SMOOTHED PROJECTIONS AND MIXED BOUNDARY CONDITIONS

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ABSTRACT. We discuss mixed boundary conditions in finite element exterior calculus. Our major contribution are smoothed projections from Sobolev de Rham complexes onto finite element de Rham complexes which commute with the exterior derivative, preserve homogeneous boundary conditions along a fixed boundary part, and satisfy uniform bounds for shape-regular families of triangulations and bounded polynomial degree. The existence of such projections implies stability and quasi-optimal convergence of mixed finite element methods for the Hodge Laplace equation with mixed boundary conditions. In addition, we contribute a proof of the density of smooth differential forms in Sobolev spaces of differential forms over weakly Lipschitz domains with partial boundary conditions.

1. INTRODUCTION

In this article, we address the numerical analysis of the Hodge Laplace equation when *mixed boundary conditions* are imposed. Here, we speak of mixed boundary conditions when essential boundary conditions are imposed on one part of the boundary, while natural boundary conditions are imposed on the complementary boundary part. Special cases are the Poisson equation with mixed Dirichlet and Neumann boundary conditions [47], and the vector Laplace equation with mixed tangential and normal boundary conditions [27]. It is known in the theory of partial differential equations that the Hodge Laplace equation with mixed boundary conditions arises from Sobolev de Rham complexes with *partial boundary conditions* [33, 28]. These are composed of spaces of Sobolev differential forms in which boundary conditions are imposed only on a part of the boundary (corresponding to the essential boundary conditions).

Moving towards the numerical analysis of mixed finite element methods for the Hodge Laplace equation with mixed boundary conditions, we adopt the framework of *finite element exterior calculus* (FEEC, [3]). Smoothed projections from Sobolev de Rham complexes onto finite element de Rham complexes play a central role in the a priori error analysis within FEEC. Previous contributions [3, 5] provided the corresponding smoothed projections in the special cases of either fully essential or fully natural boundary conditions in FEEC, but the general case of mixed boundary conditions has remained open. In order to overcome this limitation and enable the

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abstract Galerkin theory of FEEC, we need a smoothed projection that preserves partial boundary conditions.

Constructing such a smoothed projection is the main contribution of this article. The abstract Galerkin theory of Hilbert complexes then provides quasi-optimal convergence of a large class of mixed finite element methods.

Mixed boundary conditions for vector-valued partial differential equations are a non-trivial topic, and even more so in numerical analysis. We outline the topic and prior research, starting with the Poisson problem with mixed boundary conditions. Suppose that Ω is a bounded Lipschitz domain with outward normal field $\vec{n}$ along $\partial\Omega$. We assume that $\Gamma_D \subseteq \partial\Omega$ is a subset of the boundary with positive surface measure and let $\Gamma_N := \partial\Omega \setminus \Gamma_D$. Given a function f, the Poisson problem with mixed boundary conditions is finding the solution u of

$$-\Delta u = f, \quad u_{|\Gamma_D} = 0, \quad \nabla u_{|\Gamma_N} \cdot \vec{n} = 0. \tag{1.1}$$

Here, we impose a homogeneous *Dirichlet boundary condition* along Γ_D and a homogeneous *Neumann boundary condition* along Γ_N. If $f \in L^2(\Omega)$, then a weak formulation characterizes the solution as the unique minimizer of the energy

$$\mathcal{J}(v) := \frac{1}{2}\int_\Omega |\operatorname{grad} v|^2 \, \mathrm{d}x - \int_\Omega fv \, \mathrm{d}x \tag{1.2}$$

over $H^1(\Omega, \Gamma_D)$, the subspace of $H^1(\Omega)$ whose members satisfy the (essential) Dirichlet boundary condition along Γ_D. The well-posedness of this variational problem follows by a Friedrichs inequality with partial boundary conditions [47]. Moreover, the compactness of the embedding $H^1(\Omega, \Gamma_D) \to L^2(\Omega)$ is crucial in proving the compactness of the solution operator. A typical finite element method seeks a discrete approximation of u by minimizing $\mathcal{J}$ over a space of Lagrange elements in $H^1(\Omega, \Gamma_D)$. The discussion of this Galerkin method is standard [11], but still we cannot approach the Poisson problem with mixed boundary conditions by the current means of FEEC due to the lack of a smoothed projection.

The natural generalization to vector-valued problems in three dimensions is given by the vector Laplace equation with mixed boundary conditions. This equation appears in electromagnetism or fluid dynamics. The analysis of this vector-valued partial differential equation, however, is considerably more complex. Given the vector field $\vec{f}$, we seek a vector field $\vec{u}$ that solves

$$\operatorname{curl}\operatorname{curl}\vec{u} - \operatorname{grad}\operatorname{div}\vec{u} = \vec{f}, \tag{1.3}$$

over the domain Ω. Moreover we assume that $\partial\Omega = \Gamma_T \cup \Gamma_N$ is an essentially disjoint partition of the boundary into relatively open subsets; geometric details are discussed later in this article. The boundary conditions on $\vec{u}$ are

$$\vec{u}_{|\Gamma_T} \times \vec{n} = 0, \quad (\operatorname{div}\vec{u})_{|\Gamma_T} = 0, \quad \vec{u}_{|\Gamma_N} \cdot \vec{n} = 0, \quad (\operatorname{curl}\vec{u})_{|\Gamma_N} \times \vec{n} = 0. \tag{1.4}$$

Here we impose homogeneous *tangential boundary conditions* on $\vec{u}$ along a boundary part Γ_T, and homogeneous *normal boundary conditions* on $\vec{u}$ along the complementary boundary part Γ_N. When $\vec{f} \in L^2(\Omega, \mathbb{R}^3)$, then a variational formulation seeks the solution by minimizing the energy functional

$$\mathcal{J}(\vec{v}) := \frac{1}{2}\int_\Omega |\operatorname{div}\vec{v}|^2 + |\operatorname{curl}\vec{v}|^2 \, \mathrm{d}x - \int_\Omega \vec{f} \cdot \vec{v} \, \mathrm{d}x \tag{1.5}$$

over the space $H(\mathrm{div},\Omega,\Gamma_N)\cap H(\mathrm{curl},\Omega,\Gamma_T)$. Here $H(\mathrm{div},\Omega,\Gamma_N)$ is the subspace of $H(\mathrm{div},\Omega)$ satisfying normal boundary conditions along Γ_N, and $H(\mathrm{curl},\Omega,\Gamma_T)$ is the subspace of $H(\mathrm{curl},\Omega)$ satisfying tangential boundary conditions along Γ_T.

The additional complexity in comparison to the scalar-valued case begins with the correct definition of tangential and normal boundary conditions in a setting of low regularity [53, 13, 14, 54, 28]. When non-mixed boundary conditions are imposed, so that either $\Gamma_T=\emptyset$ or $\Gamma_T=\partial\Omega$, then Rellich-type compact embeddings $H(\mathrm{div},\Omega,\Gamma_N)\cap H(\mathrm{curl},\Omega,\Gamma_T)\to L^2(\Omega,\mathbb{R}^3)$ and vector-valued Poincaré-Friedrichs inequalities have been known for a long time [52, 48, 56, 18]. Mixed boundary conditions in vector analysis, however, have only recently been addressed systematically in pure analysis [34, 37, 35, 1, 7].

Additional difficulties arise in numerical analysis. Minimizing (1.5) over a finite element subspace of $H(\mathrm{div},\Omega,\Gamma_N)\cap H(\mathrm{curl},\Omega,\Gamma_T)$ generally does not lead to a consistent finite element method [19, 5]. But mixed finite element methods, which introduce either $\mathrm{div}\,\vec{u}$ or $\mathrm{curl}\,\vec{u}$ as auxiliary variables, have been studied with great success [10, 32, 46, 20]. Mixed boundary conditions for the vector Laplace equation, however, have not yet received much attention in numerical analysis (but see [49, 31]). In a mixed finite element method for the vector Laplace equation with mixed boundary conditions we only incorporate the essential boundary conditions along Γ_T into the finite element space.

We attend particularly to a phenomenon that significantly affects the theoretical and numerical analysis of the vector Laplace equation but remains absent in the scalar-valued theory: the presence of non-trivial harmonic vector fields in $H(\mathrm{div},\Omega,\Gamma_N)\cap H(\mathrm{curl},\Omega,\Gamma_T)$. Specifically, let $\vec{\mathcal{H}}(\Omega,\Gamma_T,\Gamma_N)$ denote the subspace of $H(\mathrm{div},\Omega,\Gamma_N)\cap H(\mathrm{curl},\Omega,\Gamma_T)$ whose members have vanishing curl and vanishing divergence. This space has physical relevance: in fluid dynamics, for example, those vector fields describe the incompressible irrotational flows that satisfy given boundary conditions. In the case of non-mixed boundary conditions, their dimension corresponds to topological properties of the domain [42], and in particular that dimension is zero on contractible domains. But in the case of mixed boundary conditions, this dimension depends on the topology of both the domain Ω and the boundary part Γ_T. Thus $\vec{\mathcal{H}}(\Omega,\Gamma_T,\Gamma_N)$ may have positive dimension if Γ_T has a sufficiently complicated topology even if Ω itself is contractible [36, 28]. This dimension can be calculated exactly from a given triangulation of Ω and Γ_T. In a finite element method, the subspace $\vec{\mathcal{H}}(\Omega,\Gamma_T,\Gamma_N)$ must be approximated by discrete harmonic fields, i.e., the kernel of the finite element vector Laplacian.

It is instructive to study these partial differential equations in a unified manner using the calculus of differential forms. Both the Poisson problem and the vector Laplace equation with mixed boundary conditions are special cases of the Hodge Laplace equation with mixed boundary conditions. The Hodge Laplace equation has been studied extensively over Sobolev spaces of differential forms [53, 12, 51, 37, 44, 43, 6, 45, 54]. The case of mixed boundary conditions has been a recent subject of research in the field of analysis on manifolds [28, 33]. The theoretical foundation are de Rham complexes with partial boundary conditions. Let $L^2\Lambda^k(\Omega)$ be the space of differential forms with square-integrable coefficients, and let $H\Lambda^k(\Omega,\Gamma_T)$ be the subspace of $L^2\Lambda^k(\Omega)$ whose members have exterior derivatives in $L^2\Lambda^{k+1}(\Omega)$ and satisfy boundary conditions along Γ_T. Then we consider the differential complex

$$\dots \xrightarrow{\mathsf{d}} H\Lambda^k(\Omega,\Gamma_T) \xrightarrow{\mathsf{d}} H\Lambda^{k+1}(\Omega,\Gamma_T) \xrightarrow{\mathsf{d}} \dots \tag{1.6}$$

where d denotes the exterior derivative. The widely studied special cases $\Gamma_T = \emptyset$ and $\Gamma_T = \partial\Omega$ correspond to either imposing no essential boundary conditions at all or essential boundary conditions along the entire boundary.

The calculus of differential forms has attracted interest as a unifying framework for mixed finite element methods [32, 3, 5, 25, 2, 21]. The numerical analysis of mixed finite element methods for the Hodge Laplace equation can be formulated in terms of finite element de Rham complexes, which mimic the differential complex (1.6) on a discrete level. Finite element exterior calculus is a formalization of that approach. But whereas the special cases of non-mixed boundary conditions are standard applications, mixed boundary conditions (corresponding to more general choices of Γ_T) have not been addressed in this context yet.

We outline the corresponding finite element de Rham complexes. We let $\mathcal{T}$ be a triangulation of Ω that also contains a triangulation $\mathcal{U}$ of Γ_T. Moreover, we let $\mathcal{P}\Lambda^k(\mathcal{T})$ denote a space of piecewise polynomial differential forms in $H\Lambda^k(\Omega)$ of the type $\mathcal{P}_r\Lambda^k(\mathcal{T})$ or $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ as described in [3, 4]. Then the finite element space with essential boundary conditions is $\mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) := \mathcal{P}\Lambda^k(\mathcal{T}) \cap H\Lambda^k(\Omega,\Gamma_T)$. We classify a family of finite element de Rham complexes

$$\dots \xrightarrow{\mathsf{d}} \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}} \mathcal{P}\Lambda^{k+1}(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}} \dots \tag{1.7}$$

that feature these essential boundary conditions and guide the construction of stable mixed finite element methods, completely analogous to the classification of finite element de Rham complexes in the case of non-mixed boundary conditions [3].

To relate the continuous and discrete levels and enable the abstract Galerkin theory of FEEC ([5]), we need a projection $\pi^k : H\Lambda^k(\Omega,\Gamma_T) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$ onto the finite element space that is uniformly L^2 bounded and commutes with the differential operator. In particular, the following diagram commutes:

$$\begin{array}{ccccccc} \dots & \xrightarrow{\mathsf{d}} & H\Lambda^k(\Omega,\Gamma_T) & \xrightarrow{\mathsf{d}} & H\Lambda^{k+1}(\Omega,\Gamma_T) & \xrightarrow{\mathsf{d}} & \dots \\ & & \downarrow{\scriptstyle \pi^k} & & \downarrow{\scriptstyle \pi^{k+1}} & & \\ \dots & \xrightarrow{\mathsf{d}} & \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}} & \mathcal{P}\Lambda^{k+1}(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}} & \dots \end{array} \tag{1.8}$$

Given such a projection, we obtain a priori convergence results for mixed finite element methods [5]. A specific example are the smoothed projections which have been developed in finite element exterior calculus [3, 17, 39] for non-mixed boundary conditions.

The main innovation of this article is devising the smoothed projection π^k for de Rham complexes with partial boundary conditions. Moreover, we bound the operator norm in terms of the mesh quality and the polynomial degree of the finite element spaces.

Continuing the research in [39], we assume minimal geometric regularity and conduct our construction over *weakly Lipschitz domains*, which is a class of domains generalizing classical (strongly) Lipschitz domains. In order to define mixed boundary conditions, we partition the boundary of the domain into two complementary parts on which we impose essential or natural boundary conditions, respectively. We assume only minimal regularity for the boundary partition. This choice of geometric ambient has favorable properties. On the one hand, the class of weakly Lipschitz domains is broad enough to contain (strongly) Lipschitz domains and a large class of three-dimensional polyhedral domains that fail to be strongly Lipschitz, such as the crossed brick domain [46, Figure 3.1]. On the other hand, many analytical results that are known for strongly Lipschitz domains, such as the Rellich embedding theorem for differential forms, still hold true over weakly Lipschitz domains (see [6, 7]). Notably, a restriction to strongly Lipschitz domains does not simplify the mathematical derivations in this article. In continuity with [39], we define the smoothed projections over differential forms with coefficients in L^p spaces for $p \in [1, \infty]$, and specifically consider the $W^{p,q}$ classes of differential forms [29].

We give an outline of the stages that compose the smoothed projection. Let $u \in L^2\Lambda^k(\Omega)$. First, an operator $E^k : L^2\Lambda^k(\Omega) \to L^2\Lambda^k(\Omega^{\mathrm{e}})$ extends u over a neighborhood Ω^{e} of Ω. The basic idea is extending the differential form by reflection, but along Γ_T we extend it by zero over a "bulge" attached to the domain. E^k commutes with the exterior derivative on $H\Lambda^k(\Omega, \Gamma_T)$. Next, we construct a distortion $\mathfrak{D} : \Omega^{\mathrm{e}} \to \Omega^{\mathrm{e}}$ which moves a neighborhood of the bulge into the latter but is the identity away from the bulge. We locally control the amount of distortion. The pullback $\mathfrak{D}^* E^k u$ of $E^k u$ along $\mathfrak{D}$ vanishes then in a neighborhood of Γ_T and commutes with the exterior derivative. Subsequently, we introduce a mollification operator $R^k : L^2\Lambda^k(\Omega^{\mathrm{e}}) \to C^\infty\Lambda^k(\overline{\Omega})$ which produces a smoothing of $\mathfrak{D}^* E^k u$ that still vanishes in a neighborhood of Γ_T. This is based on the idea of taking the convolution with a smooth bump function. In our case, however, the mollification radius is locally controlled.

We apply the canonical finite element interpolant $I^k : C^\infty\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T})$ to the regularized differential form $R^k \mathfrak{D}^* E^k u$. Since the latter vanishes near Γ_T, the resulting differential form is an element of $\mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{U})$. In combination, this yields an operator $Q^k : L^2\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{U})$ that commutes with the exterior derivative on $H\Lambda^k(\Omega, \Gamma_T)$ and satisfies uniform L^2 bounds. But Q^k is generally not idempotent. To enforce idempotence, we prove a bound on the interpolation error over the finite element space and apply the "Schöberl trick" [49]. This delivers the desired smoothed projection. In particular, we prove the following main result.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded weakly Lipschitz domain, and let $\Gamma_T \subseteq \partial\Omega$ be an admissible boundary patch (in the sense of Section 3). Let $\mathcal{T}$ be a simplicial triangulation of Ω that contains a simplicial triangulation $\mathcal{U}$ of Γ_T, and let* (1.7) *be a finite element de Rham complex as in finite element exterior calculus* [3] *with essential boundary conditions along Γ_T. Then there exist bounded linear projections $\pi^k : L^2\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{U})$ such that the following diagram commutes:*

$$\begin{array}{ccccccc} H\Lambda^0(\Omega,\Gamma_T) & \xrightarrow{\mathsf{d}} & H\Lambda^1(\Omega,\Gamma_T) & \xrightarrow{\mathsf{d}} & \cdots & \xrightarrow{\mathsf{d}} & H\Lambda^n(\Omega,\Gamma_T) \\ \downarrow{\scriptstyle \pi^0} & & \downarrow{\scriptstyle \pi^1} & & & & \downarrow{\scriptstyle \pi^n} \\ \mathcal{P}\Lambda^0(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}} & \mathcal{P}\Lambda^1(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}} & \cdots & \xrightarrow{\mathsf{d}} & \mathcal{P}\Lambda^n(\mathcal{T},\mathcal{U}). \end{array} \tag{1.9}$$

Moreover, $\pi^k u = u$ for $u \in \mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{U})$. The L^2 operator norm of π^k is uniformly bounded in terms of the maximum polynomial degree of (1.7)*, the shape measure of the triangulation, and geometric properties of Ω and Γ_T.*

Commuting projections have been used in finite element analysis for a long time, and the calculus of differential forms has emerged as a unifying language for finite element methods for problems in vector analysis [32]. A bounded projection operator that commutes with the exterior derivative up to a controllable error was derived in [15]. A bounded commuting projection operator for the de Rham complex without boundary conditions has been derived in [3] in the case of quasi-uniform triangulations, which was subsequently generalized in [17] to shape-uniform triangulations and de Rham complexes with full boundary conditions. The ideas in those contributions were extended to smoothed projections over weakly Lipschitz domains in [39], which we take as our point of departure. The existence of a smoothed projection that respects partial boundary conditions has been an unproven conjecture in [9]. Commuting projections have been derived in [16] and [24] with different methods. A local bounded interpolation operator was presented in [50] in the language of classical vector analysis; this result was generalized to differential forms in [21], and a variant of this operator that preserves partial boundary condition was given in [31]. A commuting projection for spaces with weighted norms that arise in the numerical analysis of axisymmetric Maxwell's equation was given in [30]. A local commuting projection is studied in [25].

In addition to this research in numerical analysis, we contribute a result to functional analysis. Specifically, we prove that smooth differential forms over a weakly Lipschitz domain Ω which vanish near Γ_T are dense in $H\Lambda^k(\Omega, \Gamma_T)$ for $p, q \in [1, \infty)$. When Ω is a (strongly) Lipschitz domain and $\Gamma_D \subseteq \partial\Omega$ is a suitable boundary patch, then the density of $C^\infty(\overline{\Omega}) \cap H^1(\Omega, \Gamma_D)$ in $H^1(\Omega, \Gamma_D)$ (see [22, 23]) and analogous density result for differential forms with partial boundary conditions over strongly Lipschitz domains (see [33]) are known. The following generalization to weakly Lipschitz domains, however, has been not available in the literature yet.

Lemma 1.2. *Let Ω be a bounded weakly Lipschitz domain and let Γ_T be an admissible boundary patch (in the sense of Section 3). Then the smooth differential k-forms in $C^\infty\Lambda^k(\overline{\Omega})$ that vanish near Γ_T constitute a dense subset of $H\Lambda^k(\Omega, \Gamma_T)$.*

The remainder of this article is structured as follows. We review the calculus of differential forms in Section 2. In Section 3, we introduce the geometric setting and the extension operator. In Section 4, devise the distortion mapping. In Section 5, we combine these constructions to obtain the mollification operator. In Section 6, we devise the smoothed projection. Finally, Section 7 outlines applications to the convergence theory of finite element methods.

2. Lipschitz Analysis and Differential Forms

In this section we recall background material in several fields of analysis. This includes a summary of Section 3 of [39] with an additional discussion of differential forms over domains that satisfy homogeneous boundary conditions along subsets of the boundary. We draw from sources in Lipschitz analysis [41], geometric measure theory [26], the calculus of differential forms over domains [38], and differential forms with coefficients in L^p spaces [29, 28].

2.1. **Analytical Preliminaries.** We recall some basic notions. Unless mentioned otherwise, we let every subset $A \subseteq \mathbb{R}^n$, $n \in \mathbb{N}_0$, be equipped with the canonical Euclidean norm $\|\cdot\|$. For any set $U \subseteq \mathbb{R}^n$ and $r > 0$ we write $B_r(U)$ for the closed

Euclidean r-neighborhood of U and set $B_r(x) := B_r(\{x\})$. Moreover, for any subset $A \subseteq \mathbb{R}^n$ we let $\operatorname{vol}^m(A)$ be the m-dimensional volume of A.

Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$, and let $\Phi : U \to V$ be a mapping. For a subset $A \subseteq U$ we define $\operatorname{Lip}(\Phi, A) \in [0, \infty]$ as the minimal member of $[0, \infty]$ such that

$$\forall x, y \in A : \|\Phi(x) - \Phi(y)\| \leq \operatorname{Lip}(\Phi, A)\|x - y\|.$$

We call $\operatorname{Lip}(\Phi, A)$ the *Lipschitz constant* of Φ over A and we simply write $\operatorname{Lip}(\Phi) := \operatorname{Lip}(\Phi, A)$ if A is understood. We say that Φ is Lipschitz if $\operatorname{Lip}(\Phi, U) < \infty$. We call Φ *locally Lipschitz* or *LIP* if for every $x \in U$ there exists a relatively open neighborhood $A \subseteq U$ of x such that $\operatorname{Lip}(\Phi, A) < \infty$. If $\Phi : U \to V$ is invertible, then we call Φ *bi-Lipschitz* (*lipeomorphism*) if both Φ and Φ^{-1} are Lipschitz (locally Lipschitz). We call Φ a *LIP embedding* if $\Phi : U \to \Phi(U)$ is a lipeomorphism.

2.2. **Differential Forms.** Let $U \subseteq \mathbb{R}^n$ be an open set. We let $M(U)$ denote the space of Lebesgue-measurable functions over U. For $k \in \mathbb{Z}$ we let $M\Lambda^k(U)$ be the space of differential k-forms over U with coefficients in $M(U)$. For $u \in M\Lambda^k(U)$ and $v \in M\Lambda^l(U)$ we let $u \wedge v \in M\Lambda^{k+l}(U)$ denote the *exterior product* of u and v; we recall that $u \wedge v = (-1)^{kl} v \wedge u$. We let $C^\infty\Lambda^k(U)$ be the space of smooth differential forms over U, we let $C^\infty\Lambda^k(\overline{U})$ denote the space of smooth differential forms over U that are restrictions of members of $C^\infty\Lambda^k(\mathbb{R}^n)$, and we let $C_c^\infty\Lambda^k(U)$ be the subspace of $C^\infty\Lambda^k(\overline{U})$ whose members have compact support.

We let $\mathsf{d}x^1, \dots, \mathsf{d}x^n \in M\Lambda^1(U)$ be the constant 1-forms that represent the n coordinate directions. We let $\langle u, v \rangle \in M(U)$ denote the pointwise ℓ^2 product of two measurable differential k-forms $u, v \in M\Lambda^k(U)$ (see Equation (3.3) in [39] for a definition). Accordingly, we let $|u| = \sqrt{\langle u, u \rangle} \in M(U)$ be the pointwise ℓ^2 norm of $u \in M(U)$. We recall that the *Hodge star operator* $\star : M\Lambda^k(U) \to M\Lambda^{n-k}(U)$ is a linear mapping that is uniquely defined by the identity

$$u \wedge \star v = \langle u, v \rangle \, \mathsf{d}x^1 \wedge \cdots \wedge \mathsf{d}x^n \quad u, v \in M\Lambda^k(U). \tag{2.1}$$

We recall the *exterior derivative* $\mathsf{d} : C^\infty\Lambda^k(U) \to C^\infty\Lambda^{k+1}(U)$ of smooth differential forms. More generally, if $u \in M\Lambda^k(U)$ and $w \in M\Lambda^{k+1}(U)$ such that

$$\int_U w \wedge v = (-1)^{k+1} \int_U u \wedge \mathsf{d}v, \quad v \in C_c^\infty\Lambda^{n-k-1}(U), \tag{2.2}$$

then we call w the *weak exterior derivative* of u and write $\mathsf{d}u := w$. Note that w is unique up to equivalence almost everywhere in U, and that the weak exterior derivative of $u \in C^\infty\Lambda^k(\overline{U})$ agrees with the (strong) exterior derivative almost everywhere. In the sequel, weak exterior derivatives are simply called exterior derivatives. The Hodge star enters the definition of the *exterior codifferential*, which is a differential operator given (in the strong sense) by

$$\delta : C^\infty\Lambda^k(U) \to C^\infty\Lambda^{k-1}(U), \quad u \mapsto (-1)^{k(n-k)+1} \star \mathsf{d} \star u.$$

A weak exterior codifferential can be defined analogously.

2.3. $W^{p,q}$ **differential forms.** We work with differential forms whose coefficients are contained in Lebesgue spaces. We let $L^p(U)$ denote the Lebesgue space over U with exponent $p \in [1, \infty]$ and let $L^p\Lambda^k(U)$ as the Banach space of differential k-forms with coefficients in $L^p(U)$, together with the norm

$$\|u\|_{L^p\Lambda^k(U)} := \big\| |u| \big\|_{L^p(U)}, \quad u \in L^p\Lambda^k(U), \quad p \in [1, \infty].$$

For $p, q \in [1, \infty]$, we let $W^{p,q}\Lambda^k(U)$ be the space of differential k-forms in $L^p\Lambda^k(U)$ which have a weak exterior derivative in $L^q\Lambda^{k+1}(U)$. This is a Banach space with the norm

$$\|u\|_{W^{p,q}\Lambda^k(U)} = \|u\|_{L^p\Lambda^k(U)} + \|\mathsf{d}u\|_{L^q\Lambda^{k+1}(U)}. \tag{2.3}$$

Since the exterior derivative of an exterior derivative is zero, we verify for every choice of $p, q, r \in [1, \infty]$ the inclusion

$$\mathsf{d}W^{p,q}\Lambda^k(U) \subseteq W^{q,r}\Lambda^{k+1}(U).$$

Example 2.1. The space $W^{1,1}\Lambda^k(U)$ contains all integrable differential k-forms over U with integrable weak exterior derivative. If U is bounded, then $W^{1,1}\Lambda^k(U)$ contains all other spaces $W^{p,q}\Lambda^k(U)$ as embedded subspaces. The space $W^{2,2}\Lambda^k(U)$, consisting of square-integrable differential k-forms with square-integrable exterior derivative, has a Hilbert space structure equivalent to its Banach space structure, and is often written $H\Lambda^k(U)$ in the literature. The space $W^{\infty,\infty}\Lambda^k(U)$ consists of essentially bounded differential k-forms with essentially bounded exterior derivative, which are called *flat differential forms* in geometric measure theory [26, 55].

In this article we are particularly interested in spaces of differential forms that satisfy homogeneous boundary conditions along a subset Γ of the boundary ∂U. We call these *partial boundary conditions*. We define boundary conditions in the manner of Definition 3.3 of [28], which is based on the intuition that a differential form with weak exterior derivative satisfies homogeneous boundary conditions along the boundary part Γ if its extension by zero still has a weak exterior derivative along that boundary part.

Formally, when $\Gamma \subseteq \partial U$ is a relatively open subset of ∂U, then we define the space $W^{p,q}\Lambda^k(U, \Gamma)$ as the subspace of $W^{p,q}\Lambda^k(U)$ whose members adhere to the following condition: we have $u \in W^{p,q}\Lambda^k(U, \Gamma)$ if and only if for all $x \in \Gamma$ there exists $r > 0$ such that

$$\int_{U \cap B_r(x)} u \wedge \mathsf{d}v = \int_{U \cap B_r(x)} \mathsf{d}u \wedge v, \quad v \in C_c^\infty\Lambda^{n-k-1}\left(\mathring{B}_r(x)\right). \tag{2.4}$$

The definition implies that $W^{p,q}\Lambda^k(U, \Gamma)$ is a closed subspace of $W^{p,q}\Lambda^k(U)$, and hence a Banach space of its own. We also say that $u \in W^{p,q}\Lambda^k(U, \Gamma)$ satisfies partial boundary conditions along Γ. One consequence of the definition is

$$\mathsf{d}W^{p,q}\Lambda^k(U, \Gamma) \subseteq W^{q,r}\Lambda^{k+1}(U, \Gamma), \quad p, q, r \in [1, \infty]. \tag{2.5}$$

In other words, a differential form which satisfies boundary conditions along Γ has an exterior derivative satisfying boundary conditions along Γ.

Remark 2.2. Note that the identity (2.4) resembles the integration-by-parts identity in the definition of the weak exterior derivative. The trivial extension of any $u \in W^{p,q}\Lambda^k(\Omega, \Gamma)$ outside of U has a weak exterior derivative over $\mathbb{R}^n \setminus (\partial U \setminus \Gamma)$. For example, $W^{p,q}\Lambda^k(U, \partial U)$ is the subspace of $W^{p,q}\Lambda^k(U)$ whose member's extension to $\mathbb{R}^n$ by zero gives a member of $W^{p,q}\Lambda^k(\mathbb{R}^n)$. If the domain boundary is sufficiently regular, then an equivalent notion of homogeneous boundary conditions uses generalized trace operators [43, 54]. This article does not address inhomogeneous boundary conditions.

We finish this section with some results on the behavior of differential forms under bi-Lipschitz coordinate changes. Suppose that $U, V \subseteq \mathbb{R}^n$ are connected open sets and let $\Phi : U \to V$ be a bi-Lipschitz mapping. The pullback of $u \in M\Lambda^k(V)$ along Φ is the differential form $\Phi^* u \in M\Lambda^k(U)$ given at $x \in U$ by

$$\Phi^* u_{|x}(\nu_1, \dots, \nu_k) := u_{|\Phi(x)}(\mathrm{D}\,\Phi_{|x} \cdot \nu_1, \dots, \mathrm{D}\,\Phi_{|x} \cdot \nu_k), \quad \nu_1, \dots, \nu_k \in \mathbb{R}^n. \tag{2.6}$$

Whenever $u \in M\Lambda^k(U)$ has a weak exterior derivative, then

$$\mathsf{d}\Phi^* u = \Phi^* \mathsf{d} u. \tag{2.7}$$

The pullback along bi-Lipschitz mappings preserves the L^p classes of differential forms. We henceforth write $1/\infty := 0$. One can show that for $p \in [1, \infty]$ and $u \in L^p\Lambda^k(V)$ we have $\Phi^* u \in L^p\Lambda^k(U)$ with

$$\begin{aligned} \|\Phi^* u\|_{L^p\Lambda^k(U)} &\leq \|\mathrm{D}\,\Phi\|^k_{L^\infty(U,\mathbb{R}^{n\times n})} \left\|\det \mathrm{D}\,\Phi^{-1}\right\|^{\frac{1}{p}}_{L^\infty(V)} \|u\|_{L^p\Lambda^k(V)} \\ &\leq \|\mathrm{D}\,\Phi\|^k_{L^\infty(U,\mathbb{R}^{n\times n})} \left\|\mathrm{D}\,\Phi^{-1}\right\|^{\frac{n}{p}}_{L^\infty(V,\mathbb{R}^{n\times n})} \|u\|_{L^p\Lambda^k(V)}. \end{aligned} \tag{2.8}$$

Moreover, the $W^{p,q}$ classes of differential forms are preserved under pullbacks. More precisely, for $p, q \in [1, \infty]$ and $u \in W^{p,q}\Lambda^k(V)$ we have $\Phi^* u \in W^{p,q}\Lambda^k(U)$. This is immediately clear when combining (2.7) and (2.8).

3. Boundary Partitions of Weakly Lipschitz Domains

In this section we discuss weakly Lipschitz domains and describe boundary partitions with minimal regularity assumptions. Besides setting the geometric background, we introduce a commuting extension operator that takes into account partial boundary conditions. Our discussion of weakly Lipschitz domains and boundary partitions is based on [28].

3.1. **Weakly Lipschitz Domains and Boundary Partitions.** Let $\Omega \subseteq \mathbb{R}^n$ be a domain. We call Ω a *weakly Lipschitz domain* if every $x \in \partial\Omega$ has a closed neighborhood $U_x \subseteq \mathbb{R}^n$ for which there exists a bi-Lipschitz mapping $\varphi_x : U_x \to [-1,1]^n$ such that $\varphi_x(x) = 0$ and

$$\varphi_x(\Omega \cap U_x) = [-1,1]^{n-1} \times [-1, 0), \tag{3.1a}$$

$$\varphi_x(\partial\Omega \cap U_x) = [-1,1]^{n-1} \times \{0\}, \tag{3.1b}$$

$$\varphi_x(\overline{\Omega}^c \cap U_x) = [-1,1]^{n-1} \times (0, 1]. \tag{3.1c}$$

Remark 3.1. In other words, Ω is a weakly Lipschitz domain if its boundary can be flattened locally by a bi-Lipschitz mapping. For example, every Lipschitz domain (a domain whose boundary can be written locally as the graph of a Lipschitz function) is also a weakly Lipschitz domain. The converse is generally false, and a well-known counter example are the "crossed bricks" [46, p.39]. But every polyhedral domain in $\mathbb{R}^3$ is a weakly Lipschitz domain [39, Theorem 4.1].

Next we introduce the geometric background for the discussion of boundary conditions. We assume that $\Gamma_T \subseteq \partial\Omega$ is a topological submanifold of $\partial\Omega$ of dimension $n-1$ with boundary. The complement $\Gamma_N := \partial\Omega \setminus \overline{\Gamma}_T^c$ is again a topological submanifold of $\partial\Omega$ of dimension $n-1$ with boundary. We write $\Gamma_I := \partial\Gamma_T$ for the boundary of Γ_T, which is also the boundary of Γ_N. We call Γ_T an *admissible boundary patch* if for any $x \in \Gamma_I$ we can choose a bi-Lipschitz mapping $\varphi_x : U_x \to [-1,1]$ such that (3.1) holds and additionally

$$\varphi_x(\Gamma_T \cap U_x) = [-1,1]^{n-2} \times [-1, 0) \times \{0\}, \tag{3.2a}$$

$$\varphi_x(\Gamma_I \cap U_x) = [-1,1]^{n-2} \times \{0\} \times \{0\}, \tag{3.2b}$$

$$\varphi_x(\Gamma_N \cap U_x) = [-1,1]^{n-2} \times (0,1] \times \{0\}. \tag{3.2c}$$

Note that Γ_T is an admissible boundary patch if and only if Γ_N is an admissible boundary patch, and that Γ_I is their common boundary. We also call the triple $(\Gamma_T, \Gamma_I, \Gamma_N)$ an *admissible boundary partition*.

Remark 3.2. A weakly Lipschitz domain is a locally flat n-dimensional Lipschitz submanifold of $\mathbb{R}^n$ with boundary. In particular, $\partial\Omega$ is a locally flat Lipschitz submanifold of dimension $n-1$ without boundary. The tuple $(\Gamma_T, \Gamma_I, \Gamma_N)$ being an admissible boundary partition means that Γ_T and Γ_N are locally flat Lipschitz submanifolds of dimension $n-1$ of $\partial\Omega$ with common boundary $\Gamma_I := \partial\Gamma_T = \partial\Gamma_N$. In turn, Γ_I is a Lipschitz submanifold of dimension $n-2$ without boundary of $\partial\Omega$. This is in accordance with Definition 3.7 of [28], when Remark 3.2 in that reference is taken into account.

3.2. **Commuting Extension Operators.** In the remainder of this section we construct a commuting extension operator. The basic idea comprises two steps. First, any differential form over Ω is extended by zero onto a bulge domain Υ attached to Ω along Γ_T. Second, the thus extended differential form over this enlarged domain is extended again to an even larger domain Ω^{e} via a reflection along the boundary.

We begin with some geometric definitions and results. A tubular neighborhood of Ω with Lipschitz regularity is obtained from Theorem 2.3 of [39]. Specifically, we obtain a LIP embedding $\Psi^0 : \partial\Omega \times [-1,1] \to \mathbb{R}^n$ satisfying

$$\forall x \in \partial\Omega : \Psi^0(x,0) = x, \tag{3.3a}$$

$$\Psi^0(\partial\Omega \times [-1,0)) \subseteq \Omega, \quad \Psi^0(\partial\Omega \times (0,1]) \subseteq \overline{\Omega}^c. \tag{3.3b}$$

We then introduce the auxiliary domains

$$\Upsilon := \Psi^0(\Gamma_T \times [0, 1/2)), \quad \Omega^{\mathrm{b}} := \Omega \cup \Gamma_T \cup \Upsilon. \tag{3.4}$$

We think of Υ as a *bulge* attached to the domain Ω along Γ_T, which results in the combined domain Ω^{b}. It is easily verified that both Υ and Ω^{b} are again weakly Lipschitz domains; see also Figure 1 for a visualization of the construction.

The fact that Ω^{b} is a weakly Lipschitz domain allows us to apply Theorem 2.3 of [39] again to construct a tubular neighborhood with Lipschitz regularity: we obtain a LIP embedding $\Psi^{\mathrm{b}} : \partial\Omega^{\mathrm{b}} \times [-1,1] \to \mathbb{R}^n$ that satisfies

$$\forall x \in \partial\Omega^{\mathrm{b}} : \Psi^{\mathrm{b}}(x,0) = x, \tag{3.5a}$$

$$\Psi^{\mathrm{b}}\left(\partial\Omega^{\mathrm{b}} \times [-1,0)\right) \subseteq \Omega^{\mathrm{b}}, \quad \Psi^{\mathrm{b}}\left(\partial\Omega^{\mathrm{b}} \times (0,1]\right) \subseteq \overline{\Omega^{\mathrm{b}}}^c. \tag{3.5b}$$

We now define the additional auxiliary domains

$$\mathcal{C}\Omega^{\mathrm{b}} := \Psi^{\mathrm{b}}(\partial\Omega \times (-1,1)), \quad \Omega^{\mathrm{e}} := \Omega^{\mathrm{b}} \cup \Psi^{\mathrm{b}}\left(\partial\Omega^{\mathrm{b}} \times [0,1)\right), \tag{3.6}$$

which by Theorem 2.3 of [39] can be assumed to be weakly Lipschitz domains without loss of generality. We are now in the position to define the main result of this section: a commuting extension operator preserving partial boundary conditions.

Theorem 3.3. *There exists a bounded linear operator*

$$E^k : L^p\Lambda^k(\Omega) \to L^p\Lambda^k(\Omega^{\mathrm{e}}), \quad p \in [1,\infty],$$

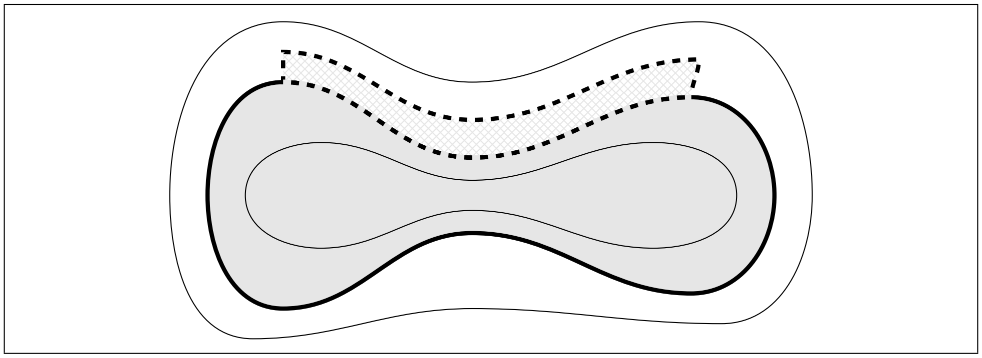

FIGURE 1. Domain Ω (gray) with a bulge Υ (shaded) attached. The thick line is the boundary part Γ_N of the original domain, and the contact line between Ω and Υ is the boundary part Γ_T. The thinner lines inside and outside of the domain indicate the inner and outer boundaries, respectively, of a tubular neighborhood Ψ^0.

such that $E^k u$ vanishes over Υ for every $u \in L^p\Lambda^k(\Omega)$. Moreover, for all $p, q \in [1,\infty]$ and $u \in W^{p,q}\Lambda^k(\Omega, \Gamma_T)$ we have

$$E^k u \in W^{p,q}(\Omega^{\mathrm{e}}), \quad \mathsf{d}E^k u = E^{k+1}\mathsf{d}u.$$

Lastly, there exist $L_E \geq 1$ and $C_E \geq 1$ such that for every $p \in [1,\infty]$, every $\delta > 0$, and every measurable $A \subseteq \overline{\Omega}$ we have

$$\|E^k u\|_{L^p\Lambda^k(B_\delta(A)\cap\Omega^{\mathrm{e}})} \leq \left(1 + C_E^{k+\frac{n}{p}}\right) \|u\|_{L^p\Lambda^k\left(B_{\delta L_E}(A)\cap\overline{\Omega}\right)}, \quad u \in L^p\Lambda^k(\Omega). \tag{3.7}$$

Proof. For any $u \in M\Lambda^k(\Omega)$ we let $E_0^k u \in M\Lambda^k(\Omega^{\mathrm{b}})$ denote the extension of u by zero to Ω^{b}. Moreover, we let $E_r^k : M\Lambda^k(\Omega^{\mathrm{b}}) \to M\Lambda^k(\Omega^{\mathrm{e}})$ be the extension operator based on reflection that is obtained by applying the results of Subsection 7.1 in [39] to our domain Ω^{b} and the tubular neighborhood described by Ψ^{b}.

We set $E^k := E_r^k E_0^k$. We first observe that $E^k u$ vanishes over Υ for every $u \in M\Lambda^k(\Omega)$, since by construction E_0^k vanishes over Υ and $E_r^k E_0^k u$ agrees with $E_0^k u$ over Ω^{b}. Next, by Lemma 7.1 of [39] we have for every $p \in [1,\infty]$ a bounded mapping $E_r^k : L^p\Lambda^k(\Omega^{\mathrm{b}}) \to L^p\Lambda^k(\Omega^{\mathrm{e}})$. In combination, for every $p \in [1,\infty]$ we have a bounded mapping $E^k : L^p\Lambda^k(\Omega) \to L^p\Lambda^k(\Omega^{\mathrm{e}})$.

Assume that $p, q \in [1,\infty]$ and $u \in W^{p,q}\Lambda^k(\Omega, \Gamma_T)$. By definition $E_0^k u \in W^{p,q}(\Omega^{\mathrm{b}})$ with $E_0^k \mathsf{d}u = \mathsf{d}E_0^k u$. Moreover, by Lemma 7.4 of [39], we have $E_r^k E_0^k u \in W^{p,q}(\Omega^{\mathrm{e}})$ with $E_r^k E_0^k \mathsf{d}u = E_r^k \mathsf{d}E_0^k u$.

The last statement follows with a combination of Lemma 7.2 of [39] applied to the extension operator E_r together with the fact that E_0 is an extension by zero. □

Remark 3.4. An outline of the above extension operator can be found in [50]. The technical details are elaborated upon in [31] with the additional assumption that the domain is (strongly) Lipschitz and that the boundary partition has a piecewise C^1-interface. By contrast, we consider weakly Lipschitz domains and admissible boundary patches in the sense of [28]. Our extension operator in this section generalizes the extension operator introduced in [39].

4. Distortion of Domain Boundaries

In this section we discuss a geometric result that enters the construction of the smoothed projection but which is also of independent interest. The basic idea is as follows: given a domain $\Upsilon \subseteq \mathbb{R}^n$, we search for a homeomorphism of $\mathbb{R}^n$ that moves $\partial\Upsilon$ into Υ and that is the identity outside of a neighborhood of $\partial\Upsilon$. Moreover, we want to locally control how far the homeomorphism moves the boundary into the domain. Specifically, we prove the following result.

Theorem 4.1. *Let $\Upsilon \subseteq \mathbb{R}^n$ be a bounded weakly Lipschitz domain. There exist $\epsilon_D > 0$ and $L_D > 0$ such that for any non-negative function $\varrho : \mathbb{R}^n \to \mathbb{R}$ satisfying*

$$\operatorname{Lip}(\varrho, \mathbb{R}^n) < \epsilon_D, \quad \max_{x \in \mathbb{R}^n} \varrho(x) < \epsilon_D, \tag{4.1}$$

there exists a bi-Lipschitz mapping $\mathfrak{D}_\varrho : \mathbb{R}^n \to \mathbb{R}^n$ with the following properties. We have

$$\operatorname{Lip}(\mathfrak{D}_\varrho) \leq L_D \left(1 + \operatorname{Lip}(\varrho)\right), \quad \operatorname{Lip}(\mathfrak{D}_\varrho^{-1}) \leq L_D \left(1 + \operatorname{Lip}(\varrho)\right). \tag{4.2a}$$

We have

$$\mathfrak{D}_\varrho(\Upsilon) \subseteq \Upsilon. \tag{4.2b}$$

For all $x \in \mathbb{R}^n$ we have

$$\|x - \mathfrak{D}_\varrho(x)\| \leq L_D \varrho(x). \tag{4.2c}$$

For every $x \in \mathbb{R}^n$ we have $x = \mathfrak{D}_\varrho(x)$ if

$$\operatorname{dist}(x, \partial\Upsilon) \geq L_D \varrho(x). \tag{4.2d}$$

For all $x \in \partial\Upsilon$ we have

$$\mathfrak{D}_\varrho\left(B_{\varrho(x)/L_D}(x)\right) \subseteq \Upsilon. \tag{4.2e}$$

Remark 4.2. We discuss the meaning and application of Theorem 4.1 before we give the proof. The mapping $\mathfrak{D}_\varrho$ is a distortion of $\mathbb{R}^n$ which moves Υ into itself. The function ϱ controls the amount of distortion near Υ. The distortion $\mathfrak{D}_\varrho$ contracts a neighborhood of Υ into the domain. Specifically, we interpret the properties (4.2) in the following manner. Property (4.2b) formalizes that the distortion moves $\partial\Upsilon$ into Υ; in particular, Υ is mapped into itself. Property (4.2d) formalizes that the homeomorphism is the identity outside of a neighborhood of $\partial\Upsilon$. By Property (4.2c) the amount of distortion is locally bounded by ϱ, and Property (4.2e) formalizes that the distortion is proportional to ϱ near the boundary.

Proof of Theorem 4.1. Since $\Upsilon \subseteq \mathbb{R}^n$ is a bounded weakly Lipschitz domain, we can apply Theorem 2.3 of [39] to deduce the existence of a LIP embedding

$$\Xi : \partial\Upsilon \times [-1, 1] \to \mathbb{R}^n$$

such that $\Xi(x, 0) = x$ for $x \in \partial\Upsilon$ and such that $\Xi(\partial\Upsilon, [0, 1]) \subset \overline{\Upsilon}$. In particular, there exist constants $c_\Xi, C_\Xi > 0$ such that

$$\|\Xi(x_1, t_1) - \Xi(x_2, t_2)\| \leq C_\Xi \sqrt{\|x_1 - x_2\|^2 + |t_1 - t_2|^2},$$
$$\sqrt{\|x_1 - x_2\|^2 + |t_1 - t_2|^2} \leq c_\Xi \|\Xi(x_1, t_1) - \Xi(x_2, t_2)\|$$

for $x_1, x_2 \in \partial\Upsilon$ and $t_1, t_2 \in [-1, 1]$. We note in particular that

$$c_\Xi^{-1} \operatorname{Lip}(\varrho) \leq \operatorname{Lip}(\varrho\Xi) \leq C_\Xi \operatorname{Lip}(\varrho).$$

For $\alpha \in [0, 1/5]$ we consider the parametrized mappings

$$\zeta_\alpha : [-1,1] \to [-1,1], \quad t \mapsto \int_{-1}^{t} 1 + \chi_{[-2\alpha,\alpha]} - \frac{2}{3}\chi_{[\alpha,3\alpha]} \,\mathrm{d}\lambda - 1,$$

$$\zeta_\alpha^{-1} : [-1,1] \to [-1,1], \quad t \mapsto \int_{-1}^{s} 1 + \frac{2}{3}\chi_{[-2\alpha,\alpha]} - 2\chi_{[3\alpha,4\alpha]} \,\mathrm{d}\lambda - 1,$$

where χ_I is the indicator function of the interval $I \subseteq [-1,1]$. As the notation already suggests, these two mappings are mutually inverse for α fixed. We easily see that they are strictly monotonically increasing, and that their Lipschitz constants are uniformly bounded for $\alpha \in [0, 1/5]$. In particular ζ_α and ζ_α^{-1} are bi-Lipschitz. Moreover, for $\alpha \in [0, 1/5]$ we observe that

$$\zeta_\alpha(t) = \zeta_\alpha^{-1}(t) = t, \quad t \notin [-2\alpha, 4\alpha], \tag{4.3}$$

$$\zeta_\alpha([-\alpha,\alpha]) = [\alpha, 3\alpha]. \tag{4.4}$$

We now write $\zeta(t;\alpha) = \zeta_\alpha(t)$ and $\zeta^{-1}(t;\alpha) = \zeta_\alpha^{-1}(t)$ for $(t,\alpha) \in [-1,1] \times [0, 1/5]$. Assume from now on that

$$\max_{x\in\mathbb{R}^n} \varrho(x) < 1/5, \quad \operatorname{Lip}(\varrho, \mathbb{R}^n) < \min\left\{1, \operatorname{Lip}(\Xi)^{-1}\right\}.$$

This implies that $\operatorname{Lip}(\varrho\Xi) < 1$. We define homeomorphisms

$$\mathfrak{D}_\varrho : \mathbb{R}^n \to \mathbb{R}^n, \quad \mathfrak{D}_\varrho^{-1} : \mathbb{R}^n \to \mathbb{R}^n,$$

in the following manner. Assume that $x \in \mathbb{R}^n$. If there exist $x_0 \in \partial\Upsilon$ and $t \in [-1,1]$ such that $x = \Xi(x_0, t)$, then we set

$$\mathfrak{D}_\varrho(x) := \Xi(x_0, t'), \quad t' := \zeta(t; \varrho(x_0)/8),$$

$$\mathfrak{D}_\varrho^{-1}(x) := \Xi(x_0, t''), \quad t'' := \zeta^{-1}(t; \varrho(x_0)/8).$$

Otherwise, we set $\mathfrak{D}_\varrho(x) := x$. It follows from the construction that $\mathfrak{D}_\varrho$ and $\mathfrak{D}_\varrho^{-1}$ are bi-Lipschitz and mutually inverse. In particular, (4.2a) is implied by

$$\operatorname{Lip}(\mathfrak{D}_\varrho) \leq 1 + c_\xi^{-1} C_\Xi \left(1 + \operatorname{Lip}(\zeta)\operatorname{Lip}(\varrho)\right),$$

$$\operatorname{Lip}\left(\mathfrak{D}_\varrho^{-1}\right) \leq 1 + c_\xi^{-1} C_\Xi \left(1 + \operatorname{Lip}\left(\zeta^{-1}\right)\operatorname{Lip}(\varrho)\right).$$

The construction shows that (4.2b) holds, since $\mathfrak{D}_\varrho$ maps $\Xi(\partial\Upsilon, [0,1])$ into itself. Moreover, $\mathfrak{D}_\varrho$ and $\mathfrak{D}_\varrho^{-1}$ act like the identity outside of $\Xi(\partial\Upsilon, [-1,1])$.

Let us assume for the remainder of this proof that that $x = \Xi(x_0, t)$ for $x_0 \in \partial\Upsilon$ and $t \in [-1,1]$. Using (4.3), we see that $\mathfrak{D}_\varrho(x) \neq x$ implies $|t| \leq \varrho(x_0)/2$, and so

$$|\varrho(x_0) - \varrho(x)| = |\varrho(\Xi(x_0, 0)) - \varrho(\Xi(x_0, t))| \leq \frac{\operatorname{Lip}(\varrho\Xi)\varrho(x_0)}{2} \leq \frac{\varrho(x_0)}{2}.$$

This implies that $\varrho(x_0) \leq 2\varrho(x)$. By the definition of ζ and $\mathfrak{D}_\varrho$ we then see

$$|x - \mathfrak{D}_\varrho(x)| \leq \operatorname{Lip}(\Xi)\left|t - \zeta\left(t; \frac{\varrho(x_0)}{8}\right)\right| \leq \frac{6}{8}\operatorname{Lip}(\Xi)\varrho(x_0) \leq \frac{3}{2}\operatorname{Lip}(\Xi)\varrho(x),$$

proving (4.2c). Furthermore, using we note that $x \neq \mathfrak{D}_\varrho(x)$ implies

$$\operatorname{dist}(x, \partial\Upsilon) \leq \|x - x_0\| \leq \operatorname{Lip}(\Xi)|t| \leq \frac{\operatorname{Lip}(\Xi)}{2}\varrho(x_0) \leq \operatorname{Lip}(\Xi)\varrho(x).$$

This means conversely that $x = \mathfrak{D}_\varrho(x)$ is implied by

$$\operatorname{dist}(x, \Gamma_T) \geq \operatorname{Lip}(\Xi)\varrho(x),$$

as required in (4.2d). It remains to prove (4.2e). Let $x_0 \in \partial\Upsilon$ and define $A \subseteq \partial\Upsilon \times [-1,1]$ by

$$A := \left(B_{\varrho(x_0)/8}(x_0) \cap \partial\Upsilon \right) \times (-7\varrho(x_0)/64, 7\varrho(x_0)/64).$$

If $y \in \partial\Upsilon$ with $\|x_0 - y\| \leq \varrho(x_0)/8$, then $\|\varrho(x_0) - \varrho(y)\| \leq \varrho(x_0)/8$ since we assume $\operatorname{Lip}(\varrho) < 1$. In particular $\varrho(y) \geq 7\varrho(x_0)/8$ follows. Via (4.3) we thus find $\mathfrak{D}_\varrho(\Xi(A)) \subseteq \Upsilon$. We observe that A contains a ball around x_0 of radius $7\varrho(x_0)/64$ in $\partial\Upsilon \times [-1,1]$. Whence $\Xi(A)$ contains a ball around x_0 of radius $c_\Xi^{-1} 7\varrho(x_0)/64$. This shows (4.2e). The proof is complete. □

5. Mollification with Partial Boundary Conditions

In this section we construct a mollification operator for differential forms on the weakly Lipschitz domain Ω that respects partial boundary conditions along the admissible boundary patch Γ_T. We define the mollified differential form at each point by averaging the coefficients in a small neighborhood of that point. A technical difference to the classical mollification operator is that we let the mollification radius vary across the domain.

Throughout this section we let $\varrho : \mathbb{R}^n \to \mathbb{R}$ be a non-negative smooth function that assumes a positive minimum over $\overline{\Omega}$. We introduce the mapping

$$\Phi_\varrho : \mathbb{R}^n \times B_1(0) \to \mathbb{R}^n, \quad (x,y) \mapsto x + \varrho(x) y. \tag{5.1}$$

Regarding the second variable as a parameter, we have a family of mappings

$$\Phi_{\varrho,y} : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto \Phi_\varrho(x,y).$$

It is easily seen that $\Phi_{\varrho,y}$ is Lipschitz with Lipschitz constant bounded by $1+\operatorname{Lip}(\varrho)$. Moreover, if $\operatorname{Lip}(\varrho) < 1/2$, then $\Phi_{\varrho,y}$ is bi-Lipschitz and its inverse has a finite Lipschitz constant bounded by 2.

The *standard mollifier* is the function

$$\mu : \mathbb{R}^n \to \mathbb{R}, \quad y \mapsto \begin{cases} C_\mu \exp\left(-(1 - \|y\|^2)^{-1} \right) & \text{if} \quad \|y\| \leq 1, \\ 0 & \text{if} \quad \|y\| > 1, \end{cases} \tag{5.2}$$

where $C_\mu > 0$ is chosen such that μ has unit integral. Note that μ is smooth and has compact support in $B_1(0)$. For $\epsilon > 0$ we then define the *scaled mollifier*

$$\mu_\epsilon : \mathbb{R}^n \to \mathbb{R}, \quad y \mapsto \epsilon^{-n} \mu(y/\epsilon).$$

For every $u \in L^p\Lambda^k(\Omega^{\mathrm{e}})$, $p \in [1,\infty]$, we then define

$$R^k_\varrho u_{|x} := \int_{\mathbb{R}^n} \mu(y) (\Phi^*_{\varrho,y} u)_{|x} \mathrm{d}y, \quad x \in \overline{\Omega}. \tag{5.3}$$

This mapping R^k_ϱ has range in $C^\infty\Lambda^k(\overline{\Omega})$, commutes with the exterior derivative, and satisfies local bounds in the supremum norm.

Lemma 5.1. *Assume that $\Phi_{\varrho,y} : \overline{\Omega} \to \Omega^{\mathrm{e}}$ is a LIP embedding for all $y \in B_1(0)$. Then the operator*

$$R^k_\varrho : L^p\Lambda^k(\Omega^{\mathrm{e}}) \to C^\infty\Lambda^k(\overline{\Omega}), \quad p \in [1,\infty],$$

is well-defined and linear. Moreover, for every $p \in [1,\infty]$, $u \in L^p\Lambda^k(\Omega^{\mathrm{e}})$, and measurable $A \subseteq \overline{\Omega}$ we have

$$\|R^k_\varrho u\|_{C\Lambda^k(A)} \leq (1 + \operatorname{Lip}(\varrho))^k \inf_A(\varrho)^{-\frac{n}{p}} \|u\|_{L^p\Lambda^k(\Phi_\varrho(A,B_1))}. \tag{5.4}$$

For every $u \in W^{p,q}\Lambda^k(\Omega^{\mathrm{e}})$ *we have* $\mathsf{d}R^k_\varrho u = R^{k+1}_\varrho \mathsf{d}u$. □

Proof. This is Lemma 7.4 in [39]. □

We are now in the position to combine the extension operator of Section 3, the distortion operator of Section 4, and mollification operator of this section. We let $\delta > 0$ be a small parameter to be determined below. We then define

$$M^k_\varrho : L^p\Lambda^k(\Omega) \to C^\infty\Lambda^k(\Omega), \quad u \mapsto R^k_{\delta\varrho}\mathfrak{D}^*_\varrho E^k u, \quad p \in [1,\infty]. \tag{5.5}$$

The properties of the mapping M^k_ϱ are summarized as follows.

Theorem 5.2. *Assume that* $\varrho : \mathbb{R}^n \to \mathbb{R}$ *satisfies the conditions of Lemma 5.1 and Theorem 4.1 applied to* Υ. *Assume also that* $\delta \in (0,1)$ *with* $2\delta L_D < 1$. *Then* M^k_ϱ *is well-defined. Moreover, there exist* $C^M_{n,k,p} > 0$ *and* $L_M > 0$, *not depending on* ϱ, *such that for all measurable* $A \subseteq \overline{\Omega}$ *and* $u \in L^p\Lambda^k(\Omega)$ *we have*

$$\|M^k_\varrho u\|_{C\Lambda^k(A)} \leq C^M_{n,k,p} \frac{(1+\operatorname{Lip}(\varrho))^{k+\frac{n}{p}}}{\inf_A(\varrho)^{\frac{n}{p}}} \|u\|_{L^p\Lambda^k(B_{L_M(1+\operatorname{Lip}(\varrho))\sup_A(\varrho)}(A)\cap\Omega)}. \tag{5.6}$$

Additionally, if $p,q \in [1,\infty]$ *and* $u \in W^{p,q}\Lambda^k(\Omega,\Gamma_T)$ *then*

$$M^{k+1}_\varrho \mathsf{d}u = \mathsf{d}M^k_\varrho u, \tag{5.7}$$

and $M^k_\varrho u$ *vanishes in a neighborhood of* Γ_T.

Constants. We may assume that $L_M \leq L_D L_E$ and $C^M_{n,k,p} \leq \delta^{-\frac{n}{p}} L_D^{k+\frac{n}{p}} \left(1 + C_E^{k+\frac{n}{p}}\right)$.

Proof. We combine Theorem 3.3, Theorem 4.1 together with (2.7), and Theorem 5.1. We find that M^k_ϱ as given by (5.5) is well-defined. By the same token we immediately deduce (5.7).

Next we prove the local estimate (5.6). Assume that $u \in L^p\Lambda^k(\Omega)$ and that $A \subseteq \Omega$ is measurable. Via Theorem 5.1 we find

$$\|R^k_{\delta\varrho}\mathfrak{D}^*_\varrho E^k u\|_{C\Lambda^k(A)} \leq \frac{(1+\operatorname{Lip}(\varrho))^k}{(\delta \inf_A(\varrho))^{\frac{n}{p}}} \|\mathfrak{D}^*_\varrho E^k u\|_{L^p\Lambda^k(\Phi_{\delta\varrho}(A,B_1))}.$$

By Theorem 4.1 and the pullback estimate (2.8) we have

$$\|\mathfrak{D}^*_\varrho E^k u\|_{L^p\Lambda^k(\Phi_{\delta\varrho}(A,B_1))} \leq L_D^{k+\frac{n}{p}} (1+\operatorname{Lip}(\varrho))^{k+\frac{n}{p}} \|E^k u\|_{L^p\Lambda^k(\mathfrak{D}_\varrho\Phi_{\delta\varrho}(A,B_1))}.$$

For $x \in A$ and $y \in B_{\delta\varrho(x)}(x)$ we find

$$|\varrho(y) - \varrho(x)| \leq \operatorname{Lip}(\varrho)\|y-x\| \leq \delta \operatorname{Lip}(\varrho)\varrho(x),$$

and via (4.2c) we obtain

$$\|\mathfrak{D}_\varrho(y) - y\| \leq L_D\varrho(y) \leq L_D\big(\varrho(x) + \delta\operatorname{Lip}(\varrho)\varrho(x)\big).$$

Consequently,

$$\begin{aligned} \mathfrak{D}_\varrho\Phi_{\delta\varrho}(A,B_1) &\subseteq \Phi_{L_D(1+\delta\operatorname{Lip}(\varrho))\varrho}(A,B_1) \cap \Omega^{\mathrm{e}} \\ &\subseteq B_{L_D(1+\delta\operatorname{Lip}(\varrho))\sup_A(\varrho)}(A) \cap \Omega^{\mathrm{e}}. \end{aligned}$$

The desired inequality (5.6) follows with Lemma 3.3.

$$\|E^k u\|_{L^p\Lambda^k(\mathfrak{D}_\varrho\Phi_{\delta\varrho}(A,B_1))} \leq \left(1 + C_E^{k+\frac{n}{p}}\right) \|u\|_{L^p\Lambda^k\left(B_{L_E L_D(1+\delta\operatorname{Lip}(\varrho))\sup_A(\varrho)}(A)\cap\Omega\right)}.$$

Finally, let $x \in \Gamma_T$ and consider $A = B_{\operatorname{Lip}(\varrho)^{-1}\varrho(x)}(x) \cap \overline{\Omega}$. For all $y \in A$ we have

$$\varrho(y) \leq \varrho(x) + \operatorname{Lip}(\varrho)\|y - x\| \leq 2\varrho(x),$$

and thus

$$B_{\delta \sup_A(\varrho)}(A) \subseteq B_{2\delta\varrho(x)}(x).$$

In particular,

$$\|R^k_{\delta\varrho}\mathfrak{D}^*_\varrho E^k u\|_{C\Lambda^k(A)} \leq \frac{(1 + \delta \operatorname{Lip}(\varrho))^k}{(\delta \inf_A(\varrho))^{\frac{n}{p}}} \|\mathfrak{D}^*_\varrho E^k u\|_{L^p\Lambda^k\left(B_{2\delta\varrho(x)}(x)\right)}.$$

Another use of the pullback estimate (2.8) gives

$$\|\mathfrak{D}^*_\varrho E^k u\|_{L^p\Lambda^k\left(B_{2\delta\varrho(x)}(x)\right)} \leq L_D^{k+\frac{n}{p}}(1 + \operatorname{Lip}(\varrho))^{k+\frac{n}{p}} \|E^k u\|_{L^p\Lambda^k\left(\mathfrak{D}_\varrho B_{2\delta\varrho(x)}(x)\right)}.$$

We now assume $2\delta < 1/L_D$. In combination with Theorem 4.1 we conclude

$$\mathfrak{D}_\varrho B_{2\delta\varrho(x)}(x) \subseteq \Upsilon.$$

Hence $M^k_\varrho u$ vanishes in a neighborhood of Γ_T relatively open in $\overline{\Omega}$. The proof is complete. □

Although the focus of this research is numerical analysis, the results up to this point allow a contribution to functional analysis. Specifically, we prove the density result mentioned in the introduction.

Lemma 5.3. *Let Ω be a bounded weakly Lipschitz domain and let Γ_T be an admissible boundary patch. Then the smooth differential k-forms in $C^\infty\Lambda^k(\overline{\Omega})$ that vanish near Γ_T constitute a dense subset of $W^{p,q}\Lambda^k(\Omega, \Gamma_T)$ for all $p, q \in [1, \infty)$.*

Proof. Let $p, q \in [1, \infty)$ and $u \in W^{p,q}\Lambda^k(\Omega, \Gamma_T)$. With some abuse of notation, we let $\epsilon : \mathbb{R}^n \to \mathbb{R}$ denote the constant function with value $\epsilon > 0$. For $\epsilon > 0$ small enough, Theorem 5.2 provides a well-defined operator $M^k_\epsilon : L^p\Lambda^k(\Omega) \to C^\infty\Lambda^k(\Omega)$. We define

$$Y_\epsilon := \Omega \cap B_{L_D\epsilon}(\partial\Omega), \quad Z_\epsilon := \Omega \setminus Y_\epsilon.$$

On the one hand, for some $C > 0$ not depending on u or ϵ we easily find

$$\left\|u - M^k_\epsilon u\right\|_{L^p\Lambda^k(Y_\epsilon)} \leq C\, \|u\|_{L^p\Lambda^k(\Omega\cap Y_{C\epsilon})}.$$

Using that Ω is a weakly Lipschitz domain, one can show that $\operatorname{vol}^n(Y_{C\epsilon})$ converges to zero as ϵ converges to zero. This implies that the L^p norm of u over $Y_\epsilon \cap \Omega$ converges to zero as ϵ converges to zero. On the other hand, we have

$$\left\|u - M^k_\epsilon u\right\|_{L^p\Lambda^k(Z_\epsilon)} = \|u - \mu_{\delta\epsilon} \star u\|_{L^p\Lambda^k(Z_\epsilon)} \leq \|u - \mu_{\delta\epsilon} \star u\|_{L^p\Lambda^k(\Omega)}.$$

By basic results on mollifications, the last expression converges to zero as ϵ converges to zero. Since $\Omega = Y_\epsilon \cup Z_\epsilon$ for all $\epsilon > 0$, and since $M^{k+1}_\epsilon \mathsf{d}u = \mathsf{d}M^k_\epsilon u$, the combination of both observations provides

$$\lim_{\epsilon\to 0} \left\|u - M^k_\epsilon u\right\|_{W^{p,q}\Lambda^k(\Omega)} = 0.$$

We recall that $M^k_\epsilon u \in C^\infty\Lambda^k(\overline{\Omega})$ with support away from Γ_T for all $\epsilon > 0$. The proof is complete. □

6. Finite Element Projection with Partial Boundary Conditions

In this section, we prove the main result of this paper: we construct a uniformly bounded commuting smoothed projection from a Sobolev de Rham complex with partial boundary conditions onto a conforming finite element de Rham complex. Throughout this section, we let $\Omega \subseteq \mathbb{R}^n$ be a fixed weakly Lipschitz domain and we let $\Gamma_T \subseteq \partial\Omega$ be an admissible boundary patch.

We henceforth let $\mathtt{h}$ be as in Lemma ??. Having set up these technical definitions, we begin the discussion of the main results of this section. For $\epsilon > 0$ small enough, we define the *smoothed interpolant* as

$$Q^k_\epsilon : L^p\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \subseteq L^p\Lambda^k(\Omega), \quad u \mapsto I^k M^k_{\epsilon\mathtt{h}} u, \quad p \in [1,\infty]. \tag{6.1}$$

The operator Q^k_ϵ has the following properties.

Theorem 6.1. *Let $\epsilon > 0$ be small enough. For $p \in [1,\infty]$, the linear operator $Q^k_\epsilon : L^p\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \subseteq L^p\Lambda^k(\Omega)$ is bounded, and there exists a uniformly bounded constant $C_{Q,p} > 0$ such that*

$$\|Q^k_\epsilon u\|_{L^p\Lambda^k(T)} \leq C_{Q,p}\epsilon^{-\frac{n}{p}}\|u\|_{L^p\Lambda^k(\mathcal{T}(T))}, \quad u \in L^p\Lambda^k(\Omega), \quad T \in \mathcal{T}^n, \tag{6.2}$$

$$\|Q^k_\epsilon u\|_{L^p\Lambda^k(\Omega)} \leq C_{\mathrm{N}}^{1/p} C_{Q,p}\epsilon^{-\frac{n}{p}}\|u\|_{L^p\Lambda^k(\Omega)}, \quad u \in L^p\Lambda^k(\Omega). \tag{6.3}$$

Moreover, we have

$$\mathsf{d}Q^k_\epsilon u = Q^k_\epsilon \mathsf{d}u, \quad u \in W^{p,q}\Lambda^k(\Omega,\Gamma_T), \quad p,q \in [1,\infty]. \tag{6.4}$$

Constants. It suffices that $\epsilon > 0$ is so small that $L_{\mathrm{M}}(1+\epsilon L_h)C_h\epsilon < \epsilon_h$ and that Theorem 5.2 applies for $\varrho = \epsilon\mathtt{h}$. With the notation as in the following proof, we may assume $C_{Q,p} \leq C^k_{\mathrm{M}} c^k_{\mathrm{M}} C_I C_h C^M_{n,k,p}\left(1+\epsilon L_h\right)^{\frac{n}{p}}$.

Proof. The proof is very similar to the proof of Theorem 7.7 in [39]. We let $u \in L^p\Lambda^k(\Omega)$ and $T \in \mathcal{T}^n$. By (2.8), (??) and $\mathrm{vol}^n(T) \leq h_T^n$ we get

$$\|Q^k_\epsilon u\|_{L^p\Lambda^k(T)} \leq C^k_{\mathrm{M}} h_T^{\frac{n}{p}-k}\|\varphi_T^* I^k M^k_{\epsilon\mathtt{h}} u\|_{L^\infty\Lambda^k(\Delta^n)}.$$

By estimate (6.17) of [39] and discussions in that reference we know about the existence of a uniformly bounded constant $C_I > 0$ such that

$$\|\varphi_T^* I^k M^k_{\epsilon\mathtt{h}} u\|_{L^\infty\Lambda^k(\Delta^n)} \leq C_I c^k_{\mathrm{M}} h^k_T \|M^k_{\epsilon\mathtt{h}} u\|_{C\Lambda^k(T)}.$$

Assuming that ϵ is small enough, we can apply Theorem 5.2 to find

$$\|M^k_{\epsilon\mathtt{h}} u\|_{C\Lambda^k(T)} \leq C^M_{n,k,p}\frac{C_h(1+\epsilon L_h)^{k+\frac{n}{p}}}{(\epsilon h_T)^{\frac{n}{p}}}\|u\|_{L^p\Lambda^k\left(B_{L_{\mathrm{M}}(1+\epsilon L_h)C_h\epsilon h_T}(T)\cap\Omega\right)}.$$

If $L_{\mathrm{M}}(1+\epsilon L_h)C_h\epsilon < \epsilon_h$, then $B_{L_{\mathrm{M}}(1+\epsilon L_h)C_h\epsilon h_T}(T) \subseteq \mathcal{T}(T)$. Thus the local bound (6.2) is proven. The global bound (6.3) follows easily. Moreover, $M^k_{\epsilon\mathtt{h}} E^k u$ vanishes near every $F \in \mathcal{T}$ with $F \subseteq \Gamma_T$. Finally, (6.4) follows from Theorem 3.3, Theorem 5.1, and our assumptions on I^k. The proof is complete. □

We have proven uniform local bounds for the smoothed interpolant Q^k_ϵ. Even though Q^k_ϵ is generally not a projection onto $\mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$, the interpolation error over $\mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$ can be uniformly bounded for $\epsilon > 0$ small enough. Over the finite element space, Q^k_ϵ can be brought arbitrarily close the identity.

Theorem 6.2. *For $\epsilon > 0$ small enough, there exists uniformly bounded $C_{e,p} > 0$ for every $p \in [1,\infty]$ such that*

$$\|u - Q^k_\epsilon u\|_{L^p\Lambda^k(T)} \leq \epsilon C_{e,p} \|u\|_{L^p\Lambda^k(\mathcal{T}(T))}, \quad u \in \mathcal{P}\Lambda^k(\mathcal{T}), \quad T \in \mathcal{T}^n.$$

Constants. With the notation as in the following proof, it suffices that $\epsilon > 0$ is small enough such that Theorem 6.1 applies and that $L_E \mathcal{L}\epsilon < \epsilon_h$ and $3\mathcal{L}\epsilon < \lambda$. In addition, we may assume $C_{e,p} \leq C_{\mathrm{M}}^{2k+1+\frac{n}{p}} c_{\mathrm{M}}^{2k+1} C_I \left(1 + C_E^{k+1+\frac{n}{p}}\right) C_{\flat,p} \mathcal{L} \widehat{C}$.

Proof. The proof is very similar to the proof of Theorem 7.9 in [39], to which the reader is referred at all times for certain technical steps in the present proof. Let $u \in \mathcal{P}\Lambda^k(\mathcal{T})$ and let $T \in \mathcal{T}^n$. Using (2.8), (**??**), and definitions, we verify

$$\begin{aligned} \|u - Q^k_\epsilon u\|_{L^p\Lambda^k(T)} &\leq h_T^{\frac{n}{p}} \|E^k u - Q^k_\epsilon u\|_{L^\infty\Lambda^k(T)} \\ &\leq C^k_{\mathrm{M}} h_T^{\frac{n}{p}-k} \|\varphi_T^* I^k (E^k u - R^k_{\delta\epsilon\mathbf{h}} \mathfrak{D}^*_{\epsilon\mathbf{h}} E^k u)\|_{L^\infty\Lambda^k(\varphi_T^{-1}T)}. \end{aligned}$$

To proceed with the proof, we need to recall the definition of the canonical interpolant via degrees of freedom as in Section 6 of [39]. We let $\mathcal{PC}_k(F)$ denote the space of degrees of freedom associated with the subsimplex $F \in \Delta(T)$; these spaces of functionals are given by taking the trace of a differential k-form onto the subsimplex F and taking the integral against another polynomial differential form over F (see also Remark 6.1 of [39]).

Using estimate (6.16) of [39], we obtain a uniformly bounded constant $C_I > 0$ which satisfies the inequality

$$\begin{aligned} &\|\varphi_T^* I^k (E^k u - R^k_{\delta\epsilon\mathbf{h}} \mathfrak{D}^*_{\epsilon\mathbf{h}} E^k u)\|_{L^\infty\Lambda^k(\varphi_T^{-1}T)} \\ &\qquad \leq C_I \sup_{\substack{F \in \Delta(T) \\ S \in \mathcal{PC}^F_k}} \left|\varphi_{T*}^{-1} S\right|_k^{-1} \int_S E^k u - R^k_{\delta\epsilon\mathbf{h}} \mathfrak{D}^*_{\epsilon\mathbf{h}} E^k u. \end{aligned}$$

Here, $|\varphi_{T*}^{-1} S|_k$ is defined in the following manner. Let $S \in \mathcal{PC}^F_k$ is given as the integral over $F \in \Delta(T)^m$ against the smooth differential form $\eta_S \in C^\infty\Lambda^{m-k}(F)$. Let $\widehat{F} \in \Delta^n$ be the unique m-simplex that φ_T maps onto F. Then $|\varphi_{T*}^{-1} S|_k$ equals the L^1 norm of $\varphi_T^* \eta_S$ over $\widehat{F}$. This is equivalent to the definition of $|\varphi_{T*}^{-1} S|_k$ via the mass norm of k-chains as used in [39].

Fix $F \in \Delta(T)$ and $S \in \mathcal{PC}^F_k$. We have

$$\int_S E^k u - R^k_{\delta\epsilon\mathbf{h}} \mathfrak{D}^*_{\epsilon\mathbf{h}} E^k u = \int_S \int_{\mathbb{R}^n} \mu(y) \left(\mathrm{Id} - \Phi^*_{\epsilon\mathbf{h},y} \mathfrak{D}^*_{\epsilon\mathbf{h}}\right) E^k u \mathrm{d}y.$$

We then change the order of integration:

$$\int_S \int_{\mathbb{R}^n} \mu(y) \left(\mathrm{Id} - \Phi^*_{\epsilon\mathbf{h},y} \mathfrak{D}^*_{\epsilon\mathbf{h}}\right) E^k u \mathrm{d}y = \int_{\mathbb{R}^n} \mu(y) \int_S \left(\mathrm{Id} - \Phi^*_{\epsilon\mathbf{h},y} \mathfrak{D}^*_{\epsilon\mathbf{h}}\right) E^k u \mathrm{d}y.$$

Using these observations again, we have for $y \in B_1(0)$ that

$$\int_S \left(\mathrm{Id} - \Phi^*_{\epsilon\mathbf{h},y} \mathfrak{D}^*_{\epsilon\mathbf{h}}\right) E^k u = \int_{\varphi_{T*}^{-1}(\mathrm{Id} - \mathfrak{D}_{\epsilon\mathbf{h}*} \Phi_{\delta\epsilon\mathbf{h},y*})S} \varphi_T^* E^k u. \tag{6.5}$$

In the remainder of the proof we bound the last term. An auxiliary estimate bounds the difference $\mathrm{Id} - \Phi_{\delta\epsilon\mathbf{h},y} \mathfrak{D}_{\epsilon\mathbf{h}}$ uniformly in terms of ϵ and y within a small radius of $\varphi_T^{-1} F$. First we see

$$\sup_{y \in B_1(0)} \mathrm{Lip}\left(\varphi_T^{-1} \Phi_{\delta\epsilon\mathbf{h},y} \mathfrak{D}_{\epsilon\mathbf{h}} \varphi_T\right) \leq c_{\mathrm{M}} C_{\mathrm{M}} L_D \left(1 + \epsilon L_h\right)^2 := \mathfrak{L}.$$

Next, let $\lambda > 0$. For any $y \in B_1(0)$ and $\hat{x} \in B_\lambda(\varphi_T^{-1} F)$ we find that

$$\begin{aligned}
&\left\| \hat{x} - \varphi_T^{-1} \Phi_{\delta\epsilon\mathtt{h},y} \mathfrak{D}_{\epsilon\mathtt{h}} \varphi_T(\hat{x}) \right\| \\
&\quad \leq C_{\mathrm{M}} h_T^{-1} \left\| \varphi_T(\hat{x}) - \Phi_{\delta\epsilon\mathtt{h},y} \mathfrak{D}_{\epsilon\mathtt{h}} \varphi_T(\hat{x}) \right\| \\
&\quad \leq C_{\mathrm{M}} h_T^{-1} \left\| \varphi_T(\hat{x}) - \mathfrak{D}_{\epsilon\mathtt{h}} \varphi_T(\hat{x}) \right\| + C_{\mathrm{M}} h_T^{-1} \left\| \mathfrak{D}_{\epsilon\mathtt{h}} \varphi_T(\hat{x}) - \Phi_{\delta\epsilon\mathtt{h},y} \mathfrak{D}_{\epsilon\mathtt{h}} \varphi_T(\hat{x}) \right\|.
\end{aligned}$$

We see that

$$\left\| \varphi_T(\hat{x}) - \mathfrak{D}_{\epsilon\mathtt{h}} \varphi_T(\hat{x}) \right\| \leq L_D \epsilon \mathtt{h} \left(\varphi_T(\hat{x}) \right).$$

We also see that

$$\begin{aligned}
\left\| \mathfrak{D}_{\epsilon\mathtt{h}} \varphi_T(\hat{x}) - \Phi_{\delta\epsilon\mathtt{h},y} \mathfrak{D}_{\epsilon\mathtt{h}} \varphi_T(\hat{x})) \right\| &\leq \delta\epsilon\mathtt{h} \left(\mathfrak{D}_{\epsilon\mathtt{h}} \varphi_T(\hat{x}) \right) \\
&\leq \delta\epsilon\mathtt{h} \left(\varphi_T(\hat{x}) \right) + \delta\epsilon L_h \left\| \varphi_T(\hat{x}) - \mathfrak{D}_{\epsilon\mathtt{h}} \varphi_T(\hat{x}) \right\| \\
&\leq \delta\epsilon \left(1 + \delta\epsilon L_h L_D \right) \mathtt{h} \left(\varphi_T(\hat{x}) \right).
\end{aligned}$$

Let $x_F \in F$ such that $\|\hat{x} - \varphi_T^{-1}(x_F)\| \leq \lambda$. Then $\|\varphi_T(\hat{x}) - x_F\| \leq \lambda c_{\mathrm{M}} h_T$. Hence

$$\mathtt{h}(\varphi_T(\hat{x})) \leq \mathtt{h}(x_F) + L_h \lambda c_{\mathrm{M}} h_T \leq \left(C_h + L_h \lambda c_{\mathrm{M}} \right) h_T.$$

Writing $\mathcal{L} := C_{\mathrm{M}} \left(1 + L_D + L_h L_D \right) \left(C_h + L_h c_{\mathrm{M}} \right)$ and assuming $\epsilon, \lambda \leq 1$ for simplicity, we get

$$\sup_{\hat{x} \in B_\lambda(\varphi_T^{-1} F)} \sup_{y \in B_1(0)} \left\| \hat{x} - \varphi_T^{-1} \Phi_{\delta\epsilon\mathtt{h},y} \mathfrak{D}_{\epsilon\mathtt{h}} \varphi_T(\hat{x}) \right\| \leq \epsilon \mathcal{L}.$$

We continue with the main part of the proof. Let $\lambda > 0$ as above and let $\epsilon > 0$ be so small that $\mathcal{L}\epsilon < \lambda/3$. We apply Lemma 5.4 in [39] with $r = \lambda/3$ and the inverse inequality 6.15 in the same reference. We obtain that there exists $\widehat{C} > 0$ uniformly bounded such that for all $y \in B_1(0)$ we have

$$\int_{S - \mathfrak{D}_{\epsilon\mathtt{h}*} \Phi_{\epsilon\mathtt{h},y*} S} E^k u \leq \epsilon \cdot \mathcal{L} \cdot \widehat{C} \cdot |\varphi_{T*}^{-1} S|_k \cdot \|\varphi_T^* E^k u\|_{W^{\infty,\infty}\Lambda^k(B_{\mathcal{L}\epsilon}(\Delta_n))},$$

As in the proof of Theorem 7.9 of [39], we observe

$$\|\varphi_T^* E^k u\|_{W^{\infty,\infty}\Lambda^k(B_{\mathcal{L}\epsilon}(\Delta_n))} \leq \left(1 + C_E^{k+1+\frac{n}{p}} \right) c_{\mathrm{M}}^{k+1} C_{\mathrm{M}}^{k+1} \|\varphi_T^* u\|_{W^{\infty,\infty}\Lambda^k(\varphi_T^{-1}\mathcal{T}(T))}$$

for ϵ so small that $L_E \mathcal{L} \epsilon < \epsilon_h$. Now the inverse inequality 6.17 of [39] gives

$$\|\varphi_T^* u\|_{W^{\infty,\infty}\Lambda^k(\varphi_T^{-1}\mathcal{T}(T))} \leq C_{\flat,p} \|\varphi_T^* u\|_{L^p\Lambda^k(\varphi_T^{-1}\mathcal{T}(T))}.$$

Transforming back from the reference geometry yields

$$\|\varphi_T^* u\|_{L^p\Lambda^k(\varphi_T^{-1}\mathcal{T}(T))} \leq c_{\mathrm{M}}^k C_{\mathrm{M}}^{\frac{n}{p}} h_T^{k-\frac{n}{p}} \|u\|_{L^p\Lambda^k(\mathcal{T}(T))}.$$

The combination of these inequalities completes the proof. □

For $\epsilon > 0$ small enough, the mapping $Q_\epsilon^k : \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$ is close enough to the identity operator to be invertible. We can then construct the smoothed projection.

Theorem 6.3. *Let $\epsilon > 0$ be small enough. There exists a bounded linear operator*

$$\pi_\epsilon^k : L^p\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \subseteq L^p\Lambda^k(\Omega), \quad p \in [1,\infty],$$

such that

$$\pi_\epsilon^k u = u, \quad u \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}),$$

such that

$$\mathsf{d}\pi_\epsilon^k u = \pi_\epsilon^{k+1}\mathsf{d}u, \quad u \in W^{p,q}\Lambda^k(\Omega,\Gamma_T), \quad p,q \in [1,\infty],$$

and such that for all $p \in [1,\infty]$ *there exist uniformly bounded* $C_{\pi,p} > 0$ *with*

$$\|\pi_\epsilon^k u\|_{L^p\Lambda^k(\mathcal{T})} \leq C_{\pi,p}\epsilon^{-\frac{n}{p}}\|u\|_{L^p\Lambda^k(\Omega)}, \quad u \in L^p\Lambda^k(\Omega).$$

Constants. It suffices that $\epsilon > 0$ is so small that Theorem 6.1 and Theorem 6.2 apply, and that $C_{e,p}\epsilon < 2$. We may assume $C_{\pi,p} \leq 2C_{Q,p}C_{\mathrm{N}}^{1/p}$.

Proof. This is almost verbatim the proof of Theorem 7.11 of [39]. □

7. Applications

We conclude this article with an outline of the theoretical and numerical analysis of the Hodge-Laplace equation with mixed boundary conditions. We develop a mixed finite element method on the basis of a saddle point formulation of the Hodge Laplace equation. The smoothed projection, whose construction has been the major goal of the preceding sections, is the decisive technical component in proving the stability and convergence of the mixed finite element method.

We briefly review the analytical background (see [28]) and the formulation of mixed finite element methods in FEEC (see [5]). We eventually discuss examples in the language of vector analysis. Whereas this is similar to the theory of Hodge-Laplace equation with non-mixed boundary conditions, the analytical results are relatively recent and some non-trivial aspects include the harmonic forms with mixed boundary conditions and their discrete counterparts.

7.1. **Hodge Laplacian with Mixed Boundary Conditions.** Let us assume that Ω is a bounded weakly Lipschitz domain and that $(\Gamma_T,\Gamma_I,\Gamma_N)$ is an admissible boundary partition. To start with, we review the L^2 de Rham complex with partial boundary conditions. We write

$$H_T\Lambda^k(\Omega) := W^{2,2}\Lambda^k(\Omega,\Gamma_T), \quad H_N^\star\Lambda^k(\Omega,\Gamma_N) := \star W^{2,2}\Lambda^{n-k}(\Omega,\Gamma_N). \tag{7.1}$$

These spaces are naturally Hilbert spaces. We know by Proposition 4.4 and Proposition 4.3(i) of [28] that the unbounded linear operators

$$\mathsf{d} : H_T\Lambda^k(\Omega) \subseteq L^2\Lambda^k(\Omega) \to H_T\Lambda^{k+1}(\Omega), \tag{7.2a}$$

$$\delta : H_N^\star\Lambda^k(\Omega) \subseteq L^2\Lambda^k(\Omega) \to H_N^\star\Lambda^{k-1}(\Omega) \tag{7.2b}$$

are densely-defined, closed, have closed range, and are mutually adjoint. Thus

$$0 \longrightarrow H_T\Lambda^0(\Omega) \xrightarrow{\ \mathsf{d}\ } \dots \xrightarrow{\ \mathsf{d}\ } H_T\Lambda^n(\Omega) \longrightarrow 0, \tag{7.3a}$$

$$0 \longleftarrow H_N^\star\Lambda^0(\Omega) \xleftarrow{\ \delta\ } \dots \xleftarrow{\ \delta\ } H_N^\star\Lambda^n(\Omega) \longleftarrow 0 \tag{7.3b}$$

are closed Hilbert complexes in the sense of [12]. Moreover, (7.3a) and (7.3b) are mutually adjoint. We call (7.3a) the L^2 *de Rham complex with tangential boundary conditions* along Γ_T, and we call (7.3b) the L^2 *de Rham complex with normal boundary conditions* along Γ_N.

The intersection $H_T\Lambda^k(\Omega)\cap H_N^\star\Lambda^k(\Omega)$ is a Hilbert space of its own which carries the intersection scalar product

$$\langle u,v\rangle_{H_T\Lambda^k(\Omega)\cap H_N^\star\Lambda^k(\Omega)} := \langle u,v\rangle_{H_T\Lambda^k(\Omega)} + \langle u,v\rangle_{H_N^\star\Lambda^k(\Omega)}.$$

The compactness of the *Rellich embedding*

$$H_T\Lambda^k(\Omega) \cap H_N^\star\Lambda^k(\Omega) \to L^2\Lambda^k(\Omega) \tag{7.4}$$

follows from Proposition 4.4 of [28].

The space of *k-th harmonic forms with mixed boundary conditions* is defined as

$$\mathfrak{H}^k(\Omega,\Gamma_T,\Gamma_N) := \left\{ p \in H_T\Lambda^k(\Omega) \cap H_N^\star\Lambda^k(\Omega) \;\middle|\; \mathsf{d}p = 0, \delta p = 0 \right\}. \tag{7.5}$$

Basic results on Hilbert spaces show that

$$\mathfrak{H}^k(\Omega,\Gamma_T,\Gamma_N) = \left(\ker \mathsf{d} : H_T\Lambda^k(\Omega) \to H_T\Lambda^{k+1}(\Omega)\right) \cap \left(\mathsf{d}H_T\Lambda^{k-1}(\Omega)\right)^\perp. \tag{7.6}$$

We also have the L^2 orthogonal *Hodge decomposition*

$$L^2\Lambda^k(\Omega) = \mathsf{d}H_T\Lambda^{k-1}(\Omega) \oplus \mathfrak{H}^k(\Omega,\Gamma_T,\Gamma_N) \oplus \delta H_N^\star\Lambda^{k+1}(\Omega). \tag{7.7}$$

From Proposition 4.3(i) in [28] we find that $\mathfrak{H}^k(\Omega,\Gamma_T,\Gamma_N)$ is finite-dimensional. The dimension of $\mathfrak{H}^k(\Omega,\Gamma_T,\Gamma_N)$ is of particular interest because it reflects topological properties of Ω and Γ_T. Specifically, by Theorem 5.3 in [28] we find that $\dim \mathfrak{H}^k(\Omega,\Gamma_T,\Gamma_N)$ equals the topological Betti number $b_k\left(\overline{\Omega},\Gamma_T\right)$ of $\overline{\Omega}$ relative to Γ_T, and one can show that

$$\dim \mathfrak{H}^k(\Omega,\Gamma_T,\Gamma_N) = b_k\left(\overline{\Omega},\Gamma_T\right) = b_{n-k}\left(\overline{\Omega},\Gamma_N\right), \quad 0 \leq k \leq n. \tag{7.8}$$

In the special cases $\Gamma_T = \emptyset$ and $\Gamma_T = \partial\Omega$, which have received the major part of attention in the literature, the Betti numbers correspond to the topological properties of the domain only, such as the number of connected components or of holes of a certain dimension. But in the presence of mixed boundary conditions the Betti numbers depend also on the topology of the boundary patch Γ_T.

Example 7.1. For every weakly Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ and admissible boundary partition $(\Gamma_T,\Gamma_I,\Gamma_N)$, the spaces $\mathfrak{H}^0(\Omega,\Gamma_T,\Gamma_N)$ and $\mathfrak{H}^n(\Omega,\Gamma_T,\Gamma_N)$ are spanned by the locally constant functions over Ω whose supports are disjoint from Γ_T and Γ_N, respectively. The other harmonic spaces have more complicated descriptions, but their dimensions, the Betti numbers, can sometimes be computed from the topology of Ω and Γ_T.

For example, consider $n = 2$ and let $\Omega = (-1,1)^2$. If Γ_T has $M \in \mathbb{N}$ connected components, then $b_1(\overline{\Omega},\Gamma_T) = \max\{0, M-1\}$. For instance, in the special case $\Gamma_T = (-1,1) \times \{0,1\}$ we have $b_1(\overline{\Omega},\Gamma_T) = 1$ and $\mathfrak{H}^1(\overline{\Omega},\Gamma_T,\Gamma_N)$ has a simple description: it corresponds to the span of the constant vector field taking the value $(1,0)$ over all of Ω.

We now introduce the Hodge Laplace operator. We let

$$\operatorname{dom}(\Delta_k) := \left\{ u \in H_T\Lambda^k(\Omega) \cap H_N^\star\Lambda^k(\Omega) \;\middle|\; \mathsf{d}u \in H_N^\star\Lambda^{k+1}(\Omega), \delta u \in H_T\Lambda^{k-1}(\Omega) \right\}.$$

The unbounded operator

$$\Delta_k : \operatorname{dom}(\Delta_k) \subseteq L^2\Lambda^k(\Omega) \to L^2\Lambda^k(\Omega), \quad u \mapsto \delta\mathsf{d}u + \mathsf{d}\delta u$$

is called the k-th *Hodge Laplacian*. One can show [28, Theorem 4.5] that Δ_k is densely-defined, closed, self-adjoint, and has closed range, and that furthermore

$$\mathfrak{H}^k(\Omega,\Gamma_T,\Gamma_N) = \ker \Delta_k, \quad \operatorname{ran} \Delta_k = \mathfrak{H}^k(\Omega,\Gamma_T,\Gamma_N)^\perp.$$

We let $G_k : L^2\Lambda^k(\Omega) \to L^2\Lambda^k(\Omega)$ denote the pseudoinverse of the Hodge Laplacian, where the pseudoinverse of an unbounded linear operator is understood in the sense of [8]. This means that

$$\operatorname{ran} G_k = \mathfrak{H}^k(\Omega,\Gamma_T,\Gamma_N)^\perp, \quad \ker G_k = \mathfrak{H}^k(\Omega,\Gamma_T,\Gamma_N),$$

that for all $f \in L^2\Lambda^k(\Omega) \cap \mathfrak{H}^k(\Omega,\Gamma_T,\Gamma_N)^\perp$ we have

$$G_k f \in \operatorname{dom}(\Delta_k), \quad f = \Delta_k G_k f,$$

and that for all $u \in \operatorname{dom}(\Delta_k) \cap \mathfrak{H}^k(\Omega,\Gamma_T,\Gamma_N)^\perp$ we have

$$u = G_k \Delta_k u.$$

One can show that the operator norm of G_k is bounded with respect to the L^2 norm. Additionally, G_k takes values in the intersection $H_T\Lambda^k(\Omega) \cap H_N^\star\Lambda^k(\Omega)$ and is bounded as an operator from $L^2\Lambda^k(\Omega)$ to $H_T\Lambda^k(\Omega) \cap H_N^\star\Lambda^k(\Omega)$.

The k-th *Hodge Laplace equation* with mixed boundary conditions is the partial differential equation

$$\Delta_k u = f, \tag{7.9}$$

where $f \in L^2\Lambda^k(\Omega)$ is given, and where $u \in \operatorname{dom}(\Delta_k)$ is the unknown. In general, we can solve the k-th Hodge-Laplace equation only in the sense of least squares whenever there exist non-trivial k-th harmonic forms; The least-squares solution is given precisely by $u = G_k f$. By definition, whenever $u \in \operatorname{dom}(\Delta_k)$ is any solution of (7.9), then $u \in H_T\Lambda^k(\Omega)$ and $\delta u \in H_T\Lambda^{k-1}(\Omega)$ satisfy tangential boundary conditions along Γ_T and $u \in H_N^*\Lambda^k(\Omega)$ and $\delta u \in H_N\Lambda^{k+1}(\Omega)$ satisfy normal boundary conditions along Γ_N.

7.2. **Variational Theory and Finite Element Approximation.** The operator Δ_k is self-adjoint with closed range. Hence it is tempting to base a finite element method for solving $\Delta_k u = f$ on minimizing the energy functional

$$\mathcal{J}(v) := \frac{1}{2}\int_\Omega |\mathsf{d}v|^2 + |\delta v|^2 - \langle f, v\rangle \,\mathrm{d}x$$

over a piecewise polynomial subspace of $H_T\Lambda^k(\Omega) \cap H_N^\star\Lambda^k(\Omega)$. While for $k = 0$ this is just the canonical approach for the Poisson problem (already mentioned in the introduction), this approach is problematic when $k > 0$. To begin with, we generally have $\dim \mathfrak{H}^k(\Omega,\Gamma_T,\Gamma_N) \neq 0$, and so we need to introduce a Lagrange multiplier in order to accommodate for the kernel of the Hodge Laplacian. In practice, this leads to a variational crime, because $\mathfrak{H}^k(\Omega,\Gamma_T,\Gamma_N)$ generally does not contain piecewise polynomial differential forms. A more severe problem is that piecewise polynomial differential forms will generally fail to approximate a member of $H_T\Lambda^k(\Omega) \cap H_N^\star\Lambda^k(\Omega)$ in the norm of that space, and hence a finite element method based on minimizing the energy $\mathcal{J}$ is generally inconsistent [19].

To avoid those difficulties, we may use a mixed finite element method based on reformulating the Hodge Laplace problem as a saddle point problem. The idea is to introduce the codifferential of the solution u as an auxiliary variable $\sigma = \delta u$. A weak formulation of the Hodge Laplace equation is stated in [5] as

$$\langle \sigma, \tau\rangle_{L^2} - \langle u, \mathsf{d}\tau\rangle_{L^2} = 0, \qquad \tau \in H_T\Lambda^{k-1}(\Omega), \tag{7.10a}$$

$$\langle \mathsf{d}\sigma, v\rangle_{L^2} + \langle \mathsf{d}u, \mathsf{d}v\rangle_{L^2} + \langle p, v\rangle_{L^2} = \langle f, v\rangle_{L^2}, \qquad v \in H_T\Lambda^k(\Omega), \tag{7.10b}$$

$$\langle u, q\rangle_{L^2} = 0, \qquad q \in \mathfrak{H}^k(\Omega,\Gamma_T,\Gamma_N), \tag{7.10c}$$

where $\sigma \in H_T\Lambda^{k-1}(\Omega)$, $u \in H_T\Lambda^k(\Omega)$, and $p \in \mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N)$ are the unknowns. These are the Euler-Lagrange equations of a saddle point functional. One can show [5, Theorem 3.1] that (7.10) has a unique solution for every $f \in L^2\Lambda^k(\Omega)$. Furthermore, by the discussion in [5, Subsection 3.2.1] we see that (7.10) is equivalent to (7.9) in the following sense: the tuple (σ, u, p) solves (7.10) if and only if

$$u = G_k f, \quad p = f - G_k f, \quad \sigma = \delta u.$$

One can show that for a constant $C_\Delta > 0$ that only depends on Ω and Γ_T we have

$$\|\sigma\|_{H\Lambda^{k-1}(\Omega)} + \|u\|_{H\Lambda^k(\Omega)} + \|p\|_{L^2\Lambda^k(\Omega)} \leq C_\Delta \|f\|_{L^2\Lambda^k(\Omega)} \tag{7.11}$$

Hence the saddle point problem is well-posed. We see that these equations are posed only over the spaces with well-defined exterior derivative. We have already seen conforming finite element spaces of $H_T\Lambda^k(\Omega)$ in previous sections, and hence it seems to be a promising approach to build a finite element method based on (7.10). It is instructive to start with a finite element de Rham complex as a discretization of the L^2 de Rham complex (7.3a) with tangential boundary conditions. Specifically, we fix a simplicial complex $\mathcal{T}$ that triangulates Ω and that contains a subtriangulations $\mathcal{U}$ of Γ_T, and moreover we fix a finite element de Rham complex with partial boundary conditions as outlined in the preceding Section.

$$0 \longrightarrow \mathcal{P}\Lambda^0(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}} \dots \xrightarrow{\mathsf{d}} \mathcal{P}\Lambda^n(\mathcal{T},\mathcal{U}) \longrightarrow 0. \tag{7.12}$$

Then the conditions of Section 6 are satisfied and we can apply Theorem 6.3. Thus there exists a bounded projection $\pi^k : L^2\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$ from the original de Rham complex (7.3a) with tangential boundary conditions along Γ_T to the finite element de Rham complex (7.12).

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_T\Lambda^0(\mathcal{T}) & \xrightarrow{\mathsf{d}} & \dots & \xrightarrow{\mathsf{d}} & H_T\Lambda^n(\mathcal{T}) & \longrightarrow & 0 \\
 & & \pi^0 \downarrow & & & & \pi^n \downarrow & & \\
0 & \longrightarrow & \mathcal{P}\Lambda^0(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}} & \dots & \xrightarrow{\mathsf{d}} & \mathcal{P}\Lambda^n(\mathcal{T},\mathcal{U}) & \longrightarrow & 0
\end{array}$$

The operator norm of π^k is bounded in terms of the polynomial degree of the finite element spaces, the mesh quality, and geometric properties of Ω.

A central concept on the analytical level which we want to mimic on the discrete level are the harmonic forms. We define the *k-th discrete harmonic space* as

$$\mathfrak{H}^k(\mathcal{T},\mathcal{U}) := \left(\ker \mathsf{d} : \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \to \mathcal{P}\Lambda^{k+1}(\mathcal{T},\mathcal{U})\right) \cap \left(\mathsf{d}\mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U})\right)^\perp.$$

We note that this definition of discrete harmonic k-forms $\mathfrak{H}^k(\mathcal{T},\mathcal{U})$ is entirely analogous to the identity satisfied by the harmonic k-forms $\mathfrak{H}^k(\overline{\Omega}, \Gamma_T)$. The dimension of $\mathfrak{H}^k(\mathcal{T},\mathcal{U})$ is the Betti number $b_k(\overline{\Omega}, \Gamma_T)$ of $\overline{\Omega}$ relative to Γ_T, as follows, e.g., by Corollary 2 in [40], but can also be shown with an adaption of methods in [5]. In particular, the dimension of $\mathfrak{H}^k(\mathcal{T},\mathcal{U})$ depends only on Ω and Γ_T.

We outline a mixed finite element method for the Hodge Laplace equation (7.10). Given $f \in L^2\Lambda^k(\Omega)$, we search for $\sigma_h \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$, $u_h \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$, and $p_h \in \mathfrak{H}^k(\mathcal{T},\mathcal{U})$ such that

$$\langle \sigma_h, \tau_h \rangle_{L^2} - \langle u_h, \mathsf{d}\tau_h \rangle_{L^2} = 0, \qquad \tau_h \in \mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U}), \tag{7.13a}$$

$$\langle \mathsf{d}\sigma_h, v_h \rangle_{L^2} + \langle \mathsf{d}u_h, \mathsf{d}v_h \rangle_{L^2} + \langle p_h, v_h \rangle_{L^2} = \langle f, v_h \rangle_{L^2}, \quad v_h \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}), \tag{7.13b}$$

$$\langle u_h, q_h \rangle_{L^2} = 0, \qquad q_h \in \mathfrak{H}^k(\mathcal{T},\mathcal{U}). \tag{7.13c}$$

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