}^{-1} is easily seen to be Lipschitz too, being given by $\varphi_{CB}^{-1}(x_1,x_2,x_3) = (x_1,x_2',x_3)$ with

$$x_2' = \begin{cases} x_2 + \frac{3}{7}(1-x_2)\min(1,|x_3|) & \text{if } x_2 \in [-{}^3\!/\!_4,1],\\ x_2 + 3(1+x_2)\min(1,|x_3|) & \text{if } x_2 \in [-1,-{}^3\!/\!_4],\\ x_2 & \text{if } x_2 \notin [-1,1].\end{cases}$$

The bi-Lipschitz mapping φ_{CB} transforms Ω_{CB} onto $\widetilde{\Omega}_{CB}$, as displayed in Figure 1. As a next step, we define the vectors

$$e_x := \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \quad e_y := (0,1,0), \quad e_z := \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \tag{2.3}$$

and for fixed $\delta > 0$ to be determined below we define the Minkowski sum

$$V_0 := [-3\delta, 3\delta]\cdot e_x + [-\delta,\delta]\cdot e_y + [-\delta,\delta]\cdot e_z.$$

We write $\overline{\Omega}_{CB}$ as the hypograph of a Lipschitz function χ near the origin in the orthogonal coordinate frame (2.3), with the graph varying in the e_x direction. We introduce $\chi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\chi(y,z) := \begin{cases} -z & \text{if } y \geq 0, \\ -z - \frac{2\sqrt{2}}{3}y & \text{if } -\frac{3}{\sqrt{2}}z \leq y \leq 0, \\ -z + \frac{2\sqrt{2}}{3}y & \text{if } \frac{3}{\sqrt{2}}z \leq y \leq 0, \\ z & \text{otherwise.} \end{cases}$$

By visual inspection of Figure 1 and a moment of reflection we see that for $\delta > 0$ chosen small enough the intersection $V_0 \cap \widetilde{\Omega}_{CB}$ coincides with the set

$$\left\{ xe_x + ye_y + ze_z \;\middle|\; -3\delta \leq x < \chi(y,z),\ (y,z) \in [-\delta,\delta]^2 \right\}.$$

In other words, we have shown that $\widetilde{\Omega}_{CB}$ is a Lipschitz graph in an orthogonal coordinate system near the origin. Next we define $\widetilde{\varphi}_0 : [-1,1]^3 \to V_0$ by

$$\widetilde{\varphi}_0(z,y,x) = x' \cdot e_x + \delta y \cdot e_y + \delta z \cdot e_z,$$

$$\text{where } x' := \begin{cases} \chi(\delta y, \delta z)(1-x) - 3\delta(x+1) & \text{if } x \in [-1,0], \\ \chi(\delta y, \delta z)(1-x) + 3\delta x & \text{if } x \in [0,1]. \end{cases}$$

One can see that $\widetilde{\varphi}_0$ is bi-Lipschitz and maps $[-1,1]^2 \times [-1,0)$ onto $V_0 \cap \widetilde{\Omega}_{CB}$. Now

$$U_0 := \varphi_{CB}^{-1}\widetilde{\varphi}_0\left([-1,1]^3\right), \quad \varphi_0 := \widetilde{\varphi}_0^{-1}\varphi_{CB|U_0} \tag{2.4}$$

is the desired bi-Lipschitz coordinate chart around the origin in which $\partial\Omega_{CB}$ is flattened.

A variant of the crossed bricks domain is displayed in the monograph of Monk [24, Figure 3.1, p.39], and another variant is discussed in [6]. For a generalization of this example, we refer to Example 2.2 in [4].

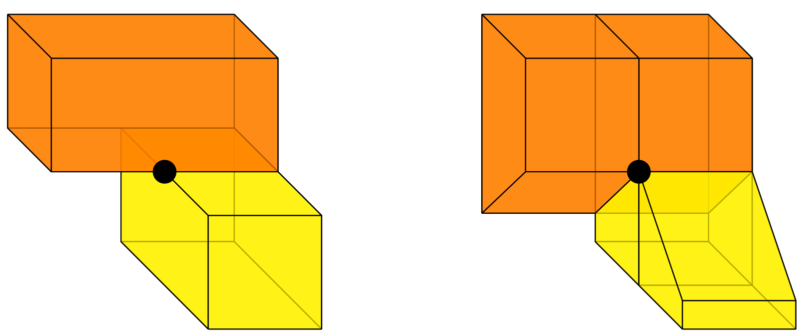

FIGURE 1. Left: polyhedral three-dimensional domain Ω_{CB} that is not the graph of a Lipschitz function at the marked point. The upper brick extends into the x-direction and the lower brick extends into the y-direction. Right: bi-Lipschitz transformation of that domain into a domain $\widetilde{\Omega}_{CB}$ that is strongly Lipschitz domain, as can be seen by visual inspection.

The remainder of this section builds up a key notion of this article. As a motivation, we recall that strongly Lipschitz domains have parametrized tubular neighborhoods, which can be constructed with a transversal vector field near the boundary

[26, 8, 18]. Generalizing this idea, the following theorem shows that weakly Lipschitz domains allow for *two-sided Lipschitz collars*.

Theorem 2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded weakly Lipschitz domain. Then there exists a LIP embedding $\Psi : \partial\Omega \times [-1,1] \to \mathbb{R}^n$ such that $\Psi(x,0) = x$ for $x \in \partial\Omega$ and such that*

$$\Psi\left(\partial\Omega, [-1,0)\right) \subseteq \Omega, \quad \Psi\left(\partial\Omega, (0,1]\right) \subseteq \overline{\Omega}^c. \tag{2.5}$$

Without loss of generality, for every $t \in (0,1)$ the sets $\Omega \setminus \Psi\left(\partial\Omega, [-t,0)\right)$ and $\overline{\Omega} \cup \Psi\left(\partial\Omega, (0,t)\right)$ are weakly Lipschitz domains.

Proof. We first prove a one-sided version of the result. From definitions we deduce that there exist a collection $\{V_i\}_{i\in\mathbb{N}}$ of relatively open subsets of $\partial\Omega$ that constitute a covering of $\partial\Omega$ and a collection $\{\psi_i\}_{i\in\mathbb{N}}$ of LIP embeddings $\psi_i : V_i \times [0,1) \to \overline{\Omega}$ such that for each $i \in \mathbb{N}$ we have $\psi_i(x,0) = x$ for each $x \in \partial\Omega$. It follows that $\{(V_i, \psi_i)\}_{i\in\mathbb{N}}$ is a *local LIP collar* in the sense of Definition 7.2 in [22]. By Theorem 7.4 in [22], and a successive reparametrization, there exists a LIP embedding $\Psi^-(x,t) : \partial\Omega \times [0,1] \to \overline{\Omega}$ such that $\Psi^-(x,0) = x$ for all $x \in \partial\Omega$.

We see that $\overline{\Omega}^c$ is a weakly Lipschitz domain too. Analogous arguments give a LIP embedding $\Psi^+(x,t) : \partial\Omega \times [0,1] \to \Omega^c$ such that $\Psi^+(x,0) = x$ for all $x \in \partial\Omega$. We combine these two LIP embeddings and let

$$\Psi : \partial\Omega \times [-1,1] \to \mathbb{R}^n, \quad (x,t) \mapsto \begin{cases} \Psi^-(x,-t) & \text{if } x \in \partial\Omega,\ t \in [-1,0), \\ x & \text{if } x \in \partial\Omega,\ t = 0, \\ \Psi^+(x,t) & \text{if } x \in \partial\Omega,\ t \in (0,1]. \end{cases}$$

Then Ψ is well-defined, bijective, and (2.5) holds. To prove that Ψ is a LIP embedding, we show the existence of a constant $C > 0$ such that

$$\|\Psi(x_1,t_1) - \Psi(x_2,t_2)\| \leq C\left(\|x_1 - x_2\| + |t_2 - t_1|\right), \tag{2.6}$$

$$\|x_1 - x_2\| + |t_2 - t_1| \leq C\,\|\Psi(x_1,t_1) - \Psi(x_2,t_2)\| \tag{2.7}$$

for all $x_1, x_2 \in \partial\Omega$ and $t_1, t_2 \in [-1,1]$. If t_1 and t_2 are both non-negative or both non-positive, then the both inequalities follow directly from Ψ^+ or Ψ^- being LIP embeddings with a constant $C \geq 1$ that depends only on Ψ^+ and Ψ^-. Hence it remains to consider the case $t_1 < 0 < t_2$. Here, (2.6) follows from

$$\begin{aligned} \|\Psi(x_1,t_1) - \Psi(x_2,t_2)\| &\leq \|\Psi(x_1,t_1) - x_1\| + \|x_1 - x_2\| + \|x_2 - \Psi(x_2,t_2)\| \\ &\leq C|t_1| + C|t_2| + \|x_1 - x_2\| \\ &\leq C\,|t_1 - t_2| + \|x_1 - x_2\| \end{aligned}$$

since Ψ^+ and Ψ^- are LIP embeddings. Furthermore, there exists $z \in \partial\Omega$ on the straight line segment from $\Psi(x_1,t_1)$ to $\Psi(x_2,t_2)$. We then find (2.7) via

$$\begin{aligned} |t_1 - t_2| + \|x_1 - x_2\| &\geq |t_1| + \|x_1 - z\| + \|z - x_2\| + |t_2| \\ &\geq C^{-1}\,\|\Psi(x_1,t_1) - z\| + C^{-1}\,\|z - \Psi(x_2,t_2)\| \\ &= C^{-1}\,\|\Psi(x_1,t_1) - \Psi(x_2,t_2)\|\,, \end{aligned}$$

again using that Ψ^+ and Ψ^- are LIP embeddings. We conclude that Ψ is a LIP embedding. Restricting and reparametrizing Ψ completes the proof. □

Remark 2.4. Our Theorem 2.3 realizes an idea from differential topology in a Lipschitz setting: any locally bi-collared surface is also globally bi-collared. Such results are well-known in the topological or smooth sense, but it seems to be only folklore in the Lipschitz sense. Notably, the result is mentioned in the unpublished preprint [13]. We have provided a proof for formal completeness.

3. Differential forms

In this section we review the calculus of differential forms in a setting of low regularity. Particular attention is given to differential forms with coefficients in L^p spaces and their transformation properties under bi-Lipschitz mappings. We adopt the notion of $W^{p,q}$ differential form of [15], to which we also refer for further details on Lebesgue spaces of differential forms. An elementary introduction to the calculus of differential forms is given in [21].

Let $U \subseteq \mathbb{R}^n$ be an open set. We let $M(U)$ denote the vector space of Lebesgue measurable functions over U up to equivalence almost everywhere. For $k \in \mathbb{Z}$ we let $M\Lambda^k(U)$ be the vector space differential k-forms over U with Lebesgue measurable coefficients. We denote by $\omega \wedge \eta \in M\Lambda^{k+l}(U)$ the exterior product of $\omega \in M\Lambda^k(U)$ and $\eta \in M\Lambda^l(U)$, and we recall that $\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega$.

Let $e_1, \dots, e_n$ be the canonical orthonormal basis of $\mathbb{R}^n$. The constant 1-forms $\mathsf{d}x^1, \dots, \mathsf{d}x^n \in M\Lambda^1(U)$ are uniquely defined by $\mathsf{d}x^i(e_j) = \delta_{ij}$, where $\delta_{ij} \in \{0, 1\}$ denotes the Kronecker delta. In the sequel, we let $\Sigma(k, n)$ denote the set of strictly ascending mappings from $\{1, \dots, k\}$ to $\{1, \dots, n\}$. Note that $\Sigma(0, n) = \{\emptyset\}$. The *basic k-alternators* are the exterior products

$$\mathsf{d}x^\sigma := \mathsf{d}x^{\sigma(1)} \wedge \dots \wedge \mathsf{d}x^{\sigma(k)} \in M\Lambda^k(U), \quad \sigma \in \Sigma(k, n),$$

and $\mathsf{d}x^\emptyset := 1$. The canonical volume n-form $\mathrm{vol}^n \in M\Lambda^n(U)$ is

$$\mathrm{vol}^n := \mathsf{d}x^1 \wedge \ldots \wedge \mathsf{d}x^n.$$

For every $\omega \in M\Lambda^k(U)$ and $\sigma \in \Sigma(k, n)$ we define $\omega_\sigma = \omega(e_{\sigma(1)}, \dots, e_{\sigma(k)}) \in M(U)$ and note that ω can be written as

$$\omega = \sum_{\sigma \in \Sigma(k,n)} \omega_\sigma \mathsf{d}x^\sigma. \tag{3.1}$$

For every n-form $\omega \in M\Lambda^n(U)$ there exists a unique $\omega_{\mathrm{vol}} \in M(U)$ such that $\omega = \omega_{\mathrm{vol}} \mathrm{vol}^n$. We define the integral of $\omega \in M\Lambda^n(U)$ over U as

$$\int_U \omega := \int_U \omega_{\mathrm{vol}} \, \mathrm{d}x \tag{3.2}$$

whenever $\omega_{\mathrm{vol}} \in M(U)$ is integrable. Note that this definition of the integral presumes that $\mathbb{R}^n$ carries the canonical orientation.

For $\omega, \eta \in M\Lambda^k(U)$ we define the pointwise ℓ^2 product $\langle \omega, \eta \rangle \in M(U)$ by

$$\langle \omega, \eta \rangle := \sum_{\sigma \in \Sigma(k,n)} \omega_\sigma \eta_\sigma. \tag{3.3}$$

For $\omega \in M\Lambda^k(U)$ we let $|\omega| = \sqrt{\langle \omega, \omega \rangle} \in M(U)$ be the pointwise ℓ^2 norm. We let $L^p(U)$ denote the Lebesgue space with exponent $p \in [1, \infty]$, and let $L^p\Lambda^k(U)$ denote the Banach space of differential k-forms with coefficients in $L^p(U)$. The topology of $L^p\Lambda^k(U)$ is generated by the norm

$$\|\omega\|_{L^p\Lambda^k(U)} := \left\| \sqrt{\langle \omega, \omega \rangle} \right\|_{L^p(U)}, \quad \omega \in L^p\Lambda^k(U).$$

We let $C\Lambda^k(\overline{U})$ be the Banach space of bounded continuous differential k-forms over $\overline{U}$, equipped with the maximum norm. We let $C^\infty\Lambda^k(U)$ be the space of smooth differential k-forms over U, we let $C^\infty\Lambda^k(\overline{U})$ be the subspace of $C^\infty\Lambda^k(U)$ whose members can be extended smoothly onto $\mathbb{R}^n$, and we let $C_c^\infty\Lambda^k(U)$ be the subspace of $C^\infty\Lambda^k(U)$ whose members have compact support in U.

The exterior derivative $\mathsf{d} : C^\infty\Lambda^k(\overline{U}) \to C^\infty\Lambda^{k+1}(\overline{U})$ is defined by

$$\mathsf{d}\omega = \sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^{n} \left(\partial_i\omega_\sigma \mathsf{d}x^i\right) \wedge \mathsf{d}x^\sigma, \quad \omega \in C^\infty\Lambda^k(\overline{U}), \tag{3.4}$$

where we use the representation (3.1). One can show that d is linear, satisfies the differential property $\mathsf{dd} = 0$, and relates to the exterior product via

$$\mathsf{d}(\omega \wedge \eta) = \mathsf{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathsf{d}\eta, \quad \omega \in C^\infty\Lambda^k(\overline{U}), \quad \eta \in C^\infty\Lambda^l(\overline{U}). \tag{3.5}$$

We are interested in defining the exterior derivative in a weak sense over differential forms of low regularity. If $\omega \in M\Lambda^k(U)$ and $\xi \in M\Lambda^{k+1}(U)$ are locally integrable such that

$$\int_U \xi \wedge \eta = (-1)^{k+1} \int_U \omega \wedge \mathsf{d}\eta, \quad \eta \in C_c^\infty\Lambda^{n-k-1}(U), \tag{3.6}$$

then ξ is the only member of $M\Lambda^{k+1}(U)$ with this property, up to equivalence almost everywhere, and we call $\mathsf{d}\omega := \xi$ the *weak exterior derivative* of ω. Note that $\mathsf{d}\omega$ has vanishing weak exterior derivative, since

$$\int_U \mathsf{d}\omega \wedge \mathsf{d}\eta = (-1)^k \int_U \omega \wedge \mathsf{dd}\eta = 0, \quad \eta \in C_c^\infty\Lambda^{n-k-1}(U). \tag{3.7}$$

Moreover, (3.5) generalizes in the obvious manner to the weak exterior derivative, provided that all expressions are well-defined.

Next we introduce a notion of Sobolev differential forms. For $p, q \in [1, \infty]$, we let $W^{p,q}\Lambda^k(U)$ be the space of differential k-forms in $L^p\Lambda^k(U)$ whose members have a weak exterior derivative in $L^q\Lambda^{k+1}(U)$. We equip $W^{p,q}\Lambda^k(U)$ with the norm

$$\|\omega\|_{W^{p,q}\Lambda^k(U)} := \|\omega\|_{L^p\Lambda^k(U)} + \|\mathsf{d}\omega\|_{L^q\Lambda^{k+1}(U)}, \quad \omega \in W^{p,q}\Lambda^k(U). \tag{3.8}$$

It is obvious that $W^{p,q}\Lambda^k(U)$ is a Banach space. Moreover, as a consequence of [15, Lemma 1.3] we know that $C^\infty\Lambda^k(\overline{U})$ is dense in $W^{p,q}\Lambda^k(U)$ for $p, q \in [1, \infty)$.

Note that $\mathsf{d}W^{p,q}\Lambda^k(U) \subset W^{q,r}\Lambda^{k+1}(U)$ for $p, q, r \in [1, \infty]$ by definition. Hence one may study de Rham complexes of the form

$$\cdots \xrightarrow{\mathsf{d}} W^{p,q}\Lambda^k(U) \xrightarrow{\mathsf{d}} W^{q,r}\Lambda^{k+1}(U) \xrightarrow{\mathsf{d}} \cdots$$

The choice of the Lebesgue exponents determines analytical and algebraic properties of these de Rham complexes. This is not a subject of the present article, but we refer to [16] for corresponding results over smooth manifolds. De Rham complexes of the above form with a Lebesgue exponent p fixed are known as L^p de Rham complexes (e.g. [23]). Two examples of such de Rham complexes are of specific relevance to us.

Example 3.1. The space $H\Lambda^k(U) := W^{2,2}\Lambda^k(U)$, consisting of those L^2 differential k-forms that have a weak exterior derivative with L^2 integrable coefficients, is a Hilbert space whose topology is induced by the scalar product

$$\langle \omega, \eta \rangle_{H\Lambda^k(U)} := \langle \omega, \eta \rangle_{L^2\Lambda^k(U)} + \langle \mathsf{d}^k \omega, \mathsf{d}^k \eta \rangle_{L^2\Lambda^{k+1}(U)}, \quad \omega, \eta \in H\Lambda^k(U).$$

In particular, the norms $\|\cdot\|_{W^{2,2}\Lambda^k(U)}$ and $\|\cdot\|_{H\Lambda^k(U)}$ are equivalent. These spaces constitute the L^2 de Rham complex

$$\cdots \xrightarrow{\mathsf{d}} H\Lambda^k(U) \xrightarrow{\mathsf{d}} H\Lambda^{k+1}(U) \xrightarrow{\mathsf{d}} \cdots$$

which has received considerable attention in global and numerical analysis.

Example 3.2. The space $W^{\infty,\infty}\Lambda^k(U)$ of *flat differential forms* is spanned by those differential forms with essentially bounded coefficients whose exterior derivative has essentially bounded coefficients. These spaces constitute the flat de Rham complex

$$\cdots \xrightarrow{\mathsf{d}} W^{\infty,\infty}\Lambda^k(U) \xrightarrow{\mathsf{d}} W^{\infty,\infty}\Lambda^{k+1}(U) \xrightarrow{\mathsf{d}} \cdots$$

Flat differential forms have been studied extensively in geometric integration theory [27]; see also Theorem 1.5 of [15].

We conclude this section with some basic results on the behavior of differential forms and their integrals under pullback by bi-Lipschitz mappings. For the remainder of this section, we let $U, V \subseteq \mathbb{R}^n$ be open sets, and let $\Phi : U \to V$ be a bi-Lipschitz mapping.

We first gather some facts on the Jacobians of bi-Lipschitz mappings. It follows from Rademacher's theorem [12, Theorem 3.1.6] that the Jacobians

$$\mathrm{D}\,\Phi : U \to \mathbb{R}^{n\times n}, \quad \mathrm{D}\,\Phi^{-1} : V \to \mathbb{R}^{n\times n}$$

exist almost everywhere. One can show that

$$\|\mathrm{D}\,\Phi\|_{L^\infty(U)} \leq \mathrm{Lip}(\Phi, U), \quad \left\|\mathrm{D}\,\Phi^{-1}\right\|_{L^\infty(V)} \leq \mathrm{Lip}(\Phi^{-1}, V). \tag{3.9}$$

According to [12, Lemma 3.2.8], the identities

$$\mathrm{D}\,\Phi^{-1}_{|\Phi(x)} \cdot \mathrm{D}\,\Phi_{|x} = \mathrm{Id}, \quad \mathrm{D}\,\Phi_{|\Phi^{-1}(y)} \cdot \mathrm{D}\,\Phi^{-1}_{|y} = \mathrm{Id} \tag{3.10}$$

hold true almost everywhere over U and V, respectively. In particular, these Jacobians have full rank almost everywhere and by [12, Corollary 4.1.26] the signs of the Jacobians are essentially constant: under the condition that U and V are connected, there exists $o(\Phi) \in \{-1, 1\}$ such that

$$o(\Phi) = \operatorname{sgn}\det \mathrm{D}\,\Phi \tag{3.11}$$

almost everywhere over U. It follows from [12, Theorem 3.2.3] that

$$\int_U (\omega \circ \Phi) \cdot |\det \mathrm{D}\,\Phi| \,\mathrm{d}x = \int_V \omega(y) \,\mathrm{d}y \tag{3.12}$$

for $\omega \in M(V)$ if at least one of the integrals exists.

The pullback $\Phi^*\omega \in M\Lambda^k(U)$ of $\omega \in M\Lambda^k(V)$ under Φ is defined as

$$\Phi^*\omega_{|x}(\nu_1, \dots, \nu_k) := \omega_{|\Phi(x)}(\mathrm{D}\,\Phi_{|x} \cdot \nu_1, \dots, \mathrm{D}\,\Phi_{|x} \cdot \nu_k), \quad \nu_1, \dots, \nu_k \in \mathbb{R}^n, \quad x \in U.$$

By the discussion at the beginning of Section 2 of [15], the algebraic identity

$$\Phi^*(\omega \wedge \eta) = \Phi^*\omega \wedge \eta + (-1)^k \omega \wedge \Phi^*\eta$$

holds for $\omega \in M\Lambda^k(V)$ and $\eta \in M\Lambda^l(V)$. Next we show how the integral of n-forms transforms under pullback by bi-Lipschitz mappings:

Lemma 3.3. *If $\Phi : U \to V$ is a bi-Lipschitz mapping between connected open subsets of $\mathbb{R}^n$, then for every Lebesgue integrable function $\omega \in M(V)$ we have*

$$\int_U \Phi^* (\omega \operatorname{vol}^n) = o(\Phi) \int_V \omega \operatorname{vol}^n . \tag{3.13}$$

Proof. Using (3.11), (3.12), and the definition of the pullback, we find

$$\begin{aligned}\int_U \Phi^* (\omega \operatorname{vol}^n) &= \int_U (\omega \circ \Phi) \cdot \det \mathrm{D}\,\Phi \operatorname{vol}^n = \int_U (\omega \circ \Phi) \cdot \det \mathrm{D}\,\Phi \,\mathrm{d}x \\ &= \int_U (\omega \circ \Phi) \cdot \operatorname{sgn} \det \mathrm{D}\,\Phi \cdot |\det \mathrm{D}\,\Phi| \,\mathrm{d}x \\ &= o(\Phi) \int_U (\omega \circ \Phi) \cdot |\det \mathrm{D}\,\Phi| \,\mathrm{d}x = o(\Phi) \int_V \omega \,\mathrm{d}x = o(\Phi) \int_V \omega \operatorname{vol}^n .\end{aligned}$$

This shows the desired identity. □

It can be shown that the pullback under bi-Lipschitz mappings commutes with the exterior derivative and preserves the L^p and $W^{p,q}$ classes of differential forms.

Lemma 3.4 (Theorem 2.2 of [15]). *Let $\Phi : U \to V$ be a bi-Lipschitz mapping between open subsets of $\mathbb{R}^n$. If $p \in [1, \infty]$ and $\omega \in L^p\Lambda^k(V)$, then $\Phi^*\omega \in L^p\Lambda^k(U)$. If $p, q \in [1, \infty]$ and $\omega \in W^{p,q}\Lambda^k(V)$, then $\Phi^*\omega \in W^{p,q}\Lambda^k(U)$ and $\Phi^*\mathsf{d}\omega = \mathsf{d}\Phi^*\omega$.*

We refine the preceding statement and give an explicit estimate for the operator norm of the pullback operation. Here and in the sequel, $n/\infty = 0$ for $n \in \mathbb{N}$.

Theorem 3.5. *Let $U, V \subseteq \mathbb{R}^n$ be open sets and let $\Phi : U \to V$ be a bi-Lipschitz mapping. For every $p \in [1, \infty]$ and every $\omega \in L^p\Lambda^k(U)$ we then have*

$$\begin{aligned}\|\Phi^*\omega\|_{L^p\Lambda^k(U)} &\leq \|\mathrm{D}\,\Phi\|^k_{L^\infty(U)} \|\det \mathrm{D}\,\Phi^{-1}\|^{\frac{1}{p}}_{L^\infty(V)} \|\omega\|_{L^p\Lambda^k(V)} \\ &\leq \|\mathrm{D}\,\Phi\|^k_{L^\infty(U)} \|\mathrm{D}\,\Phi^{-1}\|^{\frac{n}{p}}_{L^\infty(V)} \|\omega\|_{L^p\Lambda^k(V)}\end{aligned} \tag{3.14}$$

Proof. Let $\Phi : U \to V$ and $p \in [1, \infty]$ be as in the statement of the theorem, and let $u \in L^p\Lambda^k(U)$. For almost every $x \in U$ we observe

$$|\Phi^*\omega|_{|x} \leq \left\|\mathrm{D}\,\Phi_{|x}\right\|^k \sqrt{\sum_{\sigma \in \Sigma(k,n)} \left(\omega_{\sigma|\Phi(x)}\right)^2} = \left\|\mathrm{D}\,\Phi_{|x}\right\|^k |\omega|_{|\Phi(x)} .$$

From this we easily get

$$\|\Phi^*\omega\|_{L^p\Lambda^k(U)} \leq \|\mathrm{D}\,\Phi\|^k_{L^\infty(U)} \left\||\omega| \circ \Phi\right\|_{L^p(U)}.$$

The desired statement follows trivially if $p = \infty$, and (3.12) gives

$$\begin{aligned}\int_U (|\omega| \circ \Phi)^p \,\mathrm{d}x &\leq \left\|\det \mathrm{D}\,\Phi^{-1}\right\|_{L^\infty(V)} \int_U (|\omega| \circ \Phi)^p \cdot |\det \mathrm{D}\,\Phi| \,\mathrm{d}x \\ &\leq \left\|\det \mathrm{D}\,\Phi^{-1}\right\|_{L^\infty(V)} \int_{\Phi(U)} |\omega|^p \,\mathrm{d}x\end{aligned}$$

if $p \in [1, \infty)$. This shows the first estimate of (3.14). The second estimate in (3.14) follows by Hadamard's inequality, which estimates the determinant of a matrix by the product of the norms of its columns. □

4. Triangulations

In this section we review simplicial triangulations of domains and related notions, most of which is standard in the literature. We assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded open set such that $\overline{\Omega}$ is a topological manifold with boundary and Ω is its interior. A *finite triangulation* of $\overline{\Omega}$ is a finite set $\mathcal{T}$ of closed simplices such that the union of the elements of $\mathcal{T}$ equals $\overline{\Omega}$, such that for any $T \in \mathcal{T}$ and any subsimplex $S \subseteq T$ we have $S \in \mathcal{T}$, and such that for all $T, T' \in \mathcal{T}$ the set $T \cap T'$ is either empty or a common subsimplex of both T and T'. We write

$$\Delta(T) := \{ \, S \in \mathcal{T} \mid S \subseteq T \, \}, \quad \mathcal{T}(T) := \{ \, S \in \mathcal{T} \mid S \cap T \neq \emptyset \, \}.$$

With some abuse of notation, we let $\mathcal{T}(T)$ also denote the closed set that is the union of the simplices of $\mathcal{T}$ adjacent to T. We write $\mathcal{T}^m$ for the set of m-dimensional simplices in $\mathcal{T}$.

Having formally introduced triangulations, we make precise and prove the introduction's claim that all polyhedral domains in $\mathbb{R}^3$ are weakly Lipschitz domains.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^3$ be an open set. Assume that $\overline{\Omega}$ is a topological submanifold of $\mathbb{R}^3$ with boundary whose interior submanifold equals Ω. If there exists a finite triangulation $\mathcal{T}$ of Ω, then Ω is a weakly Lipschitz domain.*

Proof. Let $x \in \partial\Omega$ and $\mathcal{T}$ be a finite triangulation of $\overline{\Omega}$. We seek a compact neighborhood $U_x \subseteq \mathbb{R}^3$ of x and a bi-Lipschitz mapping $\varphi_x : U_x \to [-1,1]^3$ such that $\varphi_x(x) = 0$ and (2.1) holds.

If x is not a vertex of $\mathcal{T}$, then x is either contained in the interior of a boundary triangle of $\mathcal{T}$, or in the interior of an edge between two adjacent boundary triangles of $\mathcal{T}$. In both cases, we may choose $U_x := B_r(x)$ for $r > 0$ small enough, and $\varphi_x : U_x \to [-1,1]^3$ is easily constructed.

It remains to consider the case $x \in \mathcal{T}^0$. Let $r > 0$ be so small that $B_r(x)$ intersects $T \in \mathcal{T}^3$ if and only if $x \in T$, so $\partial B_r(x) \cap \partial\Omega$ is a simple closed curve in $\partial B_r(x)$. Hence $\partial B_r(x) \cap \partial\Omega$ is *locally flat* in the sense of [22, p.100]. By the Schoenflies theorem in the Lipschitz category (see Theorem 7.8 of [22]), there exists a bi-Lipschitz mapping

$$\varphi_x^0 : \partial B_r(x) \to \partial B_1(0) \subset \mathbb{R}^3$$

which maps $\partial B_r(x) \cap \partial\Omega$ onto $\partial B_1(0) \cap \{x \in \mathbb{R}^3 \mid x_3 = 0\}$. By radial continuation, we extend this to a bi-Lipschitz mapping

$$\varphi_x^I : B_r(x) \to B_1(0) \subset \mathbb{R}^3$$

which maps $B_r(x) \cap \Omega$ onto $B_1(0) \cap \{x \in \mathbb{R}^3 \mid x_3 < 0\}$ and which satisfies $\varphi_x^I(x) = 0$. Moreover, there exists a bi-Lipschitz mapping

$$\varphi^{II} : B_1(0) \to [-1,1]^3$$

which maps $B_1(0) \cap \{x \in \mathbb{R}^3 \mid x_3 < 0\}$ onto $[-1,1]^2 \times [-1,0)$ and satisfies $\varphi^{II}(0) = 0$. Specifically, we may set $\varphi^{II}(0) = 0$ and

$$\varphi^{II}(y) := \|y\|_{\ell^2}^{-1} \|y\|_{\ell^\infty} y, \quad y \in B_1(0) \setminus \{0\}.$$

Since all norms on $\mathbb{R}^3$ are equivalent, φ^{II} is a bi-Lipschitz mapping from the unit ball in the Euclidean norm to the unit ball in the maximum norm. The theorem follows with $U_x := B_r(x)$ and $\varphi_x := \varphi^{II} \varphi_x^I$. $\square$

Remark 4.2. The class of weakly Lipschitz domains is large but still excludes several domains that are common in finite element literature. For example, the slit domain $\Omega_S = (-1,1)^2 \backslash [0,1) \times \{0\}$ is not a weakly Lipschitz domain. Note that Ω_S is not the interior of $\overline{\Omega}_S$, so Theorem 4.1 does not apply. Defining a smoothed projection over such a domain remains for future research. Another example is a cube dissected into two domains by a plane through the origin, which we exclude for analogous reasons. In the latter example, however, we may still apply the results of this article over each subdomain separately.

The remainder of this section is devoted to notions of regularity of triangulations. Let us fix a finite triangulation $\mathcal{T}$ of $\overline{\Omega}$. When $T \in \mathcal{T}^m$ is any simplex of the triangulation, then we write $h_T = \operatorname{diam}(T)$ for the diameter of T, and $|T| = \operatorname{vol}^m(T)$ for the m-dimensional volume of T. If $V \in \mathcal{T}^0$, then $|V| = 1$, and h_V is defined, by convention, as the average length of all n-simplices of $\mathcal{T}$ that are adjacent to V.

We define the *shape constant* of $\mathcal{T}$ as the minimal $C_{\mathrm{mesh}} > 0$ that satisfies

$$\forall T \in \mathcal{T}^n : h_T^n \leq C_{\mathrm{mesh}} |T|, \tag{4.1}$$

$$\forall T \in \mathcal{T}, S \in \mathcal{T}(T) : h_T \leq C_{\mathrm{mesh}} h_S. \tag{4.2}$$

Intuitively, (4.1) describes a bound on the flatness of the simplices, while (4.2) describes that the diameter of adjacent simplices are comparable. In applications, we consider families of triangulations, such as generated by successive uniform refinement or newest vertex bisection, whose shape constants are uniformly bounded.

We can bound some important quantities in terms of C_{mesh} and the geometric ambient. There exists a constant $C_{\mathrm{N}} > 0$, depending only on C_{mesh} and the ambient dimension n, such that

$$\forall T \in \mathcal{T} : |\mathcal{T}(T)| \leq C_{\mathrm{N}}. \tag{4.3}$$

This bounds the numbers of neighbors of any simplex. Furthermore, there exists a constant $\epsilon_h > 0$, depending only on C_{mesh} and Ω, such that

$$\forall T \in \mathcal{T} : B_{\epsilon_h h_T}(T) \cap \overline{\Omega} \subseteq \mathcal{T}(T). \tag{4.4}$$

In the sequel, we use affine transformations to a reference simplex. Let

$$\Delta^n = \operatorname{convex}\{0, e_1, \dots, e_n\} \subseteq \mathbb{R}^n$$

be the n-dimensional reference simplex. For each n-simplex $T \in \mathcal{T}^n$ of the triangulation, we fix an affine transformation $\varphi_T(x) = M_T x + b_T$ where $b_T \in \mathbb{R}^n$ and $M_T \in \mathbb{R}^{n \times n}$ are such that $\varphi_T(\Delta^n) = T$. Each matrix M_T is invertible, and

$$\|M_T\| \leq c_{\mathrm{M}} h_T, \quad \|M_T^{-1}\| \leq C_{\mathrm{M}} h_T^{-1} \tag{4.5}$$

for constants $c_{\mathrm{M}}, C_{\mathrm{M}} > 0$ that depend only on C_{mesh} and n.

5. Elements of Geometric Measure Theory

This section gives an outline of relevant ideas from geometric measure theory, for which we use Whitney's monograph [27] as our main reference. Our motivation for studying geometric measure theory lies in proving Theorem 7.9 later in this article. The key observation is that finite element differential forms are flat differential forms, and that the degrees of freedom are flat chains (see Lemma 5.2). This allows us to estimate Lipschitz deformations of degrees of freedom (Lemma 5.4), which is of critical importance in the construction of the smoothed projection.

We begin with basic notions of chains and cochains in geometric measure theory, which can be found in Sections 1-3 of Chapter V in [27]. Throughout this section, we fix for each simplex $S \subseteq \mathbb{R}^n$ an orientation. We may identify each positively oriented simplex S with the indicator function $\chi_S : \mathbb{R}^n \to \mathbb{R}$. Let $k \in \mathbb{Z}$. To each finite formal sum $\sum_i a_i S_i$ of (oriented) k-simplices S_i with real coefficients a_i we may associate the function $\sum_i a_i \chi_{S_i}$. We call two such finite formal sums $\sum_i a_i S_i$ and $\sum_j b_j T_j$ equivalent, written $\sum_i a_i S_i \sim \sum_j b_j T_j$, if the associated functions $\sum_i a_i \chi_{S_i}$ and $\sum_j b_j \chi_{T_j}$ agree almost everywhere with respect to the k-dimensional Hausdorff measure. The space $\mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n)$ of polyhedral k-chains in $\mathbb{R}^n$ is the vector space of finite formal sums of positively oriented k-simplices with the equivalence relation factored out. If $S \in \mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n)$, then we write $S \sim \sum_i a_i S_i$ if the latter formal sum represents S. We may identify a polyhedral k-chain $S \sim \sum_i a_i S_i$ in $\mathbb{R}^n$ with the function $\chi_S = \sum_i a_i \chi_{S_i}$ whenever convenient.

The boundary ∂S of a positively oriented k-simplex $S \subseteq \mathbb{R}^n$ is defined as

$$\partial S = \sum_{F \in \Delta(S)^{k-1}} F, \tag{5.1}$$

where each $F \in \Delta(S)^{k-1}$ carries the orientation induced by S. We define a linear operator on the finite formal sums of positively oriented k-simplices by linear extension: $\partial \sum_i a_i S_i = \sum_i a_i \partial S_i$. Furthermore, it is apparent that this operation preserves the equivalence relation. The boundary operator (5.1) gives rise to a linear mapping $\partial : \mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n) \to \mathcal{C}_{k-1}^{\mathrm{pol}}(\mathbb{R}^n)$ that satisfies $\partial\partial = 0$.

The *mass* $|S|_k$ of a polyhedral k-chain S in $\mathbb{R}^n$ is defined as the L^1 norm of the associated function χ_S with respect to the k-dimensional Hausdorff measure.[1] Hence, if $S \sim \sum_i a_i S_i$ with the simplices S_i being essentially disjoint with respect to the k-dimensional Hausdorff measure, then

$$|S|_k = \sum_i |a_i| \operatorname{vol}^k(S_i).$$

It is easy to see that $|\cdot|_k$ is a norm on the polyhedral chains, called *mass norm*. We write $\mathcal{C}_k^{\mathrm{mass}}(\mathbb{R}^n)$ for the Banach space that results from taking the completion of the polyhedral chains with respect to the mass norm.

The *flat norm* $\|S\|_{k,\flat}$ of a polyhedral k-chain $S \in \mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n)$ is defined as

$$\|S\|_{k,\flat} := \inf_{Q \in \mathcal{C}_{k+1}^{\mathrm{pol}}(\mathbb{R}^n)} \Big(|S - \partial Q|_k + |Q|_{k+1} \Big). \tag{5.2}$$

As the name already suggest, one can show that $\|\cdot\|_{k,\flat}$ is a norm on the polyhedral chains. The Banach space $\mathcal{C}_k^{\flat}(\mathbb{R}^n)$ is defined as the completion of $\mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n)$ with respect to the flat norm. It is apparent from the definition that

$$\|S\|_{k,\flat} \leq |S|_k, \quad S \in \mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n).$$

In particular, $\mathcal{C}_k^{\mathrm{mass}}(\mathbb{R}^n)$ is densely embedded in $\mathcal{C}_k^{\flat}(\mathbb{R}^n)$.

The boundary operator is bounded with respect to the flat norm. To see this, let $S \in \mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n)$, let $\epsilon > 0$, and let $Q \in \mathcal{C}_{k+1}^{\mathrm{pol}}(\mathbb{R}^n)$ such that $|S - \partial Q|_k + |Q|_{k+1} \leq$

[1] We assume the convention that the k-dimensional Hausdorff volume of a k-simplex S equals its k-dimensional volume $\operatorname{vol}^k(S)$.

$\|S\|_{k,\flat} + \epsilon$. We then observe that

$$\|\partial S\|_{k-1,\flat} \leq |\partial S - \partial(S - \partial Q)|_{k-1} + |S - \partial Q|_k = |S - \partial Q|_k \leq \|S\|_{k,\flat} + \epsilon.$$

Taking ϵ to zero gives $\|\partial S\|_{k-1,\flat} \leq \|S\|_{k,\flat}$. Using the density of $\mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n)$ in $\mathcal{C}_k^{\flat}(\mathbb{R}^n)$, we eventually verify that

$$\|\partial \alpha\|_{k-1,\flat} \leq \|\alpha\|_{k,\flat}, \quad \alpha \in \mathcal{C}_k^{\flat}(\mathbb{R}^n).$$

We remark that the boundary operator is generally not bounded with respect to the mass norm. This can be seen by shrinking a single simplex: the surface measure scales differently than the volume.

Remark 5.1. The space $\mathcal{C}_k^{\mathrm{mass}}(\mathbb{R}^n)$ is a closed subspace of the Banach space of functions over $\mathbb{R}^n$ integrable with respect to the k-dimensional Hausdorff measure. The members of $\mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n)$ play a similar role as the simple functions in the theory of the Lebesgue measure. Every polyhedral k-chain can be represented as the finite linear combination of k-simplices that are essentially disjoint with respect to the k-dimensional Hausdorff measure.

The Banach space $\mathcal{C}_k^{\flat}(\mathbb{R}^n)$ can be motivated by the following example: for $r \in (0,1)$, consider the two opposing longer sides of the rectangle $[0,r] \times [0,1]$. The mass norm of these two edges is 2 regardless of r, but their flat norm equals r, corresponding to (areal) mass of the original rectangle. In this sense, the flat norm takes into account the distance between simplices.

The chains in the space $\mathcal{C}_k^{\mathrm{mass}}(\mathbb{R}^n)$ are the most important ones in this chapter. We discuss the space $\mathcal{C}_k^{\flat}(\mathbb{R}^n)$ to utilize some technical tools in geometric measure theory that are stated for flat chains in the literature.

The Banach space $\mathcal{C}_k^{\flat}(\mathbb{R}^n)$ of flat chains has a dual space, which is called the Banach space of *flat cochains*. The space of flat cochains can be represented by a class of differential forms: to every cochain we associate a differential form such that evaluating the cochain on a simplex is equal to integrating the associated differential form over that simplex. This is another instance of a recurrent idea throughout differential geometry. Specifically, the space of flat cochains can be represented by the space of flat differential forms. Flat forms were studied in Whitney's monograph [27], there mainly as representations of flat cochains, and in functional analysis (see [15]). For the following facts, we refer to Section 2 of [15] and Chapters IX and X of Whitney's monograph [27].

Flat differential forms have well-defined traces on simplices. More precisely, for each m-simplex $S \subset \mathbb{R}^n$ there exists a bounded linear mapping

$$\mathrm{tr}_S : W^{\infty,\infty}\Lambda^k(\mathbb{R}^n) \to W^{\infty,\infty}\Lambda^k(S),$$

which extends the trace of smooth forms. In particular, for $\omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)$ the trace $\mathrm{tr}_S\,\omega$ depends only on the values of ω near S. We write

$$\int_S \omega := \int_S \mathrm{tr}_S\,\omega \tag{5.3}$$

for the integral of $\omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)$ over a k-simplex S. By linearity, (5.3) gives a bilinear pairing between $\mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n)$ and $W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)$. Via the density of the polyhedral k-chains in $C_k^{\mathrm{mass}}(\mathbb{R}^n)$ this furthermore induces a bilinear pairing between

$\mathcal{C}_k^{\mathrm{mass}}(\mathbb{R}^n)$ and $W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)$. We have

$$\left|\int_S \omega\right| \leq |S|_k \|\omega\|_{W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)}, \quad S \in \mathcal{C}_k^{\mathrm{mass}}(\mathbb{R}^n), \quad \omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n). \tag{5.4}$$

This pairing furthermore extends to flat chains. We have

$$\left|\int_\alpha \omega\right| \leq \|\alpha\|_{k,\flat} \|\omega\|_{W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)}, \quad \alpha \in \mathcal{C}_k^{\flat}(\mathbb{R}^n), \quad \omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n). \tag{5.5}$$

The exterior derivative between spaces of flat forms is dual to the boundary operator between spaces of flat chains (see Paragraph 12 of Chapter IX of [27]), and consequently we have

$$\int_{\partial\alpha} \omega = \int_\alpha \mathsf{d}\omega, \quad \alpha \in \mathcal{C}_k^{\flat}(\mathbb{R}^n), \quad \omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n), \tag{5.6}$$

as a generalized Stokes' theorem.

Many results in geometric measure theory are invariant under Lipschitz mappings. We recall some basic facts about pushforwards of chains and pullbacks of differential forms along Lipschitz mappings. Here we refer to Paragraph 7 in Chapter X of Whitney's monograph [27].

Let $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz mapping. Then there exists a mapping

$$\varphi_* : \mathcal{C}_k^{\flat}(\mathbb{R}^m) \to \mathcal{C}_k^{\flat}(\mathbb{R}^n), \tag{5.7}$$

called the *pushforward* along φ, which commutes with the boundary operator,

$$\partial\varphi_*\alpha = \varphi_*\partial\alpha, \quad \alpha \in \mathcal{C}_k^{\flat}(\mathbb{R}^m), \tag{5.8}$$

and which satisfies the norm estimates

$$\|\varphi_*\alpha\|_{k,\flat} \leq \max\left\{\mathrm{Lip}(\varphi,\mathbb{R}^m)^k, \mathrm{Lip}(\varphi,\mathbb{R}^m)^{k+1}\right\} \|\alpha\|_{k,\flat}, \quad \alpha \in \mathcal{C}_k^{\flat}(\mathbb{R}^m), \tag{5.9}$$

$$|\varphi_* S|_k \leq \mathrm{Lip}(\varphi,\mathbb{R}^m)^k |S|_k, \quad S \in \mathcal{C}_k^{\mathrm{mass}}(\mathbb{R}^m). \tag{5.10}$$

The pushforward of chains is dual to the pullback of differential forms. We recall that the latter is a mapping

$$\varphi^* : W^{\infty,\infty}\Lambda^k(\mathbb{R}^n) \to W^{\infty,\infty}\Lambda^k(\mathbb{R}^m) \tag{5.11}$$

which commutes with the exterior derivative,

$$\mathsf{d}\varphi^*\omega = \varphi^*\mathsf{d}\omega, \quad \omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n), \tag{5.12}$$

and satisfies the norm estimate

$$\|\varphi^*\omega\|_{L^\infty\Lambda^k(\mathbb{R}^m)} \leq \mathrm{Lip}(\varphi,\mathbb{R}^n)^k \|\omega\|_{L^\infty\Lambda^k(\mathbb{R}^n)}, \quad \omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n). \tag{5.13}$$

The pushforward and the pullback are related by the identity

$$\int_{\varphi_*\alpha} \omega = \int_\alpha \varphi^*\omega, \quad \omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n), \quad \alpha \in \mathcal{C}_k^{\flat}(\mathbb{R}^m). \tag{5.14}$$

Lastly, if $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ and $\psi : \mathbb{R}^l \to \mathbb{R}^m$ are Lipschitz mappings, then $\varphi\psi : \mathbb{R}^l \to \mathbb{R}^n$ is a Lipschitz mapping, and we have $(\varphi\psi)_* = \varphi_*\psi_*$ and $(\varphi\psi)^* = \psi^*\varphi^*$ over the corresponding spaces of chains and differential forms, respectively.

Having outlined basic concepts of geometric measure theory, we provide a new result which makes these notions interesting for the theory of finite element methods: the degrees of freedom in finite element exterior calculus are flat chains.

Lemma 5.2. *Let $F \subset \mathbb{R}^n$ be a closed oriented m-simplex and $\eta \in C^\infty\Lambda^{m-k}(F)$. Then there exists $\alpha(F,\eta) \in C_k^\flat(\mathbb{R}^n)$ such that for all $\omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)$ we have*

$$\int_F \operatorname{tr}_F \omega \wedge \eta = \int_{\alpha(F,\eta)} \omega. \tag{5.15}$$

Moreover, $\alpha(F,\eta) \in \mathcal{C}_k^{\mathrm{mass}}(\mathbb{R}^n)$ and $\partial\alpha(F,\eta) \in \mathcal{C}_{k-1}^{\mathrm{mass}}(\mathbb{R}^n)$.

Proof. We first assume that $\dim F = n$ and F is positively oriented. We use Theorem 15A of [27, Chapter IX] to deduce the existence of $\alpha(F,\eta) \in C_k^\flat(\mathbb{R}^n)$ such that

$$\int_F \operatorname{tr}_F \omega \wedge \eta = \int_{\alpha(F,\eta)} \omega, \quad \omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n),$$

and such that $|\alpha(F,\eta)|_k = \|\eta\|_{L^1\Lambda^{m-k}(F)}$. In particular, $\alpha(F,\eta) \in \mathcal{C}_k^{\mathrm{mass}}(\mathbb{R}^n)$.

Now assume that $\dim F = m \leq n$. There exist a positively oriented simplex $F_0 \subseteq \mathbb{R}^m$ and an isometric inclusion $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ which maps F_0 onto F. Recall that the pullback of a flat form along a Lipschitz mapping is well-defined. For $\omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)$ we have

$$\begin{aligned}\int_F \operatorname{tr}_F \omega \wedge \eta = \int_{\varphi_* F_0} \operatorname{tr}_F \omega \wedge \eta &= \int_{F_0} \varphi^* \operatorname{tr}_F \omega \wedge \varphi^*\eta \\ &= \int_{\alpha(F_0,\varphi^*\eta)} \varphi^* \operatorname{tr}_F \omega = \int_{\varphi_*\alpha(F_0,\varphi^*\eta)} \omega.\end{aligned}$$

Thus we may choose $\alpha(F,\eta) = \varphi_*\alpha(F_0,\varphi^*\eta) \in \mathcal{C}_k^{\mathrm{mass}}(\mathbb{R}^n)$. It remains to show that $\partial_{k-1}\alpha(F,\eta) \in \mathcal{C}_{k-1}^{\mathrm{mass}}(\mathbb{R}^n)$. For $\omega \in W^{\infty,\infty}\Lambda^{k-1}(\mathbb{R}^n)$, we derive

$$\begin{aligned}\int_{\partial\alpha(F,\eta)} \omega = \int_{\alpha(F,\eta)} \mathsf{d}\omega &= \int_F \operatorname{tr}_F \mathsf{d}\omega \wedge \eta \\ &= (-1)^k \int_F \operatorname{tr}_F \omega \wedge \mathsf{d}\eta + \sum_{f \in \Delta(F)^{m-1}} \int_f \operatorname{tr}_f \omega \wedge \operatorname{tr}_{F,f} \eta \\ &= (-1)^k \int_{\alpha(F,\mathsf{d}\eta)} \omega + \sum_{f \in \Delta(F)^{m-1}} \int_{\alpha(f,\operatorname{tr}_{F,f}\eta)} \omega.\end{aligned}$$

Here, $\operatorname{tr}_{F,f}\eta$ denotes the trace of $\eta \in C^\infty(F)$ onto a subsimplex $f \in \Delta(F)^{m-1}$, and each such f is assumed to carry the orientation induced by F. Moreover, we have used the generalized Stokes' theorem (5.6). We conclude that the action of the flat chain $\partial\alpha(F,\eta)$ on $W^{\infty,\infty}\Lambda^{k-1}(\mathbb{R}^n)$ can be represented as the finite sum of integrals against smooth differential forms over simplices. Hence $\partial\alpha \in \mathcal{C}_{k-1}^{\mathrm{mass}}(\mathbb{R}^n)$ by the previous observations. The proof is complete. □

Remark 5.3. In the next section we review how the degrees of freedom in finite element exterior calculus can be described in terms of integrals over simplices weighted against polynomial differential forms. Hence Lemma 5.2 can be applied to identify the degrees of freedom with flat chains.

We finish this excursion into geometric measure theory with an estimate on the deformation of flat chains by Lipschitz mappings. This result is applied later in this article and provides the rationale for considering geometric measure theory.

Lemma 5.4. *Let $F \subseteq \mathbb{R}^n$ be an m-simplex and $\eta \in C^\infty\Lambda^{m-k}(F)$. Let $\alpha(F,\eta) \in \mathcal{C}^\flat_k(\mathbb{R}^n)$ be the associated flat chain in the manner of Lemma 5.2. Let $r > 0$ be fixed and let $\varphi : \mathring{B}_{2r}(F) \to \mathring{B}_{3r}(F)$ be a Lipschitz mapping that maps $\mathring{B}_r(F)$ into $\mathring{B}_{2r}(F)$. Writing $\mathfrak{L} := \max\{\operatorname{Lip}(\varphi, B_{2r}(F)), 1\}$, we then have*

$$\|\varphi_*\alpha - \alpha\|_{k,\flat} \leq \|\varphi - \mathrm{Id}\|_{L^\infty(B_{2r}(F),\mathbb{R}^n)} \left(\mathfrak{L}^k |\alpha|_k + \mathfrak{L}^{k-1} |\partial\alpha|_{k-1} \right). \tag{5.16}$$

Proof. To prove this result, we gather several additional notions of Whitney's monograph. For any open set $U \subseteq \mathbb{R}^n$, a polyhedral chain $S \sim \sum_i a_i S_i \in \mathcal{C}^{\mathrm{pol}}_k(\mathbb{R}^n)$ is *in* U if all S_i are contained in U, and S is *of* U if there exists an open set $V \subseteq \mathbb{R}^n$ compactly contained in U such that S is a chain in V (see [27]).

The support of a flat chain $\alpha \in \mathcal{C}^\flat_k(\mathbb{R}^n)$ is the set of all points $x \in \mathbb{R}^n$ such that for all $\epsilon > 0$ there exists $\omega \in C^\infty\Lambda^k(\mathbb{R}^n)$ with support in $B_\epsilon(x)$ such that $\int_\alpha S \neq 0$. It follows from Definition (1) in Section I.13 of [27, p.52] and the discussion in Section V.10 of [27] up to Theorem V.10A that our definition of support agrees with the definition of support in [27, Section VII.3]

Having established these additional notions, the claim follows by Theorem 13A in Chapter X in [27] together with Equation VIII.1.(7) in [27, p.233]. □

6. Finite Element Spaces, Degrees of Freedom, and Interpolation

In this section we outline the discretization theory of finite element exterior calculus. We summarize basic facts on the finite element spaces and their spaces of degrees of freedom. The most important construction is the canonical finite element interpolant I^k. Moreover we consider several inverse inequalities. The reader is assumed to be familiar with the background in [2] and [1, Section 3–5]. We outline this background and additionally apply geometric measure theory in the perspective of the preceding section.

For the duration of this section, we fix a bounded weakly Lipschitz domain $\Omega \subset \mathbb{R}^n$ and a finite triangulation $\mathcal{T}$ of $\overline{\Omega}$.

The essential idea is to consider a differential complex of finite element spaces that mimics the de Rham complex on a discrete level. The finite element spaces are finite-dimensional spaces of piecewise polynomial differential forms.

Let $T \in \mathcal{T}^n$ be an n-simplex, and let $r, k \in \mathbb{Z}$. We define $\mathcal{P}_r\Lambda^k(T)$ as the space of differential k-forms whose coefficients are polynomials over T of degree at most r. We define $\mathcal{P}^-_r\Lambda^k(T) := \mathcal{P}_{r-1}\Lambda^k(T) + \vec{X} \lrcorner \mathcal{P}_{r-1}\Lambda^{k+1}(T)$, where $\vec{X}\lrcorner$ denotes contraction with the source vector field $\vec{X}(x) = x$. One can show that $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}^-_r\Lambda^k(T)$ are invariant under pullback by affine automorphisms of T. Some basic properties of these spaces are

$$\begin{gathered}
\mathcal{P}_r\Lambda^k(T) \subseteq \mathcal{P}^-_{r+1}\Lambda^k(T), \quad \mathcal{P}^-_r\Lambda^k(T) \subseteq \mathcal{P}_r\Lambda^k(T), \\
\mathsf{d}\mathcal{P}_r\Lambda^k(T) \subseteq \mathcal{P}_{r-1}\Lambda^{k+1}(T), \quad \mathsf{d}\mathcal{P}_r\Lambda^k(T) = \mathsf{d}\mathcal{P}^-_r\Lambda^k(T), \\
\mathcal{P}_r\Lambda^0(T) = \mathcal{P}^-_r\Lambda^0(T), \quad \mathcal{P}_r\Lambda^n(T) = \mathcal{P}^-_{r+1}\Lambda^n(T).
\end{gathered}$$

For any subsimplex $F \in \Delta(T)$ of T we let $\operatorname{tr}_{T,F} : C^\infty\Lambda^k(T) \to C^\infty\Lambda^k(F)$ denote the trace mapping from T onto F, and we set

$$\mathcal{P}_r\Lambda^k(F) := \operatorname{tr}_{T,F} \mathcal{P}_r\Lambda^k(T), \quad \mathcal{P}^-_r\Lambda^k(F) := \operatorname{tr}_{T,F} \mathcal{P}^-_r\Lambda^k(T). \tag{6.1}$$

Note that these two spaces do not depend on T. We define the finite element spaces

$$\mathcal{P}_r\Lambda^k(\mathcal{T}) := \left\{ \omega \in W^{\infty,\infty}\Lambda^k(\Omega) \mid \forall T \in \mathcal{T}^n : \omega_{|T} \in \mathcal{P}_r\Lambda^k(T) \right\},$$
$$\mathcal{P}_r^-\Lambda^k(\mathcal{T}) := \left\{ \omega \in W^{\infty,\infty}\Lambda^k(\Omega) \mid \forall T \in \mathcal{T}^n : \omega_{|T} \in \mathcal{P}_r^-\Lambda^k(T) \right\}.$$

These are spaces of piecewise polynomial differential forms. Membership of piecewise polynomial differential forms in $W^{\infty,\infty}\Lambda^k(\Omega)$ enforces tangential continuity along simplex boundaries. In particular, if $\omega \in \mathcal{P}_r\Lambda^k(\mathcal{T})$ and $T, T' \in \mathcal{T}$ are neighboring simplices, then the restrictions of ω to T and T' have the same trace on their common subsimplex $T \cap T'$. Thus our definition recovers precisely the finite element spaces of finite element exterior calculus [1].

From $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ we can construct finite element de Rham complexes, but the combination of spaces is not arbitrary. We single out a class of differential complexes that has been discussed by Arnold, Falk and Winther [1] and that we call *FEEC-complexes* in this article. A *FEEC-complex* is a differential complex

$$0 \to \mathcal{P}\Lambda^0(\mathcal{T}) \xrightarrow{\mathsf{d}} \mathcal{P}\Lambda^1(\mathcal{T}) \xrightarrow{\mathsf{d}} \cdots \xrightarrow{\mathsf{d}} \mathcal{P}\Lambda^n(\mathcal{T}) \to 0 \tag{6.2}$$

such that for all $k \in \mathbb{Z}$ there exists $r \in \mathbb{Z}$ with

$$\mathcal{P}\Lambda^k(\mathcal{T}) \in \left\{\mathcal{P}_r\Lambda^k(\mathcal{T}), \mathcal{P}_r^-\Lambda^k(\mathcal{T})\right\} \tag{6.3}$$

and that for all $k \in \mathbb{Z}$ we have

$$\begin{aligned} &\mathcal{P}\Lambda^k(\mathcal{T}) \in \left\{\mathcal{P}_r\Lambda^k(\mathcal{T}), \mathcal{P}_r^-\Lambda^k(\mathcal{T})\right\} \\ &\qquad \Longrightarrow \mathcal{P}\Lambda^{k+1}(\mathcal{T}) \in \left\{\mathcal{P}_{r-1}\Lambda^{k+1}(\mathcal{T}), \mathcal{P}_r^-\Lambda^{k+1}(\mathcal{T})\right\}. \end{aligned} \tag{6.4}$$

Next we introduce the degrees of freedom of finite element exterior calculus. They are represented by taking the trace of a differential form onto a simplex of $\mathcal{T}$ and then integrating against a smooth differential form. By virtue of Lemma 5.2, we introduce the degrees of freedom as chains of finite mass. Specifically, when $F \in \mathcal{T}$ and $m = \dim(F)$, then we define

$$\mathcal{P}_r\mathcal{C}_k^F := \left\{ S \in \mathcal{C}_k^\flat(\mathbb{R}^n) \;\middle|\; \exists \eta_S \in \mathcal{P}_{r+k-m}^-\Lambda^{m-k}(F) : \int_S \cdot = \int_F \eta_S \wedge \cdot \right\},$$
$$\mathcal{P}_r^-\mathcal{C}_k^F := \left\{ S \in \mathcal{C}_k^\flat(\mathbb{R}^n) \;\middle|\; \exists \eta_S \in \mathcal{P}_{r+k-m-1}\Lambda^{m-k}(F) : \int_S \cdot = \int_F \eta_S \wedge \cdot \right\}.$$

We furthermore obtain by Lemma 5.2 that the degrees of freedom are flat chains of finite mass with boundaries of finite mass. One can show that we have direct sums

$$\mathcal{P}_r\mathcal{C}_k(\mathcal{T}) := \sum_{F\in\mathcal{T}} \mathcal{P}_r\mathcal{C}_k^F, \quad \mathcal{P}_r^-\mathcal{C}_k(\mathcal{T}) := \sum_{F\in\mathcal{T}} \mathcal{P}_r^-\mathcal{C}_k^F.$$

Moreover, these flat chains have boundaries of finite mass. One can show that

$$\partial\mathcal{P}_r\mathcal{C}_k(\mathcal{T}) \subseteq \mathcal{P}_{r+1}^-\mathcal{C}_{k-1}(\mathcal{T}), \quad \mathcal{P}_r\mathcal{C}_k(\mathcal{T}) \subseteq \mathcal{P}_{r+1}^-\mathcal{C}_k(\mathcal{T}) \subseteq \mathcal{P}_{r+1}\mathcal{C}_k(\mathcal{T}).$$

Remark 6.1. To see the first inclusion, let $F \in \mathcal{T}$ and $\eta \in \mathcal{P}_r\Lambda^k(F)$. These define an element $S \in \mathcal{P}_r\mathcal{C}_k^F$, and similar as in the proof of Lemma 5.2 the generalized Stokes theorem shows for all $\omega \in W^{\infty,\infty}\Lambda^{k-1}(\mathbb{R}^n)$ that

$$\int_F \operatorname{tr}_F \mathsf{d}\omega \wedge \eta = (-1)^k \int_F \mathsf{d}\omega \wedge \eta + \sum_{f\in\Delta(F)^{m-1}} \int_f \operatorname{tr}_{T,f} \eta \wedge \operatorname{tr}_f \omega.$$

Here, each $f \in \Delta(F)^{m-1}$ carries the orientation induced by F. The first inclusion has been used implicitly in the proof of Lemma 4.24 of [1]. The second chain of inclusions follows immediately from the definition of $\mathcal{P}_r\mathcal{C}_k(\mathcal{T})$ and $\mathcal{P}_r^-\mathcal{C}_k(\mathcal{T})$.

With respect to a given FEEC-complex (6.2), we then define

$$\mathcal{P}\mathcal{C}_k(\mathcal{T}) = \begin{cases} \mathcal{P}_r\mathcal{C}_k(\mathcal{T}) & \text{if } \mathcal{P}\Lambda^k(\mathcal{T}) = \mathcal{P}_r\Lambda^k(\mathcal{T}), \\ \mathcal{P}_r^-\mathcal{C}_k(\mathcal{T}) & \text{if } \mathcal{P}\Lambda^k(\mathcal{T}) = \mathcal{P}_r^-\Lambda^k(\mathcal{T}). \end{cases} \tag{6.5}$$

for $k \in \mathbb{Z}$. Note that $\partial\mathcal{P}\mathcal{C}_{k+1}(\mathcal{T}) \subseteq \mathcal{P}\mathcal{C}_k(\mathcal{T})$ by construction. One can show that

$$\forall S \in \mathcal{P}\mathcal{C}_k(\mathcal{T}) : S \neq 0 \implies \exists \omega \in \mathcal{P}\Lambda^k(\mathcal{T}) : \int_S \omega \neq 0, \tag{6.6a}$$

$$\forall \omega \in \mathcal{P}\Lambda^k(\mathcal{T}) : \omega \neq 0 \implies \exists S \in \mathcal{P}\mathcal{C}_k(\mathcal{T}) : \int_S \omega \neq 0. \tag{6.6b}$$

We conclude that $\mathcal{P}\mathcal{C}_k(\mathcal{T})$, restricted to $\mathcal{P}\Lambda^k(\mathcal{T})$, spans the dual space of $\mathcal{P}\Lambda^k(\mathcal{T})$. Notably, the last implication can be strengthened to the following "local" result. When $T \in \mathcal{T}$ and $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$, then

$$\omega_{|T} = 0 \quad \iff \quad \forall F \in \Delta(T) : \forall S \in \mathcal{P}\mathcal{C}_k^F : \int_S \omega = 0. \tag{6.7}$$

So the value of $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ is determined uniquely by the values of the degrees of freedom associated with that simplex.

We introduce the *canonical finite element interpolant*. This linear mapping is well-defined and bounded both over $C\Lambda^k(\overline{\Omega})$ and $W^{\infty,\infty}\Lambda^k(\Omega)$. We define

$$I^k : C\Lambda^k(\overline{\Omega}) + W^{\infty,\infty}\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T}) \tag{6.8}$$

by requiring that

$$\int_S \omega = \int_S I^k\omega, \quad S \in \mathcal{P}\mathcal{C}_k(\mathcal{T}), \quad \omega \in C\Lambda^k(\overline{\Omega}) + W^{\infty,\infty}\Lambda^k(\Omega). \tag{6.9}$$

The finite element interpolant commutes with the exterior derivative, which follows easily from (6.9) and (5.6). We have

$$\int_S I^{k+1}\mathsf{d}\omega = \int_S \mathsf{d}\omega = \int_{\partial S} \omega = \int_{\partial S} I^k\omega = \int_S \mathsf{d}I^k\omega \tag{6.10}$$

for all $\omega \in W^{\infty,\infty}\Lambda^k(\Omega)$ and $S \in \mathcal{P}\mathcal{C}_{k+1}(\mathcal{T})$. In particular,

$$\begin{array}{ccccccc} \cdots & \longrightarrow & W^{\infty,\infty}\Lambda^k(\Omega) & \xrightarrow{\mathsf{d}} & W^{\infty,\infty}\Lambda^{k+1}(\Omega) & \longrightarrow & \cdots \\ & & {\scriptstyle I^k}\big\downarrow & & {\scriptstyle I^{k+1}}\big\downarrow & & \\ \cdots & \longrightarrow & \mathcal{P}\Lambda^k(\mathcal{T}) & \xrightarrow{\mathsf{d}} & \mathcal{P}\Lambda^{k+1}(\mathcal{T}) & \longrightarrow & \cdots \end{array} \tag{6.11}$$

is a commuting diagram. Furthermore I^k is idempotent, which means

$$I^k\omega = \omega, \quad \omega \in \mathcal{P}\Lambda^k(\mathcal{T}), \tag{6.12}$$

as follows directly from (6.7).

In the remainder of this section, we introduce a number of *inverse inequalities*. These rely on the equivalence of norms over finite-dimensional vector spaces.

We note that, by construction, the pullbacks $\varphi_T^*\omega_{|T}$ lie in a common finite-dimensional vector space as $\omega \in \mathcal{P}\Lambda^k(T)$ and $T \in \mathcal{T}^n$ vary. This is a fixed space of differential forms with polynomial coefficients of sufficiently high degree. Hence for each $p \in [1, \infty]$ there exists a constant $C_{\flat,p} > 0$ such that

$$\|\varphi_T^* \omega\|_{W^{\infty,\infty}\Lambda^k(\Delta^n)} \leq C_{\flat,p} \|\varphi_T^* \omega\|_{L^p\Lambda^k(\Delta^n)}, \quad \omega \in \mathcal{P}\Lambda^k(\mathcal{T}), \quad T \in \mathcal{T}^n. \tag{6.13}$$

The constant $C_{\flat,p}$ depends only on n, p, and the maximal polynomial degree in the finite element de Rham complex.

Another inverse inequality applies to the degrees of freedom. By Lemma 5.2, each degree of freedom can be identified with a flat chain of finite mass whose boundary is again a flat chain of finite mass. In general, the boundary operator is an unbounded operator as a mapping between spaces of polyhedral chains with respect to the mass norm. But in the present setting, the pushforward of the degrees of freedom onto the reference simplex takes values in a finite-dimensional vector space. We conclude that there exists $C_\partial > 0$ such that

$$|\varphi_{T*}^{-1} \partial S|_{k-1} \leq C_\partial |\varphi_{T*}^{-1} S|_k, \quad S \in \mathcal{PC}_k^F, \quad F \in \Delta(T), \quad T \in \mathcal{T}^n. \tag{6.14}$$

Again, the constant C_∂ depends only on n and the maximal polynomial degree in the finite element de Rham complex.

Finally, we observe that there exists $C_I > 0$, depending only on n and the maximal polynomial degree in the finite element de Rham complex, such that

$$\|\varphi_T^* I^k \omega\|_{L^\infty\Lambda^k(\Delta^n)} \leq C_I \sup_{F \in \Delta(T)} \sup_{S \in \mathcal{PC}_k^F} |\varphi_{T*}^{-1} S|_k^{-1} \int_{\varphi_{T*}^{-1} S} \varphi_T^* \omega \tag{6.15}$$

for all $T \in \mathcal{T}^n$ and $\omega \in C\Lambda^k(\overline{\Omega})$. Note that this inequality immediately implies

$$\|\varphi_T^* I^k \omega\|_{L^\infty\Lambda^k(\Delta^n)} \leq C_I \|\varphi_T^* \omega\|_{C\Lambda^k(\Delta^n)}, \quad \omega \in C\Lambda^k(T). \tag{6.16}$$

To see this, we recall that the integrals on the right-hand side of (6.15) can be bounded in terms of the maximum of $|\omega|$ over F and the mass of $\varphi_{T*}^{-1} S$, as follows from Lemma 5.2 and our definition of the degrees of freedom.

Remark 6.2. The existence of constants $C_{\flat,p}$, C_∂, and C_I as above follows trivially if the triangulation $\mathcal{T}$ and the sequence (6.2) are fixed. But in applications we consider families of triangulations with associated sequence (6.2), and we then demand uniform bounds for those constants. Such uniform bounds hold if the triangulations have uniformly bounded shape constants and the finite element spaces have uniformly bounded polynomial degree. The results of this article do not attend to estimates that are uniform in the polynomial degree, as would be relevant for p- and hp-methods.

7. Smoothed Projection

In this section, we construct the smoothed projection in several stages. First, we devise an extension operator E^k, applying the two-sided Lipschitz collar discussed in Section 2. We then formulate a smoothing operator $R^k_{\epsilon\mathtt{h}}$, where we use a smooth mesh size function $\mathtt{h}$ as an auxiliary construction. Successive composition with the canonical finite element interpolant I^k from Section 6 yields an uniformly bounded commuting mapping Q^k_ϵ, the *smoothed interpolant*, from differential forms with coefficients in L^p onto finite element differential forms. Q^k_ϵ is generally not idempotent on the finite element space, but the interpolation error can be controlled. After a small modification, we obtain the desired smoothed projection π^k_ϵ.

Throughout this section we assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded connected weakly Lipschitz domain and that $\mathcal{T}$ is a finite triangulation of $\overline{\Omega}$. We additionally assume that we have fixed a FEEC-complex (6.2). In the sequel, we adhere to the convention of stating each result accompanied by explicit estimates of the various constants and parameter ranges. We call a quantity *uniformly bounded* if it can be bounded in terms of the shape-constant, the geometry, and the polynomial degree of the finite element space.

7.1. **Extension.** Since Ω is a bounded weakly Lipschitz domain, we may apply Theorem 2.3 to fix a compact neighborhood $\mathcal{C}\Omega$ of $\partial\Omega$ in $\mathbb{R}^n$ and a bi-Lipschitz mapping

$$\Psi : \partial\Omega \times [-1,1] \to \mathcal{C}\Omega$$

such that $\Psi(x,0) = x$ for $x \in \partial\Omega$, and such that

$$\Psi\left(\partial\Omega \times [-1,0)\right) = \mathcal{C}\Omega \cap \Omega, \quad \Psi\left(\partial\Omega \times (0,1]\right) = \mathcal{C}\Omega \cap \overline{\Omega}^c.$$

Additionally we write

$$\mathcal{C}^-\Omega := \mathcal{C}\Omega \cap \Omega, \quad \mathcal{C}^+\Omega := \mathcal{C}\Omega \cap \overline{\Omega}^c, \quad \Omega^e := \overline{\Omega} \cup \mathcal{C}^+\Omega \tag{7.1}$$

for the *interior collar part* $\mathcal{C}^-\Omega$, the *exterior collar part* $\mathcal{C}^+\Omega$, and the *extended domain* Ω^e, respectively. Eventually, we have a well-defined bi-Lipschitz mapping

$$\mathcal{R} : \mathcal{C}^+\Omega \to \mathcal{C}^-\Omega, \quad \Psi(x,t) \mapsto \Psi(x,-t) \tag{7.2}$$

from the outer collar part into the inner collar part, called *collar reflection*.

We define the extension operator using the pullback along the collar reflection. If $\omega \in M\Lambda^k(\Omega)$ is a locally integrable k-form over Ω, then

$$E^k\omega := \left\{ \begin{array}{cl} \omega & \text{over } \Omega, \\ \mathcal{R}^*\omega & \text{over } \mathcal{C}^+\Omega, \end{array} \right. \tag{7.3}$$

is the locally integrable differential k-form constructed by extending ω onto $\mathcal{C}^+\Omega$ using the pullback along $\mathcal{R}$. For notational convenience, we define the constant

$$C_E := \max\left\{ \operatorname{Lip}(\mathcal{R}, \mathcal{C}^+\Omega), \operatorname{Lip}(\mathcal{R}^{-1}, \mathcal{C}^-\Omega) \right\}$$

We show that the linear mapping E^k satisfies local estimates and commutes with the exterior derivative.

Lemma 7.1. *Let $p \in [1,\infty]$. We have a bounded linear operator*

$$E^k : L^p\Lambda^k(\Omega) \to L^p\Lambda^k(\Omega^e), \quad \omega \mapsto E^k\omega.$$

For every measurable set $G \subseteq \mathcal{C}^+\Omega$ we have

$$\|E^k\omega\|_{L^p\Lambda^k(G)} \leq C_E^{k+n/p} \|\omega\|_{L^p\Lambda^k(\mathcal{R}(G))}, \quad \omega \in L^p\Lambda^k(\Omega). \tag{7.4}$$

Proof. Let $p \in [1,\infty]$, $G \subseteq \mathcal{C}^+\Omega$ be measurable, and $\omega \in L^p\Lambda^k(\Omega)$. Then

$$\|E^k\omega\|_{L^p\Lambda^k(G)} = \|\mathcal{R}^*\omega\|_{L^p\Lambda^k(G)} \leq \|\operatorname{D}\mathcal{R}\|^k_{L^\infty(\mathcal{C}^+\Omega)} \|\operatorname{D}\mathcal{R}^{-1}\|^{n/p}_{L^\infty(\mathcal{C}^-\Omega)} \|\omega\|_{L^p\Lambda^k(\mathcal{R}(G))}.$$

by Lemma 3.5, and hence (7.4) holds. In the case $G = \mathcal{C}^+\Omega$ we find

$$\|E^k\omega\|_{L^p\Lambda^k(\Omega^e)} \leq \|\omega\|_{L^p\Lambda^k(\Omega)} + \|E^k\omega\|_{L^p\Lambda^k(\mathcal{C}^+\Omega)} \leq \left(1 + C_E^{k+n/p}\right) \|\omega\|_{L^p\Lambda^k(\Omega)}.$$

We conclude that E^k is bounded from $L^p\Lambda^k(\Omega)$ to $L^p\Lambda^k(\Omega^e)$. $\square$

Lemma 7.2. *There exists $L_E \geq 1$, depending only on Ψ, such that for all $p \in [1,\infty]$, all $\delta \geq 0$, all measurable sets $A \subset \overline{\Omega}$, and all $\omega \in L^p\Lambda^k(\Omega)$ we have*

$$\|E^k\omega\|_{L^p\Lambda^k(B_\delta(A)\cap\Omega^e)} \leq \left(1 + C_E^{k+n/p}\right) \|\omega\|_{L^p\Lambda^k\left(B_{\delta L_E}(A)\cap\Omega\right)}. \tag{7.5}$$

Proof. Let $\delta \geq 0$, let $p \in [1,\infty]$, and let $A \subset \overline{\Omega}$ be measurable. Then

$$\|E^k\omega\|_{L^p\Lambda^k(B_\delta(A)\cap\Omega^e)} \leq \|\omega\|_{L^p\Lambda^k(B_\delta(A)\cap\Omega)} + \|E^k\omega\|_{L^p\Lambda^k(B_\delta(A)\cap\mathcal{C}^+\Omega)}.$$

We set $G^+ := B_\delta(A) \cap \mathcal{C}^+\Omega$ and $G^- = \mathcal{R}(G^+) \subseteq \Omega$. Using Lemma 7.1, we find

$$\|E^k\omega\|_{L^p\Lambda^k(G^+)} \leq C_E^{k+n/p}\|\omega\|_{L^p\Lambda^k(G^-)}.$$

Let $x \in B_\delta(A) \cap \mathcal{C}^+\Omega$ be fixed but arbitrary. There exist $z \in A$ with $\|z - x\| \leq \delta$, and $y \in \partial\Omega$ on the straight line segment between x and z. Since $x \in \mathcal{C}^+\Omega$, there exist $x_0 \in \partial\Omega$ and $t \in [0,1]$ with $x = \Psi(x_0, t)$. It is easily seen that

$$\|\mathcal{R}(x) - x\| = \|\Psi(x,-t) - \Psi(x,t)\| \leq 2\operatorname{Lip}(\Psi)\cdot t,$$
$$|t| \leq \sqrt{\|x_0 - y\|^2 + |t|^2} \leq \operatorname{Lip}(\Psi^{-1})\,\|\Psi(x_0,t) - \Psi(y,0)\| = \operatorname{Lip}(\Psi^{-1})\,\|x - y\|.$$

We then find that

$$\begin{aligned}\|\mathcal{R}(x) - z\| \leq \|\mathcal{R}(x) - x\| + \|x - z\| &\leq 2\operatorname{Lip}(\Psi)\operatorname{Lip}(\Psi^{-1})\|x - y\| + \|x - z\| \\ &\leq \left(1 + 2\cdot\operatorname{Lip}(\Psi)\operatorname{Lip}(\Psi^{-1})\right)\delta.\end{aligned}$$

We choose $L_E := \left(1 + 2\operatorname{Lip}(\Psi)\operatorname{Lip}(\Psi^{-1})\right)$. Hence $G^- \cap \Omega \subseteq B_{L_E\delta}(A) \cap \Omega$. This completes the proof. ☐

Lemma 7.3. *Let $p, q \in [1,\infty]$. If $\omega \in W^{p,q}\Lambda^k(\Omega)$, then $E^k\omega \in W^{p,q}\Lambda^k(\Omega^e)$ and $E^{k+1}\mathsf{d}\omega = \mathsf{d}E^k\omega$.*

Proof. Because Ω is bounded, it suffices to consider the case $p = q = 1$. Let $\omega \in W^{1,1}\Lambda^k(\Omega)$. We have $E^k\omega \in L^1\Lambda^k(\Omega^e)$ and $E^{k+1}\mathsf{d}^k\omega \in L^1\Lambda^{k+1}(\Omega^e)$ by Lemma 7.1. To prove that $E^k\omega \in W^{1,1}\Lambda^k(\Omega^e)$ with $E^{k+1}\mathsf{d}^k\omega = \mathsf{d}^kE^k\omega$, it suffices to show that there exists a covering $(U_i)_{i\in\mathbb{N}}$ of Ω^e by open subsets $U_i \subseteq \Omega^e$ such that $E^k\omega_{|U_i} \in W^{1,1}\Lambda^k(U_i)$ and $E^{k+1}\mathsf{d}^k\omega = \mathsf{d}^kE^k\omega$ over each U_i.

From the definition of weakly Lipschitz domains we easily see that there exists a family $(\theta_i)_{i\in\mathbb{N}}$ of LIP embeddings $\theta_i : (-1,1)^{n-1} \to \partial\Omega$ whose images cover $\partial\Omega$. We define mappings $\varphi_i : (-1,1)^n \to \mathcal{C}\Omega$ by setting $\varphi_i(y,t) \mapsto \Psi(\theta_i(y), t)$. These are a family of LIP embeddings whose images $U_i := \varphi_i((-1,1)^n)$ cover $\mathcal{C}\Omega$. Together with Ω we thus have a finite open covering of Ω^e.

We recall that $E^k\omega_{|\Omega} \in W^{1,1}\Lambda^k(\Omega)$ with $E^{k+1}\mathsf{d}^k\omega = \mathsf{d}^kE^k\omega$ over Ω. It remains to show that $E^k\omega_{|U_i} \in W^{1,1}\Lambda^k(U_i)$ and $E^{k+1}\mathsf{d}^k\omega = \mathsf{d}^kE^k\omega$ over U_i for $i \in \mathbb{N}$. To this end, we define $\omega_i := \varphi_i^*\left(E^k\omega_{|U_i}\right)$ and $\xi_i := \varphi_i^*\left(E^{k+1}\mathsf{d}^k\omega_{|U_i}\right)$. So it suffices to show $\omega_i \in W^{1,1}\Lambda^k((-1,1)^n)$ and $\mathsf{d}^k\omega_i = \xi_i$ over $(-1,1)^n$. We let

$$\mathscr{R} : (-1,1)^{n-1} \times (0,1) \to (-1,1)^{n-1} \times (-1,0)$$

be the reflection by the n-th coordinate. It is evident that

$$\omega_{i|(-1,1)^{n-1}\times(0,1)} = \mathscr{R}^*\omega_{i|(-1,1)^{n-1}\times(-1,0)}$$
$$\xi_{i|(-1,1)^{n-1}\times(0,1)} = \mathscr{R}^*\xi_{i|(-1,1)^{n-1}\times(-1,0)} = \mathscr{R}^*\mathsf{d}^k\omega_{i|(-1,1)^{n-1}\times(-1,0)}.$$

By the density of $C^\infty\Lambda^k(\overline{U})$ in $W^{1,1}\Lambda^k(U)$ there exists a sequence $(\omega_i^j)_{j\in\mathbb{N}}$ of smooth differential k-forms over $(-1,1)^{n-1} \times (-1,0)$ that converge to ω_i over $(-1,1)^{n-1} \times$

$(-1,0)$ in the $W^{1,1}\Lambda^k$ norm for $j \to \infty$. We let $\overset{\|}{\omega}{}_i^j$ be the extension of ω_i^j from $(-1,1)^{n-1} \times (-1,0)$ to $(-1,1)^n$ by pullback along $\mathscr{R}$. Then $\overset{\|}{\omega}{}_i^j$ is a locally integrable differential k-form over $(-1,1)^n$ with locally integrable weak exterior derivative. It is easy to observe that $\overset{\|}{\omega}{}_i^j$ converges to ω_i in $L^1\Lambda^k\left((-1,1)^n\right)$ and $\mathsf{d}^k \overset{\|}{\omega}{}_i^j$ converges to ξ_i in $L^1\Lambda^{k+1}\left((-1,1)^n\right)$ for $j \to \infty$. Hence $\omega_i \in W^{1,1}\Lambda^k\left((-1,1)^n\right)$ with $\mathsf{d}^k\omega_i = \xi_i$. The proof is complete. □

7.2. **Smoothing Operators.** The next step is constructing a commuting smoothing operator. We define the smoothed differential form at each point by locally averaging the original differential form. A technical difference to the classical smoothing operator is that we let the smoothing radius vary across the domain.

We first discuss such smoothing operators in a very general fashion before we focus on a specific example. We assume that $\varrho : \mathbb{R}^n \to \mathbb{R}^+$ is a non-negative smooth function that assumes a positive minimum over $\overline{\Omega}$. A concrete instance of such a function will be discussed later in this subsection. We introduce the mapping

$$\Phi_\varrho : \overline{\Omega} \times B_1(0) \to \mathbb{R}^n, \quad (x,y) \mapsto x + \varrho(x)y. \tag{7.6}$$

Regarding the second variable as a parameter, we get a family of mappings

$$\Phi_{\varrho,y} : \overline{\Omega} \to \mathbb{R}^n, \quad x \mapsto \Phi_\varrho(x,y).$$

We study some properties of $\Phi_{\varrho,y}$. This mapping is smooth and we have

$$\mathrm{D}\,\Phi_{\varrho,y} = \mathrm{Id} + y \otimes \mathsf{d}\varrho.$$

When $y \in B_1(0)$ and $x_1, x_2 \in \overline{\Omega}$, then

$$\left\|\Phi_{\varrho,y}(x_1) - \Phi_{\varrho,y}(x_2)\right\| \leq (1 + \mathrm{Lip}(\varrho)) \left\|x_1 - x_2\right\|. \tag{7.7}$$

Moreover, for any $y \in B_1(0)$ and $x \in \overline{\Omega}$ we have

$$\left\|\Phi_{\varrho,y}(x) - x\right\| \leq \varrho(x). \tag{7.8}$$

The latter inequality implies that for ϱ small enough we have

$$\Phi_{\varrho,y}\left(\overline{\Omega}\right) \subseteq \Omega^e. \tag{7.9}$$

Under the additional condition that $\mathrm{Lip}(\varrho) < 1/2$, we observe for $y \in B_1(0)$ and $x_1, x_2 \in \overline{\Omega}$ that

$$\begin{aligned} \left\|\Phi_{\varrho,y}(x_1) - \Phi_{\varrho,y}(x_2)\right\| &= \left\|x_1 - x_2 + (\varrho(x_1) - \varrho(x_2))\, y\right\| \\ &\geq \Big| \left\|x_1 - x_2\right\| - \mathrm{Lip}(\varrho) \left\|x_1 - x_2\right\| \Big| \geq \frac{1}{2} \left\|x_1 - x_2\right\|. \end{aligned} \tag{7.10}$$

We conclude that for ϱ and $\mathrm{Lip}(\varrho)$ small enough, the mapping $\Phi_{\varrho,y} : \overline{\Omega} \to \Omega^e$ is a LIP embedding for every $y \in B_1(0)$. In the sequel, we let $B_\varrho(A)$ for any $A \subseteq \Omega^e$ denote the union of the balls $B_{\varrho(x)}(x)$ for $x \in A$.

We recall the standard mollifier. This is a non-negative smooth function

$$\mu : \mathbb{R}^n \to \mathbb{R}, \quad y \mapsto \begin{cases} C_\mu \exp\left(-(1 - \|y\|^2)^{-1}\right) & \text{if} \quad \|y\| \leq 1, \\ 0 & \text{if} \quad \|y\| > 1, \end{cases} \tag{7.11}$$

with compact support, where $C_\mu > 0$ is chosen such that μ has unit integral. We set $\mu_r(y) := r^{-n}\mu(y/r)$ for $y \in \mathbb{R}^n$ and $r > 0$.

The smoothing operator in this subsection uses the standard mollifier μ as a building block and can be seen as a generalization of the classical smoothing by convolution. For every $\omega \in L^1\Lambda^k(\Omega^e)$ we define

$$R^k_\varrho \omega_{|x} := \int_{\mathbb{R}^n} \mu(y)(\Phi^*_{\varrho,y}\omega)_{|x} \mathrm{d}y, \quad x \in \overline{\Omega}. \tag{7.12}$$

We first show that R^k_ϱ maps into $C^\infty\Lambda^k(\overline{\Omega})$ and satisfies a local bound. In particular, it is a bounded mapping into $C\Lambda^k(\overline{\Omega})$ with respect to the maximum norm.

Lemma 7.4. *Assume that $\Phi_{\varrho,y} : \overline{\Omega} \to \Omega^e$ is a LIP embedding for each $y \in B_1(0)$. Then we have a well-defined linear operator*

$$R^k_\varrho : L^p\Lambda^k(\Omega^e) \to C^\infty\Lambda^k(\overline{\Omega}), \quad p \in [1,\infty].$$

Moreover, for every $p \in [1,\infty]$, $\omega \in L^p\Lambda^k(\Omega^e)$, and measurable set $A \subseteq \overline{\Omega}$ we have

$$\|R^k_\varrho \omega\|_{C\Lambda^k(A)} \leq (1+\operatorname{Lip}(\varrho))^k \left(\inf_A \varrho\right)^{-\frac{n}{p}} \|\omega\|_{L^p\Lambda^k(\Phi_\varrho(A,B_1))}. \tag{7.13}$$

In addition,

$$\mathsf{d}R^k_\varrho\omega = R^{k+1}_\varrho \mathsf{d}\omega, \quad \omega \in W^{p,q}\Lambda^k(\Omega^e), \quad p,q \in [1,\infty]. \tag{7.14}$$

Proof. Let $p \in [1,\infty]$ and $\omega \in L^p\Lambda^k(\Omega^e)$. Since $\Phi_{\varrho,y} : \overline{\Omega} \to \Omega^e$ is a LIP embedding for every $y \in B_1(0)$, we find that $\mu(y)(\Phi^*_{\varrho,y}\omega)_{|x}$ is measurable in y for every $x \in \overline{\Omega}$. Furthermore, the integral (7.12) is well-defined. Using elementary results, we get for every $x \in \overline{\Omega}$ that

$$\left|R^k_\varrho\omega\right|_{|x} \leq \operatorname{Lip}(\Phi_{\varrho,y},\overline{\Omega})^k \int_{\mathbb{R}^n} \mu(y)|\omega|_{|\Phi_{\varrho,y}(x)}\mathrm{d}y \leq (1+\operatorname{Lip}(\varrho))^k \int_{\mathbb{R}^n} \mu(y)|\omega|_{|\Phi_{\varrho,y}(x)}\mathrm{d}y.$$

A substitution of variables and Hölder's inequality give

$$\int_{\mathbb{R}^n} \mu(y)|\omega|_{|x+\varrho(x)y}\,\mathrm{d}y \leq \varrho(x)^{-\frac{n}{p}} \|\omega\|_{L^p\left(B_{\varrho(x)}(x)\right)}.$$

These estimates in combination yield (7.13). In order to prove the smoothness of $R^k_\varrho\omega$ over Ω, we first change the form of the integral. By a substitution of variables we find for $x \in \Omega^e$ that

$$\begin{aligned} R^k_\varrho\omega_{|x} &= \sum_{\sigma\in\Sigma(1:k,0:n)} \int_{\mathbb{R}^n} \mu(y)\omega_\sigma\left(x+\varrho(x)y\right)\left(\Phi^*_{\varrho,y}\mathsf{d}x^\sigma\right)_{|x}\mathrm{d}y \\ &= \sum_{\sigma\in\Sigma(1:k,0:n)} \int_{\mathbb{R}^n} \mu_{\varrho(x)}\left(y-x\right)\omega_\sigma(y)\left(\Phi^*_{\varrho,\varrho(x)^{-1}(y-x)}\mathsf{d}x^\sigma\right)_{|x}\,\mathrm{d}y. \end{aligned}$$

We know that $\omega_\sigma \in L^1(\Omega)$, that ϱ and Φ_ϱ are smooth, and that $\overline{\Omega}$ is compact. The desired smoothness of $R^k_\varrho\omega$ over Ω is now a simple consequence of the dominated convergence theorem. Furthermore, $R^k_\varrho\omega \in C^\infty\Lambda^k(\overline{\Omega})$, as can easily be seen when picking x in a sufficiently small open neighborhood of $\overline{\Omega}$.

It remains to show (7.14). Let $\eta \in C^\infty_c\Lambda^{n-k-1}(\Omega)$. By Fubini's theorem we have

$$\begin{aligned} \int_\Omega R^k_\varrho\omega \wedge \mathsf{d}\eta &= \int_\Omega \int_{\mathbb{R}^n} \mu(y)\Phi^*_{\varrho,y}\omega\,\mathrm{d}y \wedge \mathsf{d}\eta = \int_{\mathbb{R}^n} \mu(y)\int_\Omega \Phi^*_{\varrho,y}\omega \wedge \mathsf{d}\eta\,\mathrm{d}y, \\ \int_\Omega R^k_\varrho\mathsf{d}\omega \wedge \eta &= \int_\Omega \int_{\mathbb{R}^n} \mu(y)\Phi^*_{\varrho,y}\mathsf{d}\omega\,\mathrm{d}y \wedge \eta = \int_{\mathbb{R}^n} \mu(y)\int_\Omega \Phi^*_{\varrho,y}\mathsf{d}\omega \wedge \eta\,\mathrm{d}y. \end{aligned}$$

When ϱ is small enough over $\overline{\Omega}$, then $\Phi_{\varrho,y} : \overline{\Omega} \to \Omega^e$ is a LIP embedding for every $y \in B_1(0)$. Hence by Lemma 3.4 we find

$$\int_\Omega \Phi^*_{\varrho,y}\omega \wedge \mathsf{d}\eta = (-1)^{k+1} \int_\Omega \mathsf{d}\Phi^*_{\varrho,y}\omega \wedge \eta = (-1)^{k+1} \int_\Omega \Phi^*_{\varrho,y}\mathsf{d}\omega \wedge \eta.$$

By definition, $\mathsf{d}R^k_\varrho\omega = R^{k+1}_\varrho \mathsf{d}\omega$. The proof is complete. □

We will instantiate this result for a particular choice of ϱ that reflects the local mesh size. First we prove the existence of a mesh size function $\mathtt{H}$ with Lipschitz regularity, and then we prove the existence of a mesh size function $\mathtt{h}$ that is smooth.

Lemma 7.5. *There exists $L_\Omega > 0$, only depending on Ω, and a Lipschitz continuous function $\mathtt{H} : \overline{\Omega} \to \mathbb{R}^+_0$ such that*

$$\forall F \in \mathcal{T} : C^{-1}_{\mathrm{mesh}} h_F \leq \mathtt{H}_{|F} \leq C_{\mathrm{mesh}} h_F, \tag{7.15}$$

$$\mathrm{Lip}(\mathtt{H}, \overline{\Omega}) \leq C_{\mathrm{mesh}} L_\Omega, \quad \min_{T \in \mathcal{T}^n} h_T \leq \min_{\overline{\Omega}} \mathtt{H}, \quad \max_{\overline{\Omega}} \mathtt{H} \leq \max_{T \in \mathcal{T}^n} h_T. \tag{7.16}$$

Proof. Let the function $\mathtt{H} : \overline{\Omega} \to \mathbb{R}^+_0$ be defined as follows. If $V \in \mathcal{T}^0$, then we set $\mathtt{H}(V) = h_V$. We then extend $\mathtt{H}$ to each $T \in \mathcal{T}$ by affine interpolation between the vertices of T. With this definition, $\mathtt{H}$ is continuous, and (7.15) follows from (4.2). It remains to prove (7.16). Obviously, $\mathrm{Lip}(\mathtt{H}, T) \leq C_{\mathrm{mesh}}$ for $T \in \mathcal{T}^n$.

Since Ω is a bounded weakly Lipschitz domain, there exists be a finite family $(U_i)_{1 \leq i \leq N}$ of relatively open sets $U_i \subseteq \overline{\Omega}$ such that such that the union of all U_i equals $\overline{\Omega}$, and such that there exist bi-Lipschitz $\varphi_i : \overline{U_i} \to [-1,1]^n$ for each $1 \leq i \leq N$. By Lebesgue's number lemma, we may pick $\gamma > 0$ so small that for each $x \in \overline{\Omega}$ there exists $1 \leq i \leq N$ such that $B_\gamma(x) \cap \overline{\Omega} \subseteq U_i$.

First assume that $x, y \in \Omega$ with $0 < \|x - y\| \leq \gamma$. Then there exists $1 \leq i \leq N$ with $x, y \in U_i$. For $M \in \mathbb{N}$, consider a partition of the line segment in $[-1,1]^n$ from $\varphi(x)$ to $\varphi(y)$ into M subsegments of equal length with points $\varphi_i(x) = z_0, z_1, \dots, z_M = \varphi_i(x)$. Let $x_m := \varphi_i^{-1}(z_m) \in U_i$. For M large enough, the straight line segment between x_{m-1} and x_m is contained in U_i for all $1 \leq m \leq M$. After a further subpartitioning, not necessarily equidistant, we may assume to have a sequence $x = w_0, \dots, w_{M'} = y$ for some $M' \in \mathbb{N}$ such that for all $1 \leq m \leq M'$ the points w_{m-1} and w_m are connected by a straight line segment in U_i and such that there exists $F_m \in \mathcal{T}$ with $w_{m-1}, w_m \in F_m$. We first observe

$$\begin{aligned}
|\mathtt{H}(y) - \mathtt{H}(x)| &\leq \sum_{m=1}^{M'} |\mathtt{H}(w_m) - \mathtt{H}(w_{m-1})| \\
&\leq C_{\mathrm{mesh}} \sum_{m=1}^{M'} \|w_m - w_{m-1}\| = C_{\mathrm{mesh}} \sum_{m=1}^{M} \|x_m - x_{m-1}\|.
\end{aligned}$$

By the Lipschitz continuity of φ_i and φ_i^{-1} we then obtain

$$\begin{aligned}
\sum_{m=1}^{M} \|x_m - x_{m-1}\| &\leq \mathrm{Lip}(\varphi_i^{-1}) \sum_{m=1}^{M} \|\varphi_i(x_m) - \varphi_i(x_{m-1})\| \\
&\leq \mathrm{Lip}(\varphi_i^{-1}) \|\varphi_i(y) - \varphi_i(x)\| \leq \mathrm{Lip}(\varphi_i^{-1}) \,\mathrm{Lip}(\varphi_i) \|y - x\|.
\end{aligned}$$

If we instead assume that $x, y \in \Omega$ with $\|x - y\| \geq \gamma$, then

$$|\mathtt{H}(y) - \mathtt{H}(x)| \leq \gamma^{-1} \mathrm{diam}(\Omega) |\mathtt{H}(x) - \mathtt{H}(y)| \leq \gamma^{-1} \mathrm{diam}(\Omega) C_{\mathrm{mesh}} \|y - x\|,$$

since $\gamma < \mathrm{diam}(\Omega)$. Hence $\mathrm{Lip}(\mathtt{H}, \Omega) \leq C_{\mathrm{mesh}} L_\Omega$ with

$$L_\Omega := \sup\left\{ \gamma^{-1}\mathrm{diam}(\Omega),\ \mathrm{Lip}(\varphi_1^{-1})\,\mathrm{Lip}(\varphi_1), \dots,\ \mathrm{Lip}(\varphi_N^{-1})\,\mathrm{Lip}(\varphi_N) \right\}.$$

Thus $\mathrm{Lip}(\mathtt{H}, \overline{\Omega}) \leq C_{\mathrm{mesh}} L_\Omega$ because any Lipschitz continuous function is Lipschitz continuous over the closure of its domain with the same Lipschitz constant. □

Remark 7.6. The preceding result was used before in the literature, but estimating $\mathrm{Lip}(\mathtt{H})$ did not receive much attention. An interesting observation is that $\mathrm{Lip}(\mathtt{H})$ is the product of C_{mesh}, which only depends on the shape of the simplices, and L_Ω, which depends only the geometry. Conceptually, L_Ω compares the *inner path metric* of $\overline{\Omega}$ to the Euclidean metric over $\overline{\Omega}$. The equivalence of these two metrics is non-trivial in general, but holds true for bounded weakly Lipschitz domains.

Lemma 7.7. *There exist a smooth function* $\mathtt{h} : \mathbb{R}^n \to \mathbb{R}_0^+$ *and uniformly bounded constants* $C_h \geq 1$ *and* $L_h > 0$ *such that*

$$\forall F \in \mathcal{T} : \forall x \in F : C_h^{-1} h_F \leq \mathtt{h}(x) \leq C_h h_F, \tag{7.17}$$

$$\mathrm{Lip}(\mathtt{h}, \mathbb{R}^n) \leq L_h, \quad \min_{T \in \mathcal{T}^n} h_T \leq \min_{\mathbb{R}^n} \mathtt{h}, \quad \max_{\mathbb{R}^n} \mathtt{h} \leq \max_{T \in \mathcal{T}^n} h_T. \tag{7.18}$$

Constants. We may assume $C_h \leq 2C_{\mathrm{mesh}}$ and $L_h \leq C_{\mathrm{mesh}} L_\Omega$.

Proof. Let $\mathtt{H} : \overline{\Omega} \to \mathbb{R}_0^+$ be as in Lemma 7.5. We define $\mathtt{H}_0 : \mathbb{R}^n \to \mathbb{R}$ by

$$\mathtt{H}_0(x) := \inf_{x_0 \in \overline{\Omega}} \mathtt{H}(x_0) + C_{\mathrm{mesh}} L_\Omega \|x - x_0\|, \quad x \in \mathbb{R}^n.$$

By the definition of Lipschitz continuity it is easily verified that $\mathtt{H}_0(x) = \mathtt{H}(x)$ for $x \in \overline{\Omega}$. Furthermore, for $x, y \in \mathbb{R}^n$ we observe the Lipschitz condition

$$\left|\mathtt{H}_0(x) - \mathtt{H}_0(y)\right| \leq C_{\mathrm{mesh}} L_\Omega \inf_{x_0 \in \overline{\Omega}} \left| \|x - x_0\| - \|y - x_0\| \right| \leq C_{\mathrm{mesh}} L_\Omega \|x - y\|.$$

We let $M := \max_{\overline{\Omega}} \mathtt{H}$ and let $\mathtt{H}_1 : \mathbb{R}^n \to \mathbb{R}$ be defined by $\mathtt{H}_1(x) = \min\left(M, \mathtt{H}_0(x)\right)$. Then the ranges and Lipschitz constants of $\mathtt{H}_1$ and $\mathtt{H}$ agree.

Let $h_{\min} > 0$ be the smallest diameter of any simplex in $\mathcal{T}$. We let $r > 0$ be so small that $C_{\mathrm{mesh}} L_\Omega r \leq h_{\min}/2$. Thus for all $x \in \mathbb{R}^n$ and $y \in B_r(x)$ we get

$$\|\mathtt{H}_1(y) - \mathtt{H}_1(x)\| \leq C_{\mathrm{mesh}} L_\Omega r \leq h_{\min}/2 \leq \mathtt{H}_0(x)/2.$$

Consequently, the convolution $\mathtt{h} := \mathtt{H}_1 \star \mu_r$ of $\mathtt{H}_0$ with the scaled mollifier μ_r is smooth, Lipschitz-continuous with constant $C_{\mathrm{mesh}} L_\Omega$, and satisfies $\mathtt{H}(x)/2 \leq \mathtt{h}(x) \leq 2\mathtt{H}(x)$ for all $x \in \overline{\Omega}$. The desired result follows. □

We now combine the smoothed mesh size function with the general regularization operator discussed previously in this subsection. It is easily seen that there exists $\epsilon_\Omega > 0$ such that $B_{\epsilon_\Omega \mathrm{diam}(\Omega)}(\overline{\Omega}) \subseteq \Omega^e$. As an immediate consequence, we have LIP embeddings $\Phi_{\epsilon\mathtt{h},y} : \overline{\Omega} \to \Omega^e$ provided that $\epsilon > 0$ is chosen such that $\epsilon < \epsilon_\Omega$. We conclude that Lemma 7.4 applies in that case, and setting $\varrho = \epsilon\mathtt{h}$ we obtain a regularization operator $R_{\epsilon\mathtt{h}}$. This operator is used in the next subsection.

7.3. **Smoothed Interpolation and Smoothed Projection.** Towards the definition of the smoothed projection, we first define a *smoothed interpolant*. Let $\epsilon > 0$ be small enough; we assume in particular $\epsilon < \epsilon_\Omega$. Combining the extension operator, the smoothing operator, and the finite element interpolant, we define

$$Q_\epsilon^k : L^p\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T}) \subseteq L^p\Lambda^k(\Omega), \quad \omega \mapsto I^k R_{\epsilon\mathtt{h}}^k E^k \omega, \quad p \in [1, \infty]. \tag{7.19}$$

We show that the smoothed interpolant Q^k_ϵ satisfies uniform local bounds and commutes with the exterior derivative.

Theorem 7.8. *Let $\epsilon > 0$ be small enough. We have a bounded linear operator*

$$Q^k_\epsilon : L^p\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T}) \subseteq L^p\Lambda^k(\Omega), \quad p \in [1,\infty].$$

For $p \in [1,\infty]$ and there exists uniformly bounded $C_{Q,p} > 0$ such that

$$\|Q^k_\epsilon \omega\|_{L^p\Lambda^k(T)} \leq C_{Q,p}\epsilon^{-\frac{n}{p}} \|\omega\|_{L^p\Lambda^k(\mathcal{T}(T))}, \quad \omega \in L^p\Lambda^k(\Omega), \quad T \in \mathcal{T}^n, \tag{7.20}$$

$$\|Q^k_\epsilon \omega\|_{L^p\Lambda^k(\Omega)} \leq C_{\mathrm{N}}^{\frac{1}{p}} C_{Q,p}\epsilon^{-\frac{n}{p}} \|\omega\|_{L^p\Lambda^k(\Omega)}, \quad \omega \in L^p\Lambda^k(\Omega). \tag{7.21}$$

Moreover, we have

$$\mathsf{d}Q^k_\epsilon \omega = Q^k_\epsilon \mathsf{d}\omega, \quad \omega \in W^{p,q}\Lambda^k(\Omega), \quad p,q \in [1,\infty]. \tag{7.22}$$

Constants. For the following proof, it suffices that $\epsilon > 0$ is small enough to apply Lemma 7.4 and that $L_E C_h \epsilon < \epsilon_h$. We may assume

$$C_{Q,p} \leq c^k_{\mathrm{M}} C^k_{\mathrm{M}} C_I (1+\epsilon L_h)^k C_h^{\frac{n}{p}} \left(1 + C_E^{k+n/p}\right).$$

Proof. Let $\omega \in L^p\Lambda^k(\Omega)$ and let $T \in \mathcal{T}^n$. Then

$$\|Q^k_\epsilon \omega\|_{L^p\Lambda^k(T)} \leq \|I^k R^k_{\epsilon\mathbf{h}} E^k \omega\|_{L^p\Lambda^k(T)} \leq h_T^{\frac{n}{p}} \|I^k R^k_{\epsilon\mathbf{h}} E^k \omega\|_{L^\infty\Lambda^k(T)}.$$

Transforming to a reference geometry and using the estimate (6.16) gives

$$\begin{aligned}\|I^k R^k_{\epsilon\mathbf{h}} E^k \omega\|_{L^\infty\Lambda^k(T)} &\leq C^k_{\mathrm{M}} h_T^{-k} \|\varphi_T^* I^k R^k_{\epsilon\mathbf{h}} E^k \omega\|_{L^\infty\Lambda^k(\Delta^n)} \\ &\leq C_I C^k_{\mathrm{M}} h_T^{-k} \|\varphi_T^* R^k_{\epsilon\mathbf{h}} E^k \omega\|_{L^\infty\Lambda^k(\Delta^n)} \leq C_I c^k_{\mathrm{M}} C^k_{\mathrm{M}} \|R^k_{\epsilon\mathbf{h}} E^k \omega\|_{C\Lambda^k(T)}.\end{aligned}$$

Assuming that $\epsilon > 0$ is small enough, we apply Lemma 7.4,

$$\|R^k_{\epsilon\mathbf{h}} E^k \omega\|_{C\Lambda^k(T)} \leq (1+\epsilon L_h)^k \epsilon^{-\frac{n}{p}} h_T^{-\frac{n}{p}} C_h^{\frac{n}{p}} \|E^k\omega\|_{L^p\Lambda^k\left(B_{C_h \epsilon h_T}(T)\right)},$$

and find by (4.4) and Corollary 7.2 that

$$\begin{aligned}\|E^k\omega\|_{L^p\Lambda^k\left(B_{C_h \epsilon h_T}(T)\right)} &= \|E^k\omega\|_{L^p\Lambda^k\left(B_{C_h \epsilon h_T}(T) \cap \Omega^e\right)} \\ &\leq \left(1 + C_E^{k+n/p}\right) \|\omega\|_{L^p\Lambda^k\left(B_{L_E C_h \epsilon h_T}(T) \cap \overline{\Omega}\right)}.\end{aligned}$$

The local bound (7.20) is now completed with the observation

$$B_{L_E C_h \epsilon h_T}(T) \cap \overline{\Omega} \subseteq B_{\epsilon_h h_T}(T) \cap \overline{\Omega} \subseteq \mathcal{T}(T).$$

The global bound (7.21) is obtained via

$$\begin{aligned}\|Q^k_\epsilon \omega\|^p_{L^p\Lambda^k(\Omega)} &= \sum_{T\in\mathcal{T}^n} \|Q^k_\epsilon \omega\|^p_{L^p\Lambda^k(T)} \leq C^p_{Q,p} \sum_{T\in\mathcal{T}^n} \|\omega\|^p_{L^p\Lambda^k(\mathcal{T}(T))} \\ &\leq C^p_{Q,p} C_{\mathrm{N}} \sum_{T\in\mathcal{T}^n} \|\omega\|^p_{L^p\Lambda^k(T)} \leq C^p_{Q,p} C_{\mathrm{N}} \|\omega\|^p_{L^p\Lambda^k(\Omega)}\end{aligned}$$

for $p \in [1,\infty)$, and for $p = \infty$ similarly. Finally, (7.22) follows from Theorem 7.3, Theorem 7.4, and our assumptions on I^k. The proof is complete. □

The smoothed interpolant Q^k_ϵ is local and satisfies uniform bounds. Although Q^k_ϵ generally does not reduce to the identity over $\mathcal{P}\Lambda^k(\mathcal{T})$, we can show that, for $\epsilon > 0$ small enough, it is close to the identity and satisfies a local error estimate.

Theorem 7.9. *For $\epsilon > 0$ small enough, there exists uniformly bounded $C_{e,p} > 0$ for every $p \in [1,\infty]$ such that*

$$\|\omega - Q_\epsilon^k \omega\|_{L^p\Lambda^k(T)} \leq \epsilon C_{e,p} \|\omega\|_{L^p\Lambda^k(\mathcal{T}(T))}, \quad \omega \in \mathcal{P}\Lambda^k(\mathcal{T}), \quad T \in \mathcal{T}^n.$$

Constants. It suffices that $\epsilon > 0$ is small enough such that Theorem 7.8 is applicable and that $L_E \mathcal{L} \epsilon < \epsilon_h$ and $3\mathcal{L}\epsilon < \lambda$, where $\mathcal{L}$ and λ are as in the proof below. With the notation as in the following proof. Additionally we may assume

$$C_{e,p} \leq C_{\mathrm{M}}^{2k+1+\frac{n}{p}} c_{\mathrm{M}}^{2k+1} C_I \left(1 + C_E^{k+1+\frac{n}{p}}\right) C_{\flat,p} \mathcal{L} \max(1,\mathfrak{L})^k \left(1 + C_\partial\right).$$

Proof. We prove the statement by a series of inequalities. Let $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ and let $T \in \mathcal{T}^n$. First

$$\begin{aligned}\|\omega - Q_\epsilon^k \omega\|_{L^p\Lambda^k(T)} &\leq \operatorname{vol}^n(T)^{\frac{1}{p}} \|\omega - Q_\epsilon^k \omega\|_{L^\infty\Lambda^k(T)} \\ &\leq h_T^{\frac{n}{p}} \|\omega - Q_\epsilon^k \omega\|_{L^\infty\Lambda^k(T)} = h_T^{\frac{n}{p}} \|E^k\omega - Q_\epsilon^k \omega\|_{L^\infty\Lambda^k(T)}.\end{aligned}$$

With Theorem (3.5) and (4.5) we see

$$\|E^k\omega - Q_\epsilon^k\omega\|_{L^\infty\Lambda^k(T)} \leq C_{\mathrm{M}}^k h_T^{-k} \|\varphi_T^* I^k (E^k\omega - R_{\epsilon\mathtt{h}}^k E^k\omega)\|_{L^\infty\Lambda^k(\varphi_T^{-1}T)}.$$

By the estimate for the canonical interpolant (6.15) we have

$$\|\varphi_T^* I^k (E^k\omega - R_{\epsilon\mathtt{h}}^k E^k\omega)\|_{L^\infty\Lambda^k(\varphi_T^{-1}T)} \leq C_I \sup_{\substack{F\in\Delta(T)\\ S\in\mathcal{PC}_k^F}} \left|\varphi_{T*}^{-1} S\right|_k^{-1} \int_S E^k\omega - R_{\epsilon\mathtt{h}}^k E^k\omega.$$

We need to bound the last expression. Fix $F \in \Delta(T)$ and $S \in \mathcal{PC}_k^F$. We see that

$$\int_S E^k\omega - R_{\epsilon\mathtt{h}}^k E^k\omega = \int_S \int_{\mathbb{R}^n} \mu(y) \left(E^k\omega - \Phi_{\epsilon\mathtt{h},y}^* E^k\omega\right) \mathrm{d}y.$$

We want to change the order of integration between those two integrals. As a technical tool, we use Theorem VI.7A of [27], which implies that integrable continuous differential k-forms over $\mathbb{R}^n$ are densely embedded in the space of flat chains over $\mathbb{R}^n$, such that the pairing of the induced flat chain with a flat differential form is the usual scalar product between k-forms. Consider a sequence of continuous integrable differential k-forms $(\eta_i)_{i\in\mathbb{N}}$ such that $\eta_i \to S$ in $\mathcal{C}_k^\flat(\mathbb{R}^n)$. We then find with Fubini's theorem and the theorem of dominated convergence that

$$\begin{aligned}\int_S \int_{\mathbb{R}^n} \mu(y) \Phi_{\epsilon\mathtt{h},y}^* E^k\omega \,\mathrm{d}y &= \lim_{i\to\infty} \int_{\mathbb{R}^n} \left\langle \eta_i, \int_{\mathbb{R}^n} \mu(y) \Phi_{\epsilon\mathtt{h},y}^* E^k\omega \,\mathrm{d}y \right\rangle \mathrm{d}x \\ &= \int_{\mathbb{R}^n} \mu(y) \lim_{i\to\infty} \int_{\mathbb{R}^n} \left\langle \eta_i, \Phi_{\epsilon\mathtt{h},y}^* E^k\omega \right\rangle \mathrm{d}x \,\mathrm{d}y = \int_{\mathbb{R}^n} \mu(y) \int_S \Phi_{\epsilon\mathtt{h},y}^* E^k\omega \,\mathrm{d}y.\end{aligned}$$

Using these observations and (5.14) again, we have

$$\int_{\mathbb{R}^n} \mu(y) \int_S E^k\omega - \Phi_{\epsilon\mathtt{h},y}^* E^k\omega \,\mathrm{d}y = \int_{\mathbb{R}^n} \mu(y) \int_{\varphi_{T*}^{-1}S - \varphi_{T*}^{-1}\Phi_{\epsilon\mathtt{h},y*}S} \varphi_T^* E^k\omega \,\mathrm{d}y. \tag{7.23}$$

The strategy of the proof is now as follows. Within a uniformly bounded radius of $\varphi_T^{-1}F$ we bound the difference $\mathrm{Id} - \Phi_{\epsilon\mathtt{h},y}$ uniformly in terms of ϵ and y. Consequently, the right-hand side of (7.23) can be bounded by the product of the flat norm of $\varphi_T^* E^k\omega$ within a uniformly bounded radius of $\varphi_T^{-1}F$ with the flat norm of $\varphi_{T*}^{-1}S - \varphi_{T*}^{-1}\Phi_{\epsilon\mathtt{h},y*}S$. The latter can be estimated via Lemma 5.4 with a uniform choice of r and the parameter ϵ chosen sufficiently small.

We first proceed with some auxiliary estimates. Assuming $\epsilon < 1$ for simplicity, we observe that

$$\sup_{y \in B_1(0)} \operatorname{Lip}\left(\varphi_T^{-1} \Phi_{\epsilon \mathtt{h}, y} \varphi_T\right) \leq c_{\mathrm{M}} C_{\mathrm{M}} \left(1 + L_h\right) := \mathfrak{L}.$$

Next, let $\lambda > 0$. For any $y \in B_1(0)$ and $\hat{x} \in B_\lambda(\varphi_T^{-1} F)$ we find that

$$\left\| \hat{x} - \varphi_T^{-1} \Phi_{\epsilon \mathtt{h}, y}(\varphi_T(\hat{x})) \right\| \leq C_{\mathrm{M}} h_T^{-1} \left\| \varphi_T(\hat{x}) - \Phi_{\epsilon \mathtt{h}, y}(\varphi_T(\hat{x})) \right\| \leq C_{\mathrm{M}} h_T^{-1} \epsilon \mathtt{h}(\varphi_T(\hat{x})).$$

Let $x_F \in F$ such that $\|\hat{x} - \varphi_T^{-1}(x_F)\| \leq \lambda$. Then $\|\varphi_T(\hat{x}) - x_F\| \leq \lambda c_{\mathrm{M}} h_T$. Hence

$$\mathtt{h}(\varphi_T(\hat{x})) \leq \mathtt{h}(x_F) + L_h \lambda c_{\mathrm{M}} h_T \leq \left(C_h + L_h \lambda c_{\mathrm{M}}\right) h_T.$$

Assuming $\lambda \leq 1$ for simplicity and writing $\mathcal{L} := C_{\mathrm{M}} \left(C_h + L_h c_{\mathrm{M}}\right)$, we get

$$\sup_{\hat{x} \in B_\lambda(\varphi_T^{-1} F)} \sup_{y \in B_1(0)} \left\| \hat{x} - \varphi_T^{-1} \Phi_{\epsilon \mathtt{h}, y}(\varphi_T(\hat{x})) \right\| \leq \epsilon \mathcal{L}.$$

We continue with the main part of the proof. With (5.5), it follows that

$$\begin{aligned} &\int_{\varphi_{T*}^{-1} S - \varphi_{T*}^{-1} \Phi_{\epsilon \mathtt{h}, y*} S} \varphi_T^* E^k \omega \\ &\qquad \leq \sup_{y \in B_1(0)} \|\varphi_{T*}^{-1} S - \varphi_{T*}^{-1} \Phi_{\epsilon \mathtt{h}, y*} S\|_{k,\flat} \|\varphi_T^* E^k \omega\|_{W^{\infty,\infty}\Lambda^k(B_{\mathcal{L}\epsilon}(\Delta_n))}, \end{aligned}$$

We need to bound this product. To control the first factor, we apply Lemma 5.4. Let $\lambda > 0$ as above and let $\epsilon > 0$ be so small that $\mathcal{L}\epsilon < \lambda/3$. We can then apply Lemma 5.4 with $r = \lambda/3$, such that for all $y \in B_1(0)$ we obtain

$$\begin{aligned} \|\varphi_{T*}^{-1} S - \varphi_{T*}^{-1} \Phi_{\epsilon \mathtt{h}, y*} S\|_{k,\flat} &= \|\varphi_{T*}^{-1} S - \varphi_{T*}^{-1} \Phi_{\epsilon \mathtt{h}, y*} \varphi_{T*} \varphi_{T*}^{-1} S\|_{k,\flat} \\ &\leq \epsilon \cdot \mathcal{L} \cdot \max(1, \mathfrak{L})^k \cdot \left(|\varphi_{T*}^{-1} S|_k + |\partial_k \varphi_{T*}^{-1} S|_{k-1} \right). \end{aligned}$$

The inverse inequality (6.14) gives

$$|\partial_k \varphi_{T*}^{-1} S|_{k-1} \leq C_\partial |\varphi_{T*}^{-1} S|_k.$$

It remains to bound the second factor. We observe

$$\|\varphi_T^* E^k \omega\|_{W^{\infty,\infty}\Lambda^k(B_{\mathcal{L}\epsilon}(\Delta_n))} \leq \left(1 + C_E^{k+1+\frac{n}{p}}\right) c_{\mathrm{M}}^{k+1} C_{\mathrm{M}}^{k+1} \|\varphi_T^* \omega\|_{W^{\infty,\infty}\Lambda^k(\varphi_T^{-1}\mathcal{T}(T))}$$

for ϵ so small that $L_E \mathcal{L}\epsilon < \epsilon_h$. To see this, we apply Lemma 7.2 to get

$$\begin{aligned} &\|\varphi_T^* E^k \omega\|_{L^\infty\Lambda^k(B_{\mathcal{L}\epsilon}(\Delta_n))} \leq c_{\mathrm{M}}^k h_T^k \|E^k \omega\|_{L^\infty\Lambda^k(B_{\mathcal{L}\epsilon h_T}(T))} \\ &\quad \leq \left(1 + C_E^{k+\frac{n}{p}}\right) c_{\mathrm{M}}^k h_T^k \|\omega\|_{L^\infty\Lambda^k(B_{L_E\mathcal{L}\epsilon h_T}(T) \cap \overline{\Omega})} \\ &\quad \leq \left(1 + C_E^{k+\frac{n}{p}}\right) c_{\mathrm{M}}^k h_T^k \|\omega\|_{L^\infty\Lambda^k(\mathcal{T}(T))} \leq \left(1 + C_E^{k+\frac{n}{p}}\right) c_{\mathrm{M}}^k C_{\mathrm{M}}^k \|\varphi_T^* \omega\|_{L^\infty\Lambda^k(\varphi_T^{-1}\mathcal{T}(T))} \end{aligned}$$

and, similarly,

$$\|\varphi_T^* E^{k+1} \mathsf{d}\omega\|_{L^\infty\Lambda^{k+1}(B_{\mathcal{L}\epsilon}(\Delta_n))} \leq \left(1 + C_E^{k+1+\frac{n}{p}}\right) c_{\mathrm{M}}^{k+1} C_{\mathrm{M}}^{k+1} \|\varphi_T^* \mathsf{d}\omega\|_{L^\infty\Lambda^{k+1}(\varphi_T^{-1}\mathcal{T}(T))}.$$

The inverse inequality (6.13) and another pullback estimate give

$$\begin{aligned} \|\varphi_T^* \omega\|_{W^{\infty,\infty}\Lambda^k(\varphi_T^{-1}\mathcal{T}(T))} &\leq C_{\flat,p} \|\varphi_T^* \omega\|_{L^p\Lambda^k(\varphi_T^{-1}\mathcal{T}(T))} \\ &\leq C_{\flat,p} c_{\mathrm{M}}^k C_{\mathrm{M}}^{\frac{n}{p}} h_T^{k-\frac{n}{p}} \|\omega\|_{L^p\Lambda^k(\mathcal{T}(T))}. \end{aligned}$$

This completes the proof. □

Remark 7.10. Our Theorem 7.9 resembles Lemma 5.5 in [1] and Lemma 4.2 in [9]. Our method of proof, however, is different. In order to obtain the interpolation error estimate over simplices $T \in \mathcal{T}$, the authors of the aforementioned references suppose that finite element differential forms are piecewise Lipschitz near T. This holds if T is an interior simplex but not if T touches the boundary of Ω. In what appears to be a gap in the proofs, it is not clear how their method applies for such T. The reason is that their extension operator, like ours, involves a pullback along a bi-Lipschitz mapping, so the extended finite element differential form is not necessarily Lipschitz continuous anywhere outside of Ω. The extended differential form, however, is still a flat form, and this motivates our utilization of geometric measure theory to prove the desired estimate for the interpolation error.

We mention that interpolation error estimates similar to Theorem 7.9 were used earlier in an earlier publication [25], which in turn refers to a technical report for the details of the proof. This technical report, however, has not been published as of the time of this writing, and so comparing our proof of the interpolation error estimate, though desirable, is currently not possible. On the other hand, a uniformly bounded commuting projection is constructed in [8] with different techniques.

We are now in a position to prove the main result of this article. For $\epsilon > 0$ small enough, we can correct the error of the smoothed interpolation over the finite element space. The resulting smoothed projection is, however, non-local.

Theorem 7.11. *Let $\epsilon > 0$ be small enough. There exists a bounded linear operator*

$$\pi_\epsilon^k : L^p\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T}) \subseteq L^p\Lambda^k(\Omega), \quad p \in [1,\infty],$$

such that

$$\pi_\epsilon^k \omega = \omega, \quad \omega \in \mathcal{P}\Lambda^k(\mathcal{T}),$$

such that

$$\mathsf{d}\pi_\epsilon^k \omega = \pi_\epsilon^{k+1}\mathsf{d}\omega, \quad \omega \in W^{p,q}\Lambda^k(\Omega), \quad p,q \in [1,\infty],$$

and such that for all $p \in [1,\infty]$ there exist uniformly bounded $C_{\pi,p} > 0$ with

$$\|\pi_\epsilon^k \omega\|_{L^p\Lambda^k(\mathcal{T})} \leq C_{\pi,p}\epsilon^{-\frac{n}{p}}\|\omega\|_{L^p\Lambda^k(\Omega)}, \quad \omega \in L^p\Lambda^k(\Omega).$$

Constants. We may assume $C_{\pi,p} \leq 2C_{Q,p}C_{\mathrm{N}}^{\frac{1}{p}}$. It suffices that $\epsilon > 0$ is so small that $C_{e,p}\epsilon < 2$ and Theorem 7.8 and Theorem 7.9 apply.

Proof. If $\epsilon > 0$ is small enough and $p \in [1,\infty]$, then Theorem 7.9 implies that

$$\|\omega - Q_\epsilon^k \omega\|_{L^p\Lambda^k(\Omega)} \leq \frac{1}{2}\|\omega\|_{L^p\Lambda^k(\Omega)}, \quad \omega \in \mathcal{P}\Lambda^k(\mathcal{T}).$$

By standard results, the operator $Q_\epsilon^k : \mathcal{P}\Lambda^k(\mathcal{T}) \to \mathcal{P}\Lambda^k(\mathcal{T})$ is invertible. Let $J_\epsilon^k : \mathcal{P}\Lambda^k(\mathcal{T}) \to \mathcal{P}\Lambda^k(\mathcal{T})$ be its inverse. J_ϵ^k does not depend on p, since Q_ϵ^k does not depend on p. The construction of J_ϵ^k via a Neumann series reveals that

$$\|J_\epsilon^k \omega\|_{L^p\Lambda^k(\Omega)} \leq 2\|\omega\|_{L^p\Lambda^k(\Omega)}, \quad \omega \in \mathcal{P}\Lambda^k(\mathcal{T}).$$

So J_ϵ^k is bounded. Moreover, J_ϵ^k commutes with the exterior derivative because

$$\mathsf{d}J_\epsilon^k \omega = J_\epsilon^k Q_\epsilon^k \mathsf{d} J_\epsilon^k \omega = J_\epsilon^k \mathsf{d} Q_\epsilon^k J_\epsilon^k \omega = J_\epsilon^k \mathsf{d}\omega, \quad \omega \in \mathcal{P}\Lambda^k(\mathcal{T}).$$

The theorem follows with $\pi_\epsilon^k := J_\epsilon^k Q_\epsilon^k$. □

Remark 7.12. We compare our construction of the smoothed projection with previous constructions in the literature, with particular focus on the role of the mesh size function. The smoothed projection constructed in [1] applies to *quasi-uniform* families of triangulations. A family of triangulations is called quasi-uniform if for each triangulation $\mathcal{T}$ in that family we have

$$\forall T \in \mathcal{T}^n : h_T^n \leq C_{\mathrm{mesh}} |T|, \tag{7.24}$$

$$\forall T, S \in \mathcal{T} : h_T \leq C_{\mathrm{mesh}} h_S, \tag{7.25}$$

with a common constant $C_{\mathrm{mesh}} > 0$. In that case, a classical smoothing operator can be used instead of our $R^k_{\epsilon\mathbf{h}}$. That result is expanded in [9] to include *shape-uniform* families of triangulations, which means that the conditions (4.1) and (4.2) are satisfied for all triangulations $\mathcal{T}$ in that family with a common constant C_{mesh}.

The Lipschitz continuous mesh size function of Lemma 7.5 is specifically inspired by the construction in [9]. But it is easily seen that, unlike stated in [9, p.821], a regularization operator with that mesh size function does not yield a continuous differential form. For example, if $\Omega = (-1, a) \subset \mathbb{R}$ with $a > 1$ is triangulated by the two intervals $[-1, 0]$ and $[0, a]$, which have unequal lengths, and ω is a non-zero constant 1-form over Ω, then their regularization operator will generally produce discontinuous piecewise smooth differential form. This is due to the differential of the mesh size function being discontinuous in that example. As a remedy, we explicitly construct a mesh size function that is smooth.

But it is insightful to inspect the situation in more detail. The Lipschitz continuous mesh size function in Lemma 7.5 is the limit of the smoothed mesh size function in Lemma 7.7 for decreasing smoothing radius. It is natural to ask how this limit process is reflected in the regularization operator. The gradient of the original mesh size function features tangential continuity. Using this additional property, one can show that the regularization operator of [9] yields differential forms that are piecewise continuous with respect to the triangulation and that are single-valued along simplex boundaries. Consequently, their regularized differential form, though not continuous, still has well-defined degrees of freedom, and the canonical interpolant can be applied as intended. This is also seen in the simple example above. We emphasize that the main result of [9] remains unchanged.

Remark 7.13. Several estimates in this section depend on a Lebesgue exponent $p \in [1, \infty]$. We carefully observe that it suffices to consider only the case $p = 1$. Then a sufficiently small choice of $\epsilon > 0$ enables Theorem 7.11 for all $p \in [1, \infty]$ simultaneously.

Remark 7.14. Throughout this section, we have provided explicit formulas for the admissible ranges of ϵ and the various constants. With the exception of C_E, ϵ_Ω, ϵ_h and L_Ω, the quantities in those formulas are computable in terms of the ambient dimension, the polynomial degree, and the mesh regularity.

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