Chapter 6
Lagrangian and Hamiltonian Dynamics on $\mathsf{SO}(3)$

This chapter treats the Lagrangian dynamics and Hamiltonian dynamics of a rotating rigid body. A rigid body is a collection of mass particles whose relative positions do not change, that is the body does not deform when acted on by external forces. A rigid body is a useful idealization.

The most general form of rigid body motion consists of a combination of rotation and translation. In this chapter, we consider rotational motion only. Combined rotational and translational dynamics of a rigid body are studied in the subsequent chapter.

We begin by identifying the configurations of a rotating rigid body in three dimensions as elements of the Lie group $\mathsf{SO}(3)$. Equations of motion for the Lagrangian and Hamiltonian dynamics, expressed as Euler–Lagrange (or Euler) equations and Hamilton's equations, are developed for rigid body rotations in three dimensions. These results are illustrated by several examples of the rotational dynamics of a rigid body.

There are many books and research papers that treat rigid body kinematics and dynamics from both theoretical and applied perspectives. It is a common approach in the published literature to describe rigid body kinematics and dynamics in terms of rotation matrices, but not to fully exploit such geometric representations. For example, books such as [5, 10, 26, 30, 32, 70] introduce rotation matrices but make substantial use of local coordinates, such as Euler angles, in analysis and computations. The references [23, 40, 68, 77] are notable for their emphasis on rotation matrices as the primary representation for kinematics and dynamics of rigid body motion on $\mathsf{SO}(3)$ in applications to spacecraft and robotics. In the context of multi-body spacecraft control, [84] was one of the first publications formulating multi-body dynamics using the configuration manifold $(\mathsf{SO}(3))^n$.

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6.1 Configurations as Elements in the Lie Group SO(3)

Two Euclidean frames are introduced; these aid in defining the attitude configuration of a rotating rigid body. A reference Euclidean frame is arbitrarily selected; it is often selected to be an inertial frame but this is not essential. A Euclidean frame fixed to the rigid body is also introduced; this fixed frame rotates as the rigid body rotates. The origin of this body-fixed frame can be arbitrarily selected, but it is often convenient to locate it at the center of mass of the rigid body.

As a manifold embedded in $\mathsf{GL}(3)$ or $\mathbb{R}^{3\times 3}$, recall that

$$\mathsf{SO}(3) = \left\{R \in \mathsf{GL}(3) : R^T R = RR^T = I_{3\times 3},\, \mathsf{det}(R) = +1\right\},$$

has dimension three. The tangent space of $\mathsf{SO}(3)$ at $R \in \mathsf{SO}(3)$ is given by

$$\mathsf{T}_R\mathsf{SO}(3) = \left\{R\xi \in \mathbb{R}^{3\times 3} : \xi \in \mathfrak{so}(3)\right\},$$

and has dimension three. The tangent bundle of $\mathsf{SO}(3)$ is given by

$$\mathsf{TSO}(3) = \left\{(R, R\xi) \in \mathsf{SO}(3) \times \mathbb{R}^{3\times 3} : \xi \in \mathfrak{so}(3)\right\},$$

and has dimension six.

We can view $R \in \mathsf{SO}(3)$ as representing the attitude of the rigid body, so that $\mathsf{SO}(3)$ is the configuration manifold for rigid body rotational motion. An attitude matrix $R \in \mathsf{SO}(3)$ can be viewed as a linear transformation on $\mathbb{R}^3$ in the sense that a representation of a vector in the body-fixed frame is transformed into a representation of the vector in the reference frame. Thus, the transpose of an attitude matrix $R^T \in \mathsf{SO}(3)$ denotes a linear transformation from a representation of a vector in the reference frame into a representation of the vector in the body-fixed frame. These two important properties are summarized as:

- If $b \in \mathbb{R}^3$ is a representation of a vector expressed in the body-fixed frame, then $Rb \in \mathbb{R}^3$ denotes the same vector in the reference frame.
- If $x \in \mathbb{R}^3$ is a representation of a vector expressed in the reference frame, then $R^T x \in \mathbb{R}^3$ denotes the same vector in the body-fixed frame.

These are important relationships that are used extensively in the subsequent developments.

In addition, $R \in \mathsf{SO}(3)$ can be viewed as defining a rigid body rotation on $\mathbb{R}^3$ according to the rules of matrix multiplication. In this interpretation, $R \in \mathsf{SO}(3)$ is viewed as a rotation matrix that defines a linear transformation that acts on rigid body attitudes. This makes $\mathsf{SO}(3)$ a Lie group manifold using standard matrix multiplication as the group operation, as discussed in Chapter 1. Since the dimension of $\mathsf{SO}(3)$ is three, rigid body rotational motion has three degrees of freedom.

6.2 Kinematics on SO(3)

The rotational kinematics of a rotating rigid body are described in terms of the time evolution of the attitude and attitude rate of the rigid body given by $(R, \dot{R}) \in \mathsf{TSO}(3)$. As in Chapter 2, the rotational kinematics equations for a rotating rigid body are given by

$$\dot{R} = R\xi,$$

where $\xi \in \mathfrak{so}(3)$.

We make use of the isomorphism between the Lie algebra $\mathfrak{so}(3)$ and $\mathbb{R}^3$ given by $\xi = S(\omega)$ with $\xi \in \mathfrak{so}(3)$, $\omega \in \mathbb{R}^3$. This perspective is utilized in the subsequent development. This leads to the expression for the attitude or rotational kinematics given by

$$\dot{R} = RS(\omega), \tag{6.1}$$

where $\omega \in \mathbb{R}^3$ is referred to as the angular velocity vector of the rigid body expressed in the body-fixed Euclidean frame.

It is sometimes convenient to partition the rigid body attitude or rotation $R \in \mathsf{SO}(3)$ as a 3×3 matrix into its rows. We use the notation $r_i \in \mathsf{S}^2$ to denote the i-th column of $R^T \in \mathsf{SO}(3)$ for $i = 1, 2, 3$. This is equivalent to the partition

$$R = \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix}.$$

Thus, the rotational kinematics of a rotating rigid body can also be described by the three vector differential equations

$$\dot{r}_i = -\xi r_i, \quad i = 1, 2, 3,$$

or equivalently by

$$\dot{r}_i = S(r_i)\omega, \quad i = 1, 2, 3.$$

We subsequently describe the attitude configuration of a rotating rigid body by the equivalent descriptions $R \in \mathsf{SO}(3)$ or $r_i \in \mathsf{S}^2$, $i = 1, 2, 3$, depending on whichever is the most convenient description.

6.3 Lagrangian Dynamics on $\mathsf{SO}(3)$

A Lagrangian function is introduced. Euler–Lagrange equations are derived using Hamilton's principle that the infinitesimal variation of the action integral is zero. The Euler–Lagrange equations are first expressed for an arbitrary Lagrangian function; then Euler–Lagrange equations are obtained for the case that the kinetic energy term in the Lagrangian function is a quadratic function of the angular velocity vector.

6.3.1 Hamilton's Variational Principle

The Lagrangian function is defined on the tangent bundle of $\mathsf{SO}(3)$, that is $L : \mathsf{TSO}(3) \to \mathbb{R}^1$.

We identify the tangent bundle $\mathsf{TSO}(3)$ with $\mathsf{SO}(3) \times \mathfrak{so}(3)$ or with $\mathsf{SO}(3) \times \mathbb{R}^3$ using the isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$. Thus, we can express the Lagrangian as a function $L(R, \dot{R}) = L(R, R\xi) = L(R, RS(\omega))$ defined on the tangent bundle $\mathsf{TSO}(3)$. We make use of the modified Lagrangian function $\tilde{L}(R, \omega) = L(R, RS(\omega))$, where we view $\tilde{L} : \mathsf{TSO}(3) \to \mathbb{R}^1$ according to the kinematics (6.1).

In studying the dynamics of a rotating rigid body, the Lagrangian function is the difference of a kinetic energy function and a potential energy function; thus the modified Lagrangian function is

$$\tilde{L}(R, \omega) = T(R, \omega) - U(R),$$

where the kinetic energy function $T(R, \omega)$ is viewed as being defined on the tangent bundle $\mathsf{TSO}(3)$ and the potential energy function $U(R)$ is defined on $\mathsf{SO}(3)$.

The subsequent development describes variations of functions with values in the special orthogonal group $\mathsf{SO}(3)$; rather than using the abstract Lie group formalism, we obtain the results explicitly for the rotation group. In particular, we introduce variations of a rotational motion $t \to R(t) \in \mathsf{SO}(3)$, denoted by $t \to R^\epsilon(t) \in \mathsf{SO}(3)$, by using the exponential map and the isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$.

The variation of $R : [t_0, t_f] \to \mathsf{SO}(3)$ is a differentiable curve $R^\epsilon : (-c, c) \times [t_0, t_f] \to \mathsf{SO}(3)$ for $c > 0$ such that $R^0(t) = R(t)$, and $R^\epsilon(t_0) = R(t_0)$, $R^\epsilon(t_f) = R(t_f)$ for any $\epsilon \in (-c, c)$.

The variation of a rotational motion can be described using the exponential map as

$$R^\epsilon(t) = R(t)e^{\epsilon S(\eta(t))},$$

where $\epsilon \in (-c, c)$ and $\eta : [t_0, t_f] \to \mathbb{R}^3$ is a differentiable curve that vanishes at t_0 and t_f. Consequently, $S(\eta(t)) \in \mathfrak{so}(3)$ defines a differentiable curve with values in the Lie algebra of skew symmetric matrices that vanishes at t_0 and t_f, and $e^{\epsilon S(\eta(t))} \in \mathsf{SO}(3)$ defines a differentiable curve that takes values in the Lie group of rotation matrices and is the identity matrix at t_0 and t_f. Thus, the time derivative of the variation of the rotational motion of a rigid body is

$$\dot{R}^\epsilon(t) = \dot{R}(t)e^{\epsilon S(\eta(t))} + \epsilon R(t)e^{\epsilon S(\eta(t))}S(\dot{\eta}(t)).$$

Suppressing the time dependence in the subsequent notation, the varied curve satisfies

$$\begin{aligned}
\xi^\epsilon &= (R^\epsilon)^T \dot{R}^\epsilon \\
&= e^{-\epsilon S(\eta)}\xi e^{\epsilon S(\eta)} + \epsilon S(\dot{\eta}) \\
&= \xi + \epsilon\left(S(\dot{\eta}) + \xi S(\eta) - S(\eta)\xi\right) + \mathcal{O}(\epsilon^2).
\end{aligned}$$

Define the variation of the angular velocity by $\xi^\epsilon = S(\omega^\epsilon)$ and use the fact that $\xi = S(\omega)$ to obtain

$$S(\omega^\epsilon) = S(\omega) + \epsilon(S(\dot{\eta}) + S(\omega)S(\eta) - S(\eta)S(\omega)) + \mathcal{O}(\epsilon^2).$$

We use a skew symmetric matrix identity to obtain

$$S(\omega^\epsilon) = S(\omega) + \epsilon(S(\dot{\eta}) + S(\omega \times \eta)) + \mathcal{O}(\epsilon^2),$$

or equivalently

$$S(\omega^\epsilon) = S(\omega + \epsilon(\dot{\eta} + \omega \times \eta)) + \mathcal{O}(\epsilon^2).$$

Thus, the variation of the angular velocity satisfies

$$\omega^\epsilon = \omega + \epsilon\left(\dot{\eta} + \omega \times \eta\right) + \mathcal{O}(\epsilon^2).$$

From these expressions, we determine the infinitesimal variations

$$\delta R = \frac{d}{d\epsilon}R^\epsilon\bigg|_{\epsilon=0} = RS(\eta), \tag{6.2}$$

$$\delta\omega = \frac{d}{d\epsilon}\omega^\epsilon\bigg|_{\epsilon=0} = \dot{\eta} + \omega \times \eta = \dot{\eta} + S(\omega)\eta. \tag{6.3}$$

This framework allows us to introduce the action integral and Hamilton's principle to obtain Euler–Lagrange equations that describe the rotational dynamics of a rigid body.

The action integral is the integral of the Lagrangian function, or equivalently the modified Lagrangian function, along a rotational motion of the rigid body over a fixed time period. The variations are taken over all differentiable curves with values in SO(3) for which the initial and final values are fixed.

The action integral along a rotational motion of a rotating rigid body is

$$\mathfrak{G} = \int_{t_0}^{t_f} \tilde{L}(R, \omega)\, dt.$$

The action integral along a variation of a rotational motion of the rigid body is

$$\mathfrak{G}^\epsilon = \int_{t_0}^{t_f} \tilde{L}(R^\epsilon, \omega^\epsilon)\, dt.$$

The varied value of the action integral along a variation of a rotational motion of the rigid body can be expressed as a power series in ϵ as

$$\mathfrak{G}^\epsilon = \mathfrak{G} + \epsilon\delta\mathfrak{G} + \mathcal{O}(\epsilon^2),$$

where the infinitesimal variation of the action integral is

$$\delta\mathfrak{G} = \frac{d}{d\epsilon}\mathfrak{G}^\epsilon\bigg|_{\epsilon=0}.$$

Hamilton's principle states that the infinitesimal variation of the action integral along any rotational motion of the rigid body is zero:

$$\delta\mathfrak{G} = \frac{d}{d\epsilon}\mathfrak{G}^\epsilon\bigg|_{\epsilon=0} = 0, \tag{6.4}$$

for all possible infinitesimal variations $\eta : [t_0, t_f] \to \mathbb{R}^3$ satisfying $\eta(t_0) = \eta(t_f) = 0$.

6.3.2 Euler–Lagrange Equations: General Form

We first compute the infinitesimal variation of the action integral as

$$\frac{d}{d\epsilon}\mathfrak{G}^\epsilon\bigg|_{\epsilon=0} = \int_{t_0}^{t_f} \left\{ \frac{\partial \tilde{L}(R,\omega)}{\partial \omega} \cdot \delta\omega + \frac{\partial \tilde{L}(R,\omega)}{\partial R} \cdot \delta R \right\} dt.$$

Examining the first term, we obtain

$$\int_{t_0}^{t_f} \frac{\partial \tilde{L}(R,\omega)}{\partial \omega} \cdot \delta\omega \, dt = \int_{t_0}^{t_f} \frac{\partial \tilde{L}(R,\omega)}{\partial \omega} \cdot (\dot{\eta} + \omega \times \eta) \, dt$$

$$= -\int_{t_0}^{t_f} \left\{ \frac{d}{dt}\left(\frac{\partial \tilde{L}(R,\omega)}{\partial \omega} \right) + S(\omega) \frac{\partial \tilde{L}(R,\omega)}{\partial \omega} \right\} \cdot \eta \, dt,$$

where the first term is integrated by parts, using the fact that $\eta(t_0) = \eta(t_f) = 0$, and the second term is rewritten using a cross product identity.

The second term above is now rewritten. We use the notation $r_i \in \mathsf{S}^2$ and $\delta r_i \in \mathsf{T}_{r_i}\mathsf{S}^2$ to denote the i-th column of $R^T \in \mathsf{SO(3)}$ and $\delta R^T \in \mathsf{T}_R\mathsf{SO(3)}$, respectively. This is equivalent to partitioning R and δR into row vectors as

$$R = \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix}, \quad \delta R = \begin{bmatrix} \delta r_1^T \\ \delta r_2^T \\ \delta r_3^T \end{bmatrix}.$$

We use the fact that $\delta r_i = S(r_i)\eta$ to obtain

$$\int_{t_0}^{t_f} \frac{\partial \tilde{L}(R,\omega)}{\partial R} \cdot \delta R \, dt = \int_{t_0}^{t_f} \sum_{i=1}^{3} \frac{\partial \tilde{L}(R,\omega)}{\partial r_i} \cdot \delta r_i \, dt$$

$$= \int_{t_0}^{t_f} \sum_{i=1}^{3} \frac{\partial \tilde{L}(R,\omega)}{\partial r_i} \cdot S(r_i)\eta \, dt$$

$$= -\int_{t_0}^{t_f} \sum_{i=1}^{3} \left(S(r_i) \frac{\partial \tilde{L}(R,\omega)}{\partial r_i} \right) \cdot \eta \, dt. \tag{6.5}$$

Substituting, the expression for the infinitesimal variation of the action integral is obtained:

$$\left. \frac{d}{d\epsilon} \mathfrak{G}^\epsilon \right|_{\epsilon=0}$$
$$= -\int_{t_0}^{t_f} \left\{ \frac{d}{dt}\left(\frac{\partial \tilde{L}(R,\omega)}{\partial \omega} \right) + S(\omega) \frac{\partial \tilde{L}(R,\omega)}{\partial \omega} + \sum_{i=1}^{3} S(r_i) \frac{\partial \tilde{L}(R,\omega)}{\partial r_i} \right\} \cdot \eta \, dt.$$

From Hamilton's principle, the above expression for the infinitesimal variation of the action integral should be zero for all differentiable variations $\eta : [t_0, t_f] \to \mathbb{R}^3$ with fixed endpoints. The fundamental lemma of the calculus of variations leads to the Euler–Lagrange equations.

Proposition 6.1 *The Euler–Lagrange equations for a modified Lagrangian function* $\tilde{L} : \mathsf{TSO(3)} \to \mathbb{R}^1$ *are*

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}(R,\omega)}{\partial \omega} \right) + \omega \times \frac{\partial \tilde{L}(R,\omega)}{\partial \omega} + \sum_{i=1}^{3} r_i \times \frac{\partial \tilde{L}(R,\omega)}{\partial r_i} = 0. \tag{6.6}$$

This form of the Euler–Lagrange equations, together with the rotational kinematics equations (6.1), describe the Lagrangian flow of a rotating rigid body on the tangent bundle $\mathsf{TSO}(3)$ in terms of $(R, \omega) \in \mathsf{TSO}(3)$.

6.3.3 Euler–Lagrange Equations: Quadratic Kinetic Energy

We now determine a more explicit expression for the kinetic energy of a rotating rigid body. This expression is used to obtain a standard form of the Euler–Lagrange equations. For simplicity, the reference frame is assumed to be an inertial frame, and the origin of the body-fixed frame is assumed to be located at the center of mass of the rigid body.

Let $\rho \in \mathcal{B} \subset \mathbb{R}^3$ be a vector from the origin of the body-fixed frame to a mass element of the rigid body expressed in the body-fixed frame. Here $\mathcal{B}$ denotes the set of material points that constitute the rigid body in the body-fixed frame. Thus, $\dot{R}\rho$ is the velocity vector of this mass element in the inertial frame. The kinetic energy of the rotating rigid body can be expressed as the body integral

$$\begin{aligned} T(R,\omega) &= \frac{1}{2}\int_{\mathcal{B}} \|\dot{R}\rho\|^2 \, dm(\rho) \\ &= \frac{1}{2}\int_{\mathcal{B}} \|RS(\rho)\omega\|^2 \, dm(\rho) \\ &= \frac{1}{2}\omega^T \left(\int_{\mathcal{B}} S(\rho)^T S(\rho) \, dm(\rho) \right) \omega, \end{aligned}$$

where $dm(\rho)$ denotes the mass of the incremental element located at $\rho \in \mathcal{B}$. Thus, we can express the kinetic energy as a quadratic function of the angular velocity vector

$$T(R,\omega) = \frac{1}{2}\omega^T J \omega,$$

where

$$J = \int_{\mathcal{B}} S(\rho)^T S(\rho) \, dm(\rho),$$

is the 3×3 standard inertia matrix of the rigid body that characterizes the rotational inertia of the rigid body in the body-fixed frame.

The inertia matrix can be shown to be a symmetric and positive-definite matrix. It has three positive eigenvalues and three eigenvectors that form an orthonormal basis for $\mathbb{R}^3$. This special basis defines the principal axes of the rigid body and it is sometimes convenient to select the body-fixed frame to be aligned with the principal axes of the body. In this case, the inertia matrix is diagonal.

Consequently, the modified Lagrangian function has the special form

$$\tilde{L}(R,\omega) = \frac{1}{2}\omega^T J\omega - U(R). \tag{6.7}$$

This gives the standard form of the equations for a rotating rigid body, often referred to as the Euler equations for rigid body rotational dynamics, as

$$J\dot{\omega} + S(\omega)J\omega - \sum_{i=1}^{3} S(r_i)\frac{\partial U(R)}{\partial r_i} = 0. \tag{6.8}$$

These Euler equations (6.8), together with the rotational kinematics (6.1), describe the Lagrangian flow of a rotating rigid body in terms of the evolution of $(R,\omega) \in \mathsf{TSO}(3)$ on the tangent bundle $\mathsf{TSO}(3)$.

If the potential energy terms in (6.8) are globally defined on $\mathbb{R}^{3\times 3}$, then the domain of definition of the rotational kinematics (6.1) and the Euler equations (6.8) on $\mathsf{TSO}(3)$ can be extended to $\mathsf{T}\mathbb{R}^{3\times 3}$. This extension is natural and useful in that it defines a Lagrangian vector field on the tangent bundle $\mathsf{T}\mathbb{R}^{3\times 3}$ Alternatively, the manifold $\mathsf{TSO}(3)$ is an invariant manifold of this Lagrangian vector field on $\mathsf{T}\mathbb{R}^{3\times 3}$ and its restriction to this invariant manifold describes the Lagrangian flow of (6.1) and (6.8) on $\mathsf{TSO}(3)$.

6.4 Hamiltonian Dynamics on SO(3)

We introduce the Legendre transformation to obtain the angular momentum and the Hamiltonian function. We make use of Hamilton's phase space variational principle to derive Hamilton's equations for a rotating rigid body.

6.4.1 Hamilton's Phase Space Variational Principle

As in the prior section, we begin with a modified Lagrangian function $\tilde{L}$: $\mathsf{TSO}(3) \to \mathbb{R}^1$, which is a real-valued function defined on the tangent bundle of the configuration manifold $\mathsf{SO}(3)$; we assume that the modified Lagrangian function

$$\tilde{L}(R,\omega) = T(R,\omega) - U(R),$$

is given by the difference between a kinetic energy function $T(R,\omega)$ defined on the tangent bundle and a configuration dependent potential energy function $U(R)$.

The angular momentum of the rotating rigid body in the body-fixed frame is defined by the Legendre transformation

$$\Pi = \frac{\partial \tilde{L}(R, \omega)}{\partial \omega}, \tag{6.9}$$

where we assume the Lagrangian has the property that the map $\omega \in \mathfrak{so}(3) \to \Pi \in \mathfrak{so}(3)^*$ is invertible. The angular momentum is viewed as being conjugate to the angular velocity vector.

The modified Hamiltonian function $\tilde{H} : \mathsf{T}^*\mathsf{SO}(3) \to \mathbb{R}^1$ is defined on the cotangent bundle of $\mathsf{SO}(3)$ by

$$\tilde{H}(R, \Pi) = \Pi \cdot \omega - \tilde{L}(R, \omega),$$

using the Legendre transformation.

Consider the modified action integral of the form,

$$\tilde{\mathfrak{G}} = \int_{t_0}^{t_f} \left\{ \Pi \cdot \omega - \tilde{H}(R, \Pi) \right\} dt.$$

The infinitesimal variation of the action integral is given by

$$\delta\tilde{\mathfrak{G}} = \int_{t_0}^{t_f} \left\{ \Pi \cdot \delta\omega - \frac{\partial \tilde{H}(R, \Pi)}{\partial R} \cdot \delta R + \delta\Pi \cdot \left(\omega - \frac{\partial \tilde{H}(R, \Pi)}{\partial \Pi} \right) \right\} dt.$$

Recall from (6.2) and (6.3) that the infinitesimal variations can be written as

$$\delta R = R S(\eta),$$
$$\delta\omega = \dot{\eta} + S(\omega)\eta,$$

for differentiable curves $\eta : [t_0, t_f] \to \mathbb{R}^3$. Following the arguments used to obtain (6.5),

$$\int_{t_0}^{t_f} \frac{\partial \tilde{H}(R, \Pi)}{\partial R} \cdot \delta R \, dt = -\int_{t_0}^{t_f} \sum_{i=1}^{3} \left(S(r_i) \frac{\partial \tilde{H}(R, \Pi)}{\partial r_i} \right) \cdot \eta \, dt.$$

6.4.2 Hamilton's Equations: General Form

We now derive Hamilton's equations. Substitute the preceding expressions into the expression for the infinitesimal variation of the modified action integral and integrate by parts to obtain

$$\delta\tilde{\mathfrak{G}} = \int_{t_0}^{t_f} \Pi \cdot (\dot{\eta} + S(\omega)\eta) + \sum_{i=1}^{3} \left(S(r_i) \frac{\partial \tilde{H}(R, \Pi)}{\partial r_i} \right) \cdot \eta$$

$$+ \delta\Pi \cdot \left(\omega - \frac{\partial \tilde{H}(R, \Pi)}{\partial \Pi} \right) dt$$

$$= \int_{t_0}^{t_f} \left\{ -\dot{\Pi} - S(\omega)\Pi + \sum_{i=1}^{3} \left(S(r_i) \frac{\partial \tilde{H}(R, \Pi)}{\partial r_i} \right) \right\} \cdot \eta$$

$$+ \delta\Pi \cdot \left(\omega - \frac{\partial \tilde{H}(R, \Pi)}{\partial \Pi} \right) dt.$$

Invoke Hamilton's phase space variational principle that $\delta\tilde{\mathfrak{G}} = 0$ for all possible functions $\eta : [t_0, t_f] \to \mathbb{R}^3$ and $\delta\Pi : [t_0, t_f] \to \mathbb{R}^3$ that satisfy $\eta(t_0) = \eta(t_f) = 0$. This implies that the expression in each of the braces of the above equation should be zero. We thus obtain Hamilton's equations, expressed in terms of (R, Π).

Proposition 6.2 *Hamilton's equations for a modified Hamiltonian function* $\tilde{H} : \mathsf{T}^*\mathsf{SO}(3) \to \mathbb{R}^1$ *are*

$$\dot{r}_i = r_i \times \frac{\partial \tilde{H}(R, \Pi)}{\partial \Pi}, \quad i = 1, 2, 3, \tag{6.10}$$

$$\dot{\Pi} = \Pi \times \frac{\partial \tilde{H}(R, \Pi)}{\partial \Pi} + \sum_{i=1}^{3} r_i \times \frac{\partial \tilde{H}(R, \Pi)}{\partial r_i}. \tag{6.11}$$

Thus, equations (6.10) and (6.11) define Hamilton's equations of motion for the dynamics of the Hamiltonian flow in terms of the evolution of $(R, \Pi) \in \mathsf{T}^*\mathsf{SO}(3)$ on the cotangent bundle $\mathsf{TSO}(3)$.

The following property follows directly from the above formulation of Hamilton's equations on $\mathsf{SO}(3)$:

$$\begin{aligned} \frac{d\tilde{H}(R, \Pi)}{dt} &= \sum_{i=1}^{3} \frac{\partial \tilde{H}(R, \Pi)}{\partial r_i} \cdot \dot{r}_i + \frac{\partial \tilde{H}(R, \Pi)}{\partial \Pi} \cdot \dot{\Pi} \\ &= \frac{\partial \tilde{H}(R, \Pi)}{\partial \Pi} \cdot S(\Pi) \frac{\partial \tilde{H}(R, \Pi)}{\partial \Pi} \\ &= 0. \end{aligned}$$

The modified Hamiltonian function is constant along each solution of Hamilton's equation. This property does not hold if the modified Hamiltonian function has a nontrivial explicit dependence on time.

6.4.3 Hamilton's Equations: Quadratic Kinetic Energy

Suppose the kinetic energy is a quadratic in the angular velocity vector

$$\tilde{L}(R,\omega) = \frac{1}{2}\omega^T J\omega - U(R).$$

The Legendre transformation gives

$$\Pi = J\omega,$$

and the modified Hamiltonian function can be expressed as

$$\tilde{H}(R,\Pi) = \frac{1}{2}\Pi^T J^{-1}\Pi + U(R). \tag{6.12}$$

Hamilton's equations for a rotating rigid body are described on the cotangent bundle $\mathsf{T}^*\mathsf{SO}(3)$ as:

$$\dot{r}_i = r_i \times J^{-1}\Pi, \quad i = 1,2,3, \tag{6.13}$$

$$\dot{\Pi} = \Pi \times J^{-1}\Pi + \sum_{i=1}^{3} r_i \times \frac{\partial U(R)}{\partial r_i}. \tag{6.14}$$

Equations (6.13) and (6.14) define Hamilton's equations of motion for rigid body dynamics and they describe the Hamiltonian flow in terms of the evolution of $(R,\Pi) \in \mathsf{T}^*\mathsf{SO}(3)$ on the cotangent bundle $\mathsf{T}^*\mathsf{SO}(3)$.

If the potential energy terms in (6.14) are globally defined on $\mathbb{R}^{3\times 3}$, then the domain of definition of (6.13) and (6.14) on $\mathsf{T}^*\mathsf{SO}(3)$ can be extended to $\mathsf{T}^*\mathbb{R}^{3\times 3}$. This extension is natural and useful in that it defines a Hamiltonian vector field on the cotangent bundle $\mathsf{T}^*\mathbb{R}^{3\times 3}$ Alternatively, the manifold $\mathsf{T}^*\mathsf{SO}(3)$ is an invariant manifold of this Hamiltonian vector field on $\mathsf{T}^*\mathbb{R}^{3\times 3}$ and its restriction to this invariant manifold describes the Hamiltonian flow of (6.13) and (6.14) on $\mathsf{T}^*\mathsf{SO}(3)$.

6.5 Linear Approximations of Dynamics on $\mathsf{SO}(3)$

Geometric forms of the Euler–Lagrange equations and Hamilton's equations on the configuration manifold $\mathsf{SO}(3)$ have been presented. This yields equations of motion that provide insight into the geometry of the global dynamics on $\mathsf{SO}(3)$.

A linear vector field can be determined that approximates the Lagrangian vector field on $\mathsf{TSO}(3)$, at least locally in an open subset of $\mathsf{TSO}(3)$. Such linear approximations allow a straightforward analysis of local dynamics properties.

A common approach in the literature on the dynamics of rotating rigid bodies involves introducing local coordinates in the form of three angle coordinates; the most common local coordinates are Euler angles, but exponential local coordinates have some advantages as described in Appendix B. These descriptions often involve complicated trigonometric or transcendental expressions and introduce complexity in the analysis and computations.

Although our main emphasis is on global methods, we make use of local coordinates as a way of describing a linear vector field that approximates a vector field on $\mathsf{TSO}(3)$, at least in the neighborhood of an equilibrium solution. This approach is used subsequently in this chapter to study the local flow properties near an equilibrium. As further background, linearized equations are developed in local coordinates for $\mathsf{SO}(3)$ in Appendix B.

6.6 Dynamics on SO(3)

We study several physical examples of a rotating rigid body in three dimensions. In each, the configuration manifold is $\mathsf{SO}(3)$; consequently each of the dynamics has three degrees of freedom. Lagrangian and Hamiltonian formulations of the equations of motion are presented; a few simple flow properties are identified.

6.6.1 Dynamics of a Freely Rotating Rigid Body

We consider a freely rotating rigid body, also referred to as the free rigid body, in the sense that no moments act on the body. In this case, the prior development holds with zero potential energy $U(R) = 0$. This is the simplest case of a rotating rigid body in three dimensions.

An inertial Euclidean frame is selected arbitrarily. The origin of the body-fixed Euclidean frame is assumed to be located at the center of mass of the rigid body which is assumed to be fixed in the inertial frame. A schematic of a freely rotating rigid body is shown in Figure 6.1.

6.6.1.1 Euler–Lagrange Equations

The attitude kinematics equation for the free rigid body is described by

$$\dot{R} = RS(\omega). \tag{6.15}$$

The modified Lagrangian function $\tilde{L} : \mathsf{TSO}(3) \to \mathbb{R}^1$ is

$$\tilde{L}(R, \omega) = \frac{1}{2}\omega^T J \omega.$$

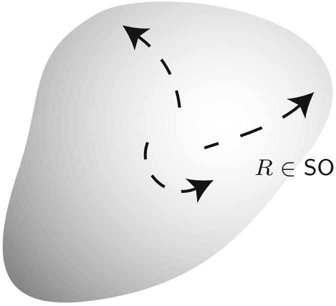

Fig. 6.1 Freely rotating rigid body

Following the results in (6.8) with zero potential energy, the Euler–Lagrange equations of motion for the free rigid body, referred to as the Euler equations, are given by

$$J\dot{\omega} + \omega \times J\omega = 0, \tag{6.16}$$

where $J = \int_{\mathcal{B}} S(\rho)^T S(\rho) dm(\rho)$ is the standard 3×3 inertia matrix of the rigid body in the body-fixed frame. These equations of motion (6.15) and (6.16) define the Lagrangian flow for the free rigid body dynamics described by the evolution of $(R, \omega) \in \mathsf{TSO}(3)$ on the tangent bundle of $\mathsf{SO}(3)$.

6.6.1.2 Hamilton's Equations

Using the Legendre transformation, let

$$\Pi = \frac{\partial \tilde{L}(R, \omega)}{\partial \omega} = J\omega$$

be the angular momentum of the free rigid body expressed in the body-fixed frame. The modified Hamiltonian is

$$\tilde{H}(R, \Pi) = \frac{1}{2} \Pi^T J^{-1} \Pi.$$

The rotational kinematics equation can be written as

$$\dot{R} = RS(J^{-1}\Pi). \tag{6.17}$$

Using (6.16), Hamilton's equations are given by

$$\dot{\Pi} = \Pi \times J^{-1}\Pi, \tag{6.18}$$

Thus, Hamilton's equations of motion (6.17) and (6.18) describe the Hamiltonian dynamics of the free rigid body as $(R, \Pi) \in \mathsf{T}^*\mathsf{SO}(3)$ as they evolve on the cotangent bundle of $\mathsf{SO}(3)$.

6.6.1.3 Conservation Properties

There are two conserved quantities, or integrals of motion, for the rotational dynamics of a free rigid body. First, the Hamiltonian, which is the rotational kinetic energy and coincides with the total energy E in this case, is conserved; that is

$$H = \frac{1}{2}\omega^T J\omega$$

is constant along each solution of the dynamical flow of the free rigid body.

In addition, there is a rotational symmetry: the Lagrangian is invariant with respect to the tangent lift of arbitrary rigid body rotations. This symmetry leads to conservation of the angular momentum in the inertial frame; that is

$$R\Pi = RJ\omega$$

is constant along each solution of the dynamical flow of the free rigid body. Consequently the magnitude of the angular momentum in the body-fixed frame is also conserved, that is

$$\|J\omega\|^2$$

is constant along each solution of the dynamical flow of the free rigid body. These results are well known for the free rigid body and they guarantee that the free rigid body is integrable [10].

There are additional conservation properties if the distribution of mass in the rigid body has a symmetry. There are many published results for such cases.

6.6.1.4 Equilibrium Properties

The equilibria or constant solutions are easily identified. The free rigid body is in equilibrium at any attitude in $\mathsf{SO}(3)$ if the angular velocity vector is zero.

To illustrate the linearization of the dynamics of a rotating rigid body, consider the equilibrium solution $(I_{3\times 3}, 0) \in \mathsf{TSO}(3)$. According to Appendix B, $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$ are exponential local coordinates for $\mathsf{SO}(3)$ in a neighborhood of $I_{3\times 3} \in \mathsf{SO}(3)$. Following the results in Appendix B, the linearized differential equations defined on the six-dimensional tangent space of $\mathsf{TSO}(3)$

at $(I_{3\times 3}, 0) \in \mathsf{TSO}(3)$ are given by

$$J\ddot{\xi} = 0.$$

These linearized differential equations approximate the rotational dynamics of the rigid body in a neighborhood of $(I_{3\times 3}, 0) \in \mathsf{TSO}(3)$. These simple linear dynamics are accurate to first-order in the perturbations expressed in local coordinates. Higher-order coupling effects are important for large perturbations of the angular velocity vector of the rigid body from equilibrium.

Solutions for which the angular velocity vector are constant can also be identified; these are referred to as relative equilibrium solutions and they necessarily satisfy

$$\omega \times J\omega = 0.$$

Thus, the relative equilibrium solutions occur when the angular velocity vector is collinear with an eigenvector of the inertia matrix J. A comprehensive treatment of relative equilibria of the free rigid body is given in [36].

6.6.2 Dynamics of a Three-Dimensional Pendulum

A three-dimensional pendulum is a rigid body supported by a fixed, frictionless pivot, acted on by uniform, constant gravity. The terminology *three-dimensional pendulum* refers to the fact that the pendulum is a rigid body, with three rotational degrees of freedom, that rotates under uniform, constant gravity. The formulation of a three-dimensional pendulum seems first to have been introduced in [87] and its dynamics developed further in [18, 20, 21, 58]. The development that follows is based on these sources.

An inertial Euclidean frame is selected so that the first two axes lie in a horizontal plane and the third axis is vertical. The origin of the inertial Euclidean frame is selected to be the location of the pendulum pivot. The body-fixed frame is selected so that its origin is located at the center of mass of the rigid body. Let m be the mass of the three-dimensional pendulum and let $\rho_0 \in \mathbb{R}^3$ be the nonzero vector from the center of mass of the body to the pivot, expressed in the body-fixed frame. Let J be the constant 3×3 inertia matrix of the rigid body described subsequently. As before, g denotes the constant acceleration of gravity. A schematic of a three-dimensional pendulum is shown in Figure 6.2.

The attitude of the rigid body is $R \in \mathsf{SO}(3)$ and $\omega \in \mathbb{R}^3$ is the angular velocity vector of the rigid body. The attitude kinematics equation for the three-dimensional pendulum is

$$\dot{R} = RS(\omega). \tag{6.19}$$

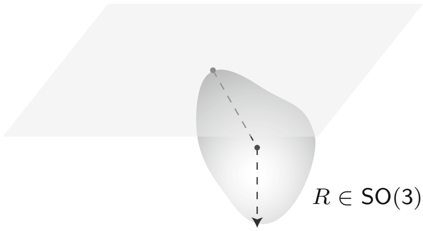

Fig. 6.2 Three-dimensional pendulum

6.6.2.1 Euler–Lagrange Equations

Let $\rho \in \mathbb{R}^3$ be a vector from the origin of the body-fixed frame to a mass element of the rigid body expressed in the body-fixed frame. Thus, $\dot{R}(-\rho_0+\rho)$ is the velocity vector of this mass element in the inertial frame. The kinetic energy of the rotating rigid body can be expressed as the body integral

$$\begin{aligned} T(R,\omega) &= \frac{1}{2}\int_{\mathcal{B}} \left\|\dot{R}(-\rho_0+\rho)\right\|^2 dm(\rho) \\ &= \frac{1}{2}\int_{\mathcal{B}} \left\|RS(-\rho_0+\rho)\omega\right\|^2 dm(\rho) \\ &= \frac{1}{2}\omega^T J\omega, \end{aligned}$$

where the moment of inertia matrix is

$$J = \int_{\mathcal{B}} S(\rho)^T S(\rho) dm(\rho) + mS^T(\rho_0)S(\rho_0).$$

The gravitational potential energy of the three-dimensional pendulum arises from the gravitational field acting on each material particle in the pendulum body. This can be expressed as

$$U(R) = -\int_{\mathcal{B}} ge_3^T R\rho \, dm(\rho) = -mge_3^T R\rho_0.$$

The modified Lagrangian function of the three-dimensional pendulum can be expressed as:

$$\tilde{L}(R,\omega) = \frac{1}{2}\omega^T J\omega + mge_3^T R\rho_0.$$

The Euler–Lagrange equations for the three-dimensional pendulum are given by

$$J\dot{\omega} + \omega \times J\omega - mg\rho_0 \times R^T e_3 = 0. \tag{6.20}$$

These equations (6.19) and (6.20) define the rotational kinematics and the Lagrangian dynamics of the three-dimensional pendulum described by $(R, \omega) \in \mathsf{TSO}(3)$.

6.6.2.2 Hamilton's Equations

Hamilton's equations of motion are easily obtained. According to the Legendre transformation,

$$\Pi = \frac{\partial \tilde{L}(R, \omega)}{\partial \omega} = J\omega$$

is the angular momentum of the three-dimensional pendulum expressed in the body-fixed frame. Thus, the modified Hamiltonian is

$$\tilde{H}(R, \Pi) = \frac{1}{2}\Pi^T J^{-1}\Pi - mgS(\rho_0)R^T e_3.$$

Hamilton's equations of motion are given by the rotational kinematics

$$\dot{R} = RS(J^{-1}\Pi). \tag{6.21}$$

and

$$\dot{\Pi} = \Pi \times J^{-1}\Pi + mg\rho_0 \times R^T e_3, \tag{6.22}$$

Thus, the Hamiltonian dynamics of the three-dimensional pendulum, described by equations (6.21) and (6.22), characterize the evolution of (R, Π) on the cotangent bundle $\mathsf{T}^*\mathsf{SO}(3)$.

6.6.2.3 Conservation Properties

There are two conserved quantities, or integrals of motion, for the three-dimensional pendulum. First, the Hamiltonian, which coincides with the total energy E in this case, is conserved, that is

$$H = \frac{1}{2}\,\omega^T J\omega - mg\rho_0^T R^T e_3,$$

and it is constant along each solution of the dynamical flow of the three-dimensional pendulum.

In addition, the modified Lagrangian is invariant with respect to the lifted action of rotations about the vertical or gravity direction. By Noether's theorem, this symmetry leads to conservation of the component of angular momentum about the vertical or gravity direction; that is

$$h = \omega^T J R^T e_3,$$

and it is constant along each solution of the dynamical flow of the three-dimensional pendulum.

6.6.2.4 Equilibrium Properties

The equilibrium or constant solutions of the three-dimensional pendulum are easily obtained. The conditions for an equilibrium solution are:

$$\omega \times J\omega - mg\rho_0 \times R^T e_3 = 0,$$
$$RS(\omega) = 0.$$

Since $R \in \mathsf{SO}(3)$ is non-singular, it follows that the angular velocity vector $\omega = 0$. Thus, an equilibrium attitude satisfies

$$\rho_0 \times R^T e_3 = 0,$$

which implies that

$$R^T e_3 = \frac{\rho_0}{\|\rho_0\|},$$

or

$$R^T e_3 = -\frac{\rho_0}{\|\rho_0\|}.$$

An attitude R is an equilibrium attitude if and only if the vertical direction or equivalently the gravity direction $R^T e_3$, resolved in the body-fixed frame, is collinear with the body-fixed vector ρ_0 from the center of mass of the rigid body to the pivot. If $R^T e_3$ is in the opposite direction to the vector ρ_0, then $(R, 0) \in \mathsf{TSO}(3)$ is an *inverted* equilibrium of the three-dimensional pendulum; if $R^T e_3$ is in the same direction to the vector ρ_0, then $(R, 0)$ is a *hanging* equilibrium of the three-dimensional pendulum.

Without loss of generality, it is convenient to assume that the constant center of mass vector, in the body-fixed frame, satisfies

$$\frac{\rho_0}{\|\rho_0\|} = -e_3.$$

Consequently, if $R \in \mathsf{SO}(3)$ defines an equilibrium attitude for the three-dimensional pendulum, then an arbitrary rotation of the three-dimensional pendulum about the vertical is also an equilibrium attitude. In summary, there are two disjoint equilibrium manifolds for the three-dimensional pendulum.

The manifold

$$\left\{ R \in \mathsf{SO}(3) : R^T e_3 = \frac{\rho_0}{\|\rho_0\|} \right\},$$

is referred to as the inverted equilibrium manifold, since the center of mass is directly above the pivot.

We now obtain linearized equations at the inverted equilibrium $(I_{3\times 3}, 0) \in \mathsf{TSO}(3)$. According to Appendix B, $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$ are exponential local coordinates for $\mathsf{SO}(3)$ in a neighborhood of $I_{3\times 3} \in \mathsf{SO}(3)$. Following the results in Appendix B, the linearized differential equations for the three-dimensional pendulum are defined on the six-dimensional tangent space of $\mathsf{TSO}(3)$ at $(I_{3\times 3}, 0) \in \mathsf{TSO}(3)$ and are given by

$$J\ddot{\xi} - mg\,\|\rho_0\| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi = 0.$$

These linearized differential equations approximate the rotational dynamics of a rotating rigid body in a neighborhood of $(I_{3\times 3}, 0) \in \mathsf{TSO}(3)$. These linear dynamics are accurate to first-order in the perturbations expressed in local coordinates.

The eigenvalues of the linearized equations can be shown to have the following pattern: two pairs of eigenvalues that are real with equal magnitudes and opposite signs and one pair of eigenvalues at the origin. Since there is a positive eigenvalue, this inverted equilibrium solution is unstable.

Next, the manifold

$$\left\{ R \in \mathsf{SO}(3) : R^T e_3 = -\frac{\rho_0}{\|\rho_0\|} \right\},$$

is referred to as the hanging equilibrium manifold, since the center of mass is directly below the pivot.

We obtain linearized differential equations at the hanging equilibrium $(-I_{3\times 3}, 0) \in \mathsf{TSO}(3)$. According to Appendix B, $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$ are exponential local coordinates for $\mathsf{SO}(3)$ in a neighborhood of $-I_{3\times 3} \in \mathsf{SO}(3)$. The linearized differential equations for the three-dimensional pendulum are defined on the six-dimensional tangent space of $\mathsf{TSO}(3)$ at $(-I_{3\times 3}, 0) \in \mathsf{TSO}(3)$ and are given by

$$J\ddot{\xi} + mg\,\|\rho_0\| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi = 0.$$

These linearized differential equations approximate the rotational dynamics of a rotating rigid body in a neighborhood of the hanging equilibrium $(-I_{3\times 3}, 0) \in \mathsf{TSO}(3)$. These linear dynamics, with two pairs of purely imaginary eigenvalues and one pair of zero eigenvalues, are accurate to first-order in the perturbations expressed in local coordinates.

Solutions for which the angular velocity vector are constant can also be identified; these are relative equilibrium solutions and they necessarily satisfy

$$\omega \times J\omega - mg\rho_0 \times R^T e_3 = 0.$$

Thus, the relative equilibrium solutions occur when the angular velocity vector is collinear with an eigenvector of the inertia matrix J, and the direction of this angular velocity vector, in the inertial frame, is collinear with the gravity direction.

6.6.3 Dynamics of a Rotating Rigid Body in Orbit

Consider the rotational motion of a rigid body in a circular orbit about a large central body. A Newtonian gravity model is used, which gives rise to a differential gravity force on each mass element of the rigid body; this gravity gradient moment is included in our subsequent analysis. The subsequent development follows the presentations in [50, 51].

Three Euclidean frames are introduced: an inertial frame whose origin is at the center of the central body, a body-fixed frame whose origin is located at the center of mass of the orbiting rigid body, and a so-called local vertical, local horizontal (LVLH) frame, whose first axis is tangent to the circular orbit, the second axis is perpendicular to the plane of the orbit, and the third axis is along the orbit radius vector. The origin of the LVLH frame is located at the center of mass of the rigid body and remains on the circular orbit, so that the LVLH frame necessarily rotates at the orbital rate. The LVLH frame is not an inertial frame, but it does have physical significance; it is used to describe the gravity gradient moment. A schematic of a rotating rigid body in a circular orbit is shown in Figure 6.3.

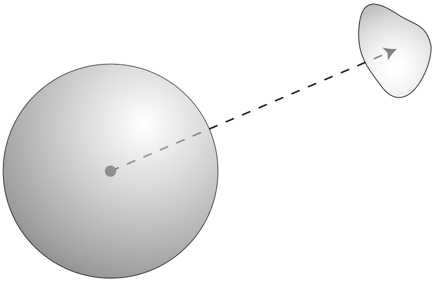

Fig. 6.3 Rotating rigid body in a circular orbit

We define three sets of rotation matrices in $\mathsf{SO}(3)$: $R^{bi} \in \mathsf{SO}(3)$ denotes a rotation matrix from the body-fixed frame to the inertial frame, $R^{li} \in \mathsf{SO}(3)$ denotes a rotation matrix from the LVLH frame to the inertial frame, and $R^{bl} \in \mathsf{SO}(3)$ denotes a rotation matrix from the body-fixed frame to the LVLH frame. Thus, the three rotation matrices satisfy $R^{bl} = (R^{li})^T R^{bi}$. We show that the dynamics of a rotating rigid body in a circular orbit can be expressed in terms of the rotation matrix $R^{bi} \in \mathsf{SO}(3)$, so that $\mathsf{SO}(3)$ is the configuration manifold.

Let $\omega \in \mathbb{R}^3$ be the angular velocity of the rigid body expressed in the body-fixed frame. The 3×3 constant matrix J is the standard inertia matrix of the rigid body in the body-fixed frame. The scalar orbital angular velocity is $\omega_0 = \sqrt{\frac{GM}{r_0^3}}$, where M denotes the mass of the central body, G is the universal gravitational constant, and r_0 is the constant radius of the circular orbit. The inertial frame is selected so that the orbital plane is orthogonal to the second inertial axis; hence the orbital angular velocity vector is $\omega_0 e_2$ in the inertial frame. The LVLH frame is selected so that the orbit radius vector of the body is $r_0 e_3$ in the LVLH frame.

6.6.3.1 Euler–Lagrange Equations

Based on the prior developments in this chapter, the on-orbit rigid body rotational kinematics equations are given as follows. The attitude of the body-fixed frame with respect to the inertial frame is described by the rotational kinematics

$$\dot{R}^{bi} = R^{bi} S(\omega),$$

the attitude of the LVLH frame with respect to the inertial frame is described by the rotational kinematics

$$\dot{R}^{li} = R^{li} S(\omega_0 e_2),$$

and the attitude of the body-fixed frame with respect to the LVLH frame is described by the rotational kinematics

$$\dot{R}^{bl} = R^{bl} S(\omega - \omega_0 R^{bl^T} e_2).$$

The modified Lagrangian $\tilde{L} : \mathsf{TSO}(3) \to \mathbb{R}^1$ is given by

$$\tilde{L}(R^{bi}, \omega) = \frac{1}{2}\omega^T J \omega - U(R^{bi}),$$

where $U(R^{bi})$ is the gravitational potential energy of the rigid body in orbit. Thus, the Euler–Lagrange equations of motion are given by

$$J\dot{\omega} + \omega \times J\omega = M^g,$$

where

$$M^g = \sum_{i=1}^{3} r_i \times \frac{\partial U(R^{bi})}{\partial r_i},$$

is the gravity gradient moment on the rigid body due to the gravity potential $U(R^{bi})$. In the gravity gradient moment expression, r_1, r_2, r_3 denote the column partitions of $(R^{bi})^T \in \mathsf{SO}(3)$.

Since the orbital angular velocity ω_0 is constant, the rotational kinematics equation for $R^{li} \in \mathsf{SO}(3)$ can be explicitly solved to obtain

$$R^{li}(t) = R^{li}(t_0)e^{S(\omega_0 e_2)(t-t_0)}.$$

This describes the rotation of the LVLH frame with respect to the inertial frame.

The gravity potential for the full orbiting rigid body is obtained by integrating the gravity potential for each element in the body over the body; this leads to

$$U(R^{bi}) = -\int_{\mathcal{B}} \frac{GM}{\|x + R^{bi}\rho\|}\, dm(\rho),$$

where $x \in \mathbb{R}^3$ is the position of the center of mass of the orbiting rigid body in the inertial frame, and $\rho \in \mathbb{R}^3$ is a vector from the center of mass of the rigid body to the body element with mass $dm(\rho)$ in the body-fixed frame.

We now derive a closed form approximation for the gravitational moment M^g using the fact that the rigid body is in a circular orbit so that the norm of x is constant. The size of the rigid body is assumed to be much smaller than the orbital radius.

Since the rigid body position vector in the LVLH frame is $r_0 e_3$, the position vector of the rigid body in the inertial frame is given by $x = r_0 R^{li} e_3$. Using this expression, the matrix of derivatives of the gravitational potential energy is

$$\begin{aligned}
\frac{\partial U(R^{bi})}{\partial R^{bi}} &= \int_{\mathcal{B}} \frac{GM\, r_0 R^{li}\, e_3 \rho^T}{\|\, r_0 e_3 + R^{bl}\rho\,\|^3}\, dm(\rho) \\
&= \frac{GM}{r_0} \int_{\mathcal{B}} \frac{\left(R^{li} e_3 \hat{\rho}^T\right) \frac{\|\rho\|}{r_0}}{\left[1 + 2\left(e_3^T R^{bl}\hat{\rho}\right)\frac{\|\rho\|}{r_0} + \frac{\|\rho\|^2}{r_0^2}\right]^{\frac{3}{2}}}\, dm(\rho),
\end{aligned}$$

where $\hat{\rho} = \frac{\rho}{\|\rho\|} \in \mathbb{R}^3$ is the unit vector along the direction of ρ. Since the size of the rigid body is significantly smaller than the orbital radius, it follows that $\frac{\|\rho\|}{r_0} \ll 1$. Using a Taylor series expansion, we obtain the second-order approximation:

$$\frac{\partial U(R^{bi})}{\partial R^{bi}} = \frac{GM}{r_0}\int_{\mathcal{B}} R^{li} e_3 \hat{\rho}^T \left\{ \frac{\|\rho\|}{r_0} - 3e_3^T R^{bl} \hat{\rho} \frac{\|\rho\|^2}{r_0^2} \right\} dm(\rho).$$

Since the body-fixed frame is located at the center of mass of the rigid body, $\int_{\mathcal{B}} \rho \; dm(\rho) = 0$. Therefore, the first term in the above equation vanishes. Since $e_3^T R^{bl}\rho$ is a scalar, it can be shown that the above partial derivative matrix can be written as

$$\frac{\partial U(R^{bi})}{\partial R^{bi}} = -3\omega_0^2 R^{li} e_3 e_3^T R^{bl} \left(\frac{1}{2}\mathrm{tr}[J]\, I_{3\times 3} - J \right).$$

This can be used to obtain an expression for the gravity gradient moment on the full rigid body:

$$M^g = \sum_{i=1}^{3} r_i \times \frac{\partial U(R^{bi})}{\partial r^i} = 3\omega_0^2 R^{bl^T} e_3 \times J R^{bl^T} e_3.$$

In summary, the Euler equations can be written as

$$J\dot{\omega} + \omega \times J\omega = 3\omega_0^2 R^{bl^T} e_3 \times J R^{bl^T} e_3, \tag{6.23}$$

and the attitude kinematics equation with respect to the LVLH frame is

$$\dot{R}^{bl} = R^{bl} S(\omega - \omega_0 R^{bl^T} e_2). \tag{6.24}$$

These rotational equations of motion (6.23) and (6.24) define the Lagrangian flow of an on-orbit rigid body as the dynamics described by $(R^{bl}, \omega) \in \mathsf{TSO(3)}$ evolve on the tangent bundle of $\mathsf{SO(3)}$. Rotational dynamics that describe the attitude of the rigid body in the inertial frame or in the body-fixed frame can be obtained from the above development.

6.6.3.2 Hamilton's Equations

Hamilton's equations are easily obtained by defining the angular momentum

$$\Pi = \frac{\partial \tilde{L}(R, \omega)}{\partial \omega} = J\omega.$$

Thus, the modified Hamiltonian function is

$$\tilde{H}(R^{bi}, \Pi) = \frac{1}{2}\Pi^T J^{-1} \Pi + U(R^{bi}).$$

Hamilton's equations of motion for the on-orbit rigid body can be written as the attitude kinematics equation with respect to the LVLH frame, namely

$$\dot{R}^{bl} = R^{bl} S(J^{-1}\Pi - \omega_0 R^{bl^T} e_2), \tag{6.25}$$

and the Euler equations

$$\dot{\Pi} = \Pi \times J^{-1}\Pi + 3\omega_0^2 R^{bl^T} e_3 \times J R^{bl^T} e_3. \tag{6.26}$$

These equations (6.25) and (6.26) define the Hamiltonian flow of the rotational dynamics of an on-orbit rigid body as described by $(R^{bl}, \Pi) \in \mathsf{T}^*\mathsf{SO}(3)$ on the cotangent bundle of $\mathsf{SO}(3)$. Rotational dynamics that describe the attitude of the rigid body in the inertial frame or in the body-fixed frame can be obtained from the above development.

6.6.3.3 Conservation Properties

The Hamiltonian, which coincides with the total energy E in this case, is

$$H = \frac{1}{2}\omega^T J \omega + U(R^{bi});$$

the Hamiltonian is constant along each solution of the dynamical flow.

6.6.3.4 Equilibrium Properties

The orbiting rigid body is in a relative equilibrium when the attitude of the body with respect to the LVLH frame is constant. The relative equilibria can be obtained by assuming that (R^{bl}, ω) are constant in (6.23) and (6.24). This leads to the requirement that the constant angular velocity of the orbiting body is

$$\omega = \omega_0 R^{bl^T} e_3,$$

and the attitude of the rigid body in the LVLH frame is such that the gravity moment on the rigid body is zero, namely

$$R^{bl^T} e_3 \times J R^{bl^T} e_3 = 0.$$

Thus, an attitude $R^{bl} \in \mathsf{SO}(3)$ is a relative equilibrium of the orbiting rigid body if $R^{bl^T} e_3 \in \mathbb{R}^3$ is an eigenvector of the inertia matrix J.

6.6.4 Dynamics of a Rigid Body Planar Pendulum

A rigid body planar pendulum is a rigid body that is constrained to rotate about an inertially fixed revolute joint under the influence of uniform, constant gravity. Since the revolute joint allows one degree of freedom rotation about its axis, each material point in the rigid body necessarily rotates along a circular arc, centered at the closest point on the axis, in a fixed two-dimensional plane. This motivates the designation of rigid body planar pendulum. This is a generalization of the lumped mass planar pendulum example that was introduced in Chapter 4 using the configuration manifold S^1.

As usual we consider an inertial Euclidean frame in $\mathbb{R}^3$ and we select a body-fixed frame. The inertial frame is selected so that the third axis is vertical. For convenience, the origin of the inertial frame is located on the axis of the revolute joint at the point on the axis that is closest to the center of mass of the rigid body; the origin of the body-fixed frame coincides with the center of mass of the rigid body. We denote the direction vector of the axis of the revolute joint, in the inertial frame, by $a \in \mathsf{S}^2$ and we denote the vector from the center of mass of the rigid body to the origin of the inertial frame, expressed in the body-fixed frame, by $\rho_0 \in \mathbb{R}^3$. The mass of the rigid body is m and the inertia matrix of the rigid body, computed subsequently, is denoted by J. A schematic of a rigid body planar pendulum is shown in Figure 6.4.

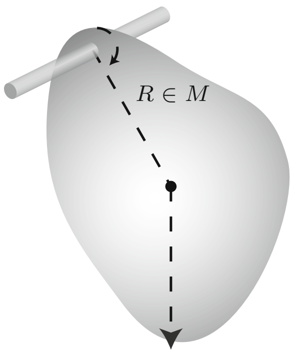

Fig. 6.4 Rigid body planar pendulum

It is an important observation that rotations of the rigid body about the axis leave material points in the rigid body located on the axis unchanged. If $R \in \mathsf{SO}(3)$ denotes the attitude of the rigid body, then it follows that $Ra = a$ expresses the fact that the direction of the revolute joint axis is unchanged under rotations about that axis. Thus, the configuration manifold for the rigid body planar pendulum is

$$M = \{R \in \mathsf{SO}(3) : Ra = a\}.$$

This is a differentiable submanifold of SO(3) with dimension one. Consequently, the rigid body planar pendulum has one degree of freedom.

6.6.4.1 Kinematics and Variations

Since the configuration manifold is a submanifold of SO(3), the kinematics and the expressions for the infinitesimal variations must be suitably modified from the prior development in this chapter.

The angular velocity vector of the rigid body $\Omega \in \mathbb{R}^3$ is introduced according to the usual rigid body kinematics

$$\dot{R} = RS(\Omega).$$

We first see that the constraint $Ra = a$ implies that $\dot{R}a = 0$; thus $S(\Omega)a = 0$, that is $\Omega \times a = 0$. This implies that Ω is collinear with a, that is there is $\omega : [t_0, t_f] \to \mathbb{R}^1$ such that

$$\Omega = \omega a,$$

where ω is the scalar angular velocity of the rigid body about its rotation axis. Thus, the rigid body angular velocity vector, in the body-fixed frame, has magnitude given by the scalar angular velocity in the direction of the axis of rotation. Thus, the rotational kinematics of the rigid body can be expressed as

$$\dot{R} = RS(\omega a). \tag{6.27}$$

From the prior analysis in this chapter, it follows that the infinitesimal variation of the rigid body attitude is

$$\delta R = RS(\eta),$$

where $\eta : [t_0, t_f] \to \mathbb{R}^3$ is a differentiable curve that vanishes at its endpoints. Since $Ra = a$, it follows that

$$\delta Ra = 0.$$

This constraint is satisfied if $S(a)\eta = 0$, or equivalently $\eta = \beta a$, where $\beta : [t_0, t_f] \to \mathbb{R}$ is a differentiable curve that vanishes at its endpoints. Thus,

$$\delta R = RS(\beta a).$$

Further, the infinitesimal variation of the angular velocity vector is

$$\begin{aligned}\delta\Omega &= \dot{\eta} + S(\omega a)\eta \\ &= \dot{\beta}a + \omega S(a)\beta a \\ &= \dot{\beta}a,\end{aligned}$$

since $S(a)a = 0$. Thus,

$$\delta\omega = a^T \delta\Omega = \dot{\beta}.$$

6.6.4.2 Euler–Lagrange Equations

We now derive Euler–Lagrange equations for the rigid body planar pendulum. The above expressions for the infinitesimal variations play a key role.

The inertial position of a material point located in the rigid body at $\rho \in \mathcal{B}$ is given by $R(-\rho_0 + \rho) \in \mathbb{R}^3$. The kinetic energy of the rigid body is

$$\begin{aligned}T &= \frac{1}{2}\int_{\mathcal{B}} \left\|\dot{R}(-\rho_0 + \rho)\right\|^2 \, dm(\rho) \\ &= \frac{1}{2}\int_{\mathcal{B}} \left\|RS(\Omega)(-\rho_0 + \rho)\right\|^2 \, dm(\rho) \\ &= \frac{1}{2}\Omega^T J \Omega,\end{aligned}$$

where the rigid body moment of inertia matrix is

$$J = \int_{\mathcal{B}} S^T(\rho)S(\rho)\, dm(\rho) + mS^T(\rho_0)S(\rho_0).$$

The gravitational potential energy of the rigid body is

$$\begin{aligned}U(R) &= \int_{\mathcal{B}} ge_3^T R(-\rho_0 + \rho)\, dm(\rho) \\ &= -mge_3^T R\rho_0.\end{aligned}$$

The modified Lagrangian function is

$$\tilde{L}(R, \Omega) = \frac{1}{2}\Omega^T J\Omega + mge_3^T R\rho_0,$$

or equivalently

$$\tilde{L}(R, \omega) = \frac{1}{2}a^T Ja\,\omega^2 + mge_3^T R\rho_0.$$

The infinitesimal variation of the action integral is

$$\left.\frac{d}{d\epsilon}\mathfrak{G}^\epsilon\right|_{\epsilon=0} = \int_{t_0}^{t_f} a^T J a\,\omega\,\delta\omega + mg\rho_0^T \delta R^T e_3\,dt.$$

Use the expression

$$\delta R^T e_3 = -S(\beta a)R^T e_3 = \beta S(R^T e_3)a$$

to obtain the infinitesimal variation of the action integral:

$$\left.\frac{d}{d\epsilon}\mathfrak{G}^\epsilon\right|_{\epsilon=0} = \int_{t_0}^{t_f} a^T J a\omega\dot{\beta} + mg\rho_0^T S(R^T e_3)a\beta\,dt.$$

Integrating by parts and using the fact that the variations vanish at the endpoints, we obtain

$$\left.\frac{d}{d\epsilon}\mathfrak{G}^\epsilon\right|_{\epsilon=0} = -\int_{t_0}^{t_f} \left\{a^T J a\dot{\omega} - mg\rho_0^T S(R^T e_3)a\right\}\cdot\beta\,dt.$$

Hamilton's principle and the fundamental lemma of the calculus of variations give the Euler–Lagrange equation

$$a^T J a\,\dot{\omega} - mga^T(\rho_0 \times R^T e_3) = 0. \tag{6.28}$$

The equations (6.27) and (6.28) describe the dynamical flow of the rigid body planar pendulum on the tangent bundle $\mathsf{T}M$.

6.6.4.3 Hamilton's Equations

According to the Legendre transformation,

$$\pi = \frac{\partial\tilde{L}(R,\omega)}{\partial\omega} = a^T J a\,\omega$$

is the scalar angular momentum of the rigid body pendulum about its axis of rotation. Thus, the modified Hamiltonian is

$$\tilde{H}(R,\pi) = \frac{1}{2}\frac{\pi^2}{a^T J a} - mge_3^T R\rho_0.$$

Hamilton's equations of motion are given by the rotational kinematics

$$\dot{R} = RS\left(\frac{\pi a}{a^T J a}\right), \tag{6.29}$$

and

$$\dot{\pi} = mga^T(\rho_0 \times R^T e_3). \tag{6.30}$$

The Hamiltonian dynamics of the rigid body planar pendulum, characterized by equations (6.29) and (6.30), are described by the evolution of (R, π) on the cotangent bundle T^*M.

6.6.4.4 Reduced Equations for the Rigid Body Planar Pendulum

As we have shown, each material point in the rigid body rotates along a planar circular arc about a center on the axis of the revolute joint. In particular, the center of mass vector ρ_0 rotates along a planar circular arc, with center at the origin of the inertial frame. The two-dimensional plane containing each such circular arc is inertially fixed and orthogonal to the axis $a \in \mathsf{S}^2$. This suggests that it should be possible to describe such rotations in terms of planar rotations in S^1 as discussed previously in Chapter 4. This connection is clarified in the following development, where the rigid body planar pendulum equations are used to obtain reduced equations that evolve on S^1.

To this end, define the direction of the position vector of the center of mass of the rigid body, expressed in the inertial frame:

$$\zeta = \frac{R\rho_0}{\|R\rho_0\|} = \frac{R\rho_0}{\|\rho_0\|},$$

which follows since $\|R\rho_0\| = \|\rho_0\|$. Thus, $\zeta \in \mathsf{S}^2$.

It is easy to see that the rotational kinematics (6.27) can be used to obtain

$$\begin{aligned}\dot{\zeta} &= \dot{R}R^T\zeta \\ &= RS(\omega a)R^T\zeta \\ &= S(R\omega a)\zeta \\ &= S(\omega a)\zeta,\end{aligned}$$

where we have used a matrix identity and the fact that $a = Ra$.

We now construct a Euclidean orthonormal basis for the inertial frame given by the ordered triple in S^2:

$$a_1,\ a_2,\ a.$$

Since $a^T\zeta = 0$, we can express

$$\zeta = q_1 a_1 + q_2 a_2,$$

where $q = (q_1, q_2) \in \mathsf{S}^1$. Substituting this into the above rotational kinematics, we obtain

$$\begin{aligned}\dot{q}_1 a_1 + \dot{q}_2 a_2 &= \omega S(a)\left\{q_1 a_1 + q_2 a_2\right\} \\ &= \omega\left\{q_1 a_2 - q_2 a_1\right\}.\end{aligned}$$

Consequently,

$$\dot{q}_1 = -\omega q_2,$$
$$\dot{q}_2 = \omega q_1.$$

In vector form, this can be written as

$$\dot{q} = \omega \mathrm{S} q, \tag{6.31}$$

where S is the constant 2×2 skew-symmetric matrix used in Chapter 4.

We now express the Euler–Lagrange equation (6.28) in a different form. Consider the expression

$$\begin{aligned} -mga^T(\rho_0 \times R^T e_3) &= mga^T S(R^T e_3)\rho_0 \\ &= mga^T R^T S(e_3) R\rho_0 \\ &= mg \left\|\rho_0\right\| a^T S(e_3)\zeta \\ &= mg \left\|\rho_0\right\| \left\{a^T S(e_3) a_1 q_1 + a^T S(e_3) a_2 q_2\right\}, \end{aligned}$$

where we have used a matrix identity and the fact that $Ra = a$. The Euler–Lagrange equation can be expressed as

$$a^T J a\, \dot{\omega} + mg \left\|\rho_0\right\| \left\{a^T S(e_3) a_1 q_1 + a^T S(e_3) a_2 q_2\right\} = 0. \tag{6.32}$$

Thus, the rotational kinematics (6.31) and the Euler–Lagrange equation (6.32) describe the dynamics of the rigid body planar pendulum in terms of $(q, \omega) \in \mathsf{TS}^1$. These are referred to as reduced equations since they describe only the dynamics of the position vector of the center of mass of the rigid body in the inertial frame.

Following a similar development, a reduced form for Hamilton's equations can be obtained that describe the evolution on the cotangent bundle T^*M. These results require introduction of the reduced Lagrangian, expressed on the tangent bundle TS^1, definition of the conjugate momentum using the Legendre transformation, and derivation of the reduced Hamilton's equations on $\mathsf{T}^*\mathsf{S}^1$. These details are not given here.

6.6.4.5 Conservation Properties

The Hamiltonian, which coincides with the total energy E in this case, is conserved. This can be expressed as

$$H = \frac{1}{2} a^T J a\, \omega^2 - mge_3^T R\rho_0,$$

which is constant along each solution of the dynamical flow of the rigid body planar pendulum.

6.6.4.6 Equilibrium Properties

The equilibrium or constant solutions of the rigid body planar pendulum occur when the angular velocity $\omega = 0$ and the rigid body attitude satisfies the algebraic equations on the configuration manifold M:

$$mga^T(\rho_0 \times R^T e_3) = 0,$$

which implies that the time derivative of the angular velocity vanishes. This requires that the direction of gravity, expressed in the body-fixed frame, and the center of mass vector ρ_0 be collinear.

6.7 Problems

6.1. In this problem, we derive an alternative expression of the moment caused by an attitude-dependent potential, summarized in Proposition 6.1.

(a) Consider two matrices $A, B \in \mathbb{R}^{3\times 3}$. Let $a_i, b_i \in \mathbb{R}^3$ be the i-th column of A^T and B^T for $i \in \{1, 2, 3\}$, respectively, such that the matrices A and B are partitioned into

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix}, \quad B = \begin{bmatrix} b_1^T \\ b_2^T \\ b_3^T \end{bmatrix}.$$

Show that

$$B^T A - A^T B = \sum_{i=1}^{3} b_i a_i^T - a_i b_i^T = \sum_{i=1}^{3} S(a_i \times b_i).$$

(b) Using the above identify, show that the moment caused by an attitude-dependent potential can be rewritten as

$$-\sum_{i=1}^{3} r_i \times \frac{\partial \tilde{L}(R,\omega)}{\partial r_i} = \left(R^T \frac{\partial \tilde{L}(R,\omega)}{\partial R} - \frac{\partial \tilde{L}(R,\omega)}{\partial R}^T R \right)^\vee,$$

where $\frac{\partial \tilde{L}(R,\omega)}{\partial R} \in \mathbb{R}^{3\times 3}$ is defined such that its i, j-th element corresponds to the derivative of $L(R, \omega)$ with respect to the i, j-th element of R for $i, j \in \{1, 2, 3\}$.

6.2. Consider the attitude dynamics of a rigid body described in Section 6.3.3. Here, we rederive the Euler–Lagrange equation given in (6.8) to include the effects of an external moment according to the Lagrange–d'Alembert principle.

Suppose that there exists an external moment $M \in \mathbb{R}^3$ acting on the rigid body. Assume it is resolved in the body-fixed frame.

(a) Let $\rho \in \mathbb{R}^3$ be the vector from the mass center of the rigid body to a mass element $dm(\rho)$. Let $dF(\rho) \in \mathbb{R}^3$ be the force acting on $dm(\rho)$. Assume that both of ρ and $dF(\rho)$ are expressed in the body-fixed frame. As there is no external force, $\int_{\mathcal{B}} dF(\rho) = 0$. Due to the external moment, we have $\int_{\mathcal{B}} \rho \times dF(\rho) = M$. Show that the virtual work due to the external moment is given by

$$\delta\mathcal{W} = \int_{\mathcal{B}} RdF(\rho) \cdot \delta R\rho = \int_{\mathcal{B}} \eta \cdot (\rho \times dF(\rho)) = \eta \cdot M,$$

where $\delta R = R\hat{\eta}$ for $\eta \in \mathbb{R}^3$.

(b) From the Lagrange–d'Alembert principle, show that the Euler–Lagrange equation is given by

$$J\dot{\omega} + S(\omega)J\omega - \sum_{i=1}^{3} S(r_i)\frac{\partial U(R)}{\partial r_i} = M.$$

6.3. Consider the dynamics of a rotating rigid body that is constrained to planar rotational motion in $\mathbb{R}^2$. That is, the configuration manifold is taken as the Lie group of 2×2 orthogonal matrices with determinant $+1$, namely $\mathsf{SO}(2)$. The rotational kinematics, expressed in terms of the rotational motion $t \to R \in \mathsf{SO}(2)$, are given by

$$\dot{R} = R\mathrm{S}\omega,$$

for some scalar angular velocity $t \to \omega \in \mathbb{R}^1$; as before, S is the standard 2×2 skew-symmetric matrix. The modified Lagrangian function is given by

$$\tilde{L}(R, \omega) = \frac{1}{2}J\omega^2 - U(R),$$

where J is the scalar moment of inertia of the rigid body and $U(R)$ is the configuration dependent potential energy function.

(a) What are expressions for the infinitesimal variations of $R \in \mathsf{SO}(2)$, $\dot{R} \in \mathsf{T}_R\mathsf{SO}(2)$, and $\omega \in \mathbb{R}^1$?
(b) Use Hamilton's principle to derive the Euler equations for the planar rotations of the rigid body.
(c) Use the Legendre transformation to derive Hamilton's equations for the planar rotations of the rigid body.
(d) What are conserved quantities of the dynamical flow on $\mathsf{TSO}(2)$?
(e) What are conditions that define equilibrium solutions of the dynamical flow on $\mathsf{TSO}(2)$?

6.4. Consider a planar pendulum, with scalar moment of inertia J, under constant, uniform gravity. Assume the configuration manifold of the planar pendulum is taken to be the Lie group $\mathsf{SO}(2)$. Use the results in the prior problem for the following.

(a) What are the Euler equations for the planar pendulum on the tangent bundle $\mathsf{TSO}(2)$?
(b) What are Hamilton's equations for the planar pendulum on the cotangent bundle $\mathsf{T}^*\mathsf{SO}(2)$?
(c) What are the conserved quantities of the dynamical flow on $\mathsf{TSO}(2)$?
(d) What are the equilibrium solutions of the dynamical flow on $\mathsf{TSO}(2)$?

6.5. Consider a double planar pendulum under constant, uniform gravity. The first link rotates about an inertially fixed one degree of freedom revolute joint. The two links are connected by another revolute joint fixed in the two links, constraining the two links to rotate in a common vertical plane. The scalar moments of inertia of the two pendulums are J_1 and J_2 about the two joint axes. Assume the configuration manifold of the planar pendulum is taken to be the Lie group product $(\mathsf{SO}(2))^2$. Use the results in the prior problem for the following.

(a) What are the Euler–Lagrange equations for the double planar pendulum on the tangent bundle $\mathsf{T}\,(\mathsf{SO}(2))^2$?
(b) What are Hamilton's equations for the double planar pendulum on the cotangent bundle $\mathsf{T}^*\,(\mathsf{SO}(2))^2$?
(c) What are the conserved quantities of the dynamical flow on $\mathsf{T}\,(\mathsf{SO}(2))^2$?
(d) What are the equilibrium solutions of the dynamical flow on $\mathsf{T}\,(\mathsf{SO}(2))^2$?

6.6. Consider the rigid body planar pendulum considered in subsection 6.6.4. The configuration manifold is $M = \{R \in \mathsf{SO}(3) : Ra = a\}$.

(a) Show that the configuration manifold M, which is a submanifold of the Lie group $\mathsf{SO}(3)$, is a one-dimensional matrix Lie group.
(b) Show that the configuration manifold M is diffeomorphic to $\mathsf{SO}(2)$.

6.7. Consider the free rotational motion of a symmetric rigid body in $\mathbb{R}^3$. Assume the moment of inertia in the body-fixed frame is $J = J_s I_{3\times 3}$, where $J_s > 0$ is a scalar.

(a) What are the Euler equations for the free rotational motion of a symmetric rigid body?
(b) Given initial conditions $\omega(t_0) = \omega_0 \in \mathbb{R}^3$, $R(t_0) = R_0 \in \mathsf{SO}(3)$, determine analytical expressions for the angular velocity and for the rigid body attitude, the latter described using exponential matrices.

6.8. Consider the free rotational motion of an asymmetric rigid body in $\mathbb{R}^3$. Assume the body-fixed frame is selected so that the moment of inertia is $J = \mathsf{diag}(J_1, J_2, J_3)$, where $J_i > 0$, $i = 1, 2, 3$, are distinct.

(a) What are the Euler equations for the free rotational motion of an asymmetric rigid body?
(b) What are the equilibrium solutions for the dynamical flow defined by the Euler equations? These equilibrium solutions of the Euler equations can be viewed as relative equilibrium solutions for the complete rotational dynamics of the asymmetric rigid body.
(c) For each equilibrium solution of the Euler equations, describe the time dependence of the resulting rigid body attitude in $\mathsf{SO}(3)$.

6.9. Consider the free rotational motion of a rigid body, with an axis of symmetry, in $\mathbb{R}^3$. Assume the body-fixed frame is selected so that the moment of inertia is $J = \mathsf{diag}(J_1, J_1, J_2)$, where $J_i > 0$, $i = 1, 2$, are distinct.

(a) What are the Euler equations for the free rotational motion of a rigid body with an axis of symmetry?
(b) What are the equilibrium solutions for dynamical flow defined by the Euler equations? These equilibrium solutions of the Euler equations can be viewed as relative equilibrium solutions for the complete rotational dynamics of the rigid body with an axis of symmetry.
(c) For each equilibrium solution of the Euler equations, describe the time dependence of the resulting rigid body attitude in $\mathsf{SO}(3)$.

6.10. Consider the rotational motion of a rigid body in $\mathbb{R}^3$. Let $b \in \mathcal{B} \subset \mathbb{R}^3$ denote the location of a material point in the body, expressed in the body-fixed frame.

(a) Assume an external force $F \in \mathbb{R}^3$, expressed in the inertial frame, acts on the rigid body at the single point in the rigid body denoted by $b \in \mathcal{B}$. Show that the component of the force $R^T F \in \mathbb{R}^3$ in the direction $b \in \mathcal{B}$ does not influence the rotational dynamics of the rigid body.
(b) What are the Euler equations for the rotational motion of a rigid body, expressed in terms of the external force acting on the rigid body in the inertial frame?
(c) Assume an external force $F \in \mathbb{R}^3$, expressed in the body-fixed frame, acts on the rigid body at the single point in the rigid body denoted by $b \in \mathcal{B}$. Show that the component of the force $F \in \mathbb{R}^3$ in the direction $b \in \mathcal{B}$ does not influence the rotational dynamics of the rigid body.
(d) What are the Euler equations for the rotational motion of a rigid body, expressed in terms of the external force acting on the rigid body in the body-fixed frame?

6.11. Consider the dynamics of a rigid body, consisting of material points denoted by $\mathcal{B}$ in a body-fixed frame, under the influence of a gravitational field. The configuration $R \in \mathsf{SO}(3)$ denotes the attitude of the rigid body. Assume the origin of the body-fixed frame is located at the center of mass of the rigid body. A gravitational force acts on each material point in the rigid body. The net moment of all of the gravity forces is obtained by integrating the gravity moment for each mass increment of the body over the whole body. The gravitational field, expressed in the inertial frame, is given by $G : \mathbb{R}^3 \to \mathsf{T}\mathbb{R}^3$. The incremental gravitational moment vector on a mass increment $dm(\rho)$ of the rigid body, located at $\rho \in \mathcal{B}$ in the body-fixed frame, is given in the inertial frame by $R\rho \times dm(\rho)G(R\rho)$ or, equivalently in the body-fixed frame, by $\rho \times dm(\rho)R^T G(R\rho)$. Thus, the net gravity moment, in the body-fixed frame, is $\int_{\mathcal{B}} \rho \times R^T G(R\rho)\, dm(\rho)$.

(a) What are the Euler equations for the rotational dynamics of the rigid body in the gravitational field?
(b) What are Hamilton's equations for the rotational dynamics of the rigid body in the gravitational field?
(c) What are the conditions for an equilibrium solution of a rotating rigid body in the gravitational field?
(d) Suppose the gravitational field $G(x) = -ge_3$ is constant. What are the simplified Euler equations for the rotational dynamics of the rigid body? What are the conditions for an equilibrium solution of a rotating rigid body in a constant gravitational field?

6.12. Consider the dynamics of a charged rigid body, consisting of material points denoted by $\mathcal{B}$ in a body-fixed frame, under the influence of an electric field and a magnetic field. The configuration $R \in \mathsf{SO(3)}$ denotes the attitude of the rigid body. Assume the origin of the body-fixed frame is located at the center of mass of the rigid body. An electric force and a magnetic force act on each material point in the rigid body. The net moment of all of the electric and magnetic forces is obtained by integrating the incremental electric and magnetic moments for each volume increment of the body over the whole body. The electric field, expressed in the inertial frame, is given by $E : \mathbb{R}^3 \to \mathsf{T}\mathbb{R}^3$; the magnetic field, expressed in the inertial frame, is given by $B : \mathbb{R}^3 \to \mathsf{T}\mathbb{R}^3$ The incremental electric and magnetic moment vector on a volume increment with charge dQ, located at $\rho \in \mathcal{B}$ in the body-fixed frame, is given in the inertial frame by $R\rho \times dQ(E(R\rho) + \dot{R}\rho \times B(R\rho))$ or, equivalently in the body-fixed frame, by $\rho \times dQR^T(E(R\rho) + \dot{R}\rho \times B(R\rho))$. Thus, the total electric and magnetic moment, in the body-fixed frame, is $\int_{\mathcal{B}} \rho \times R^T(E(R\rho) + \dot{R}\rho \times B(R\rho))\, dQ$.

(a) What are the Euler equations for the rotational dynamics of the rigid body in the electric field and the magnetic field?
(b) What are Hamilton's equations for the rotational dynamics of the rigid body in the electric field and the magnetic field?
(c) What are the conditions for an equilibrium solution of a rotating rigid body in the electric and the magnetic field?
(d) Suppose the electric field $E(x) = -Ee_3$ and the magnetic field $B(x) = Be_2$ are constant, where E and B are scalar constants. What are the simplified Euler equations for the rotational dynamics of the rigid body?

What are the conditions for an equilibrium solution of a rotating rigid body in this constant electric and magnetic field?

6.13. Consider the rotational motion of a rigid body in $\mathbb{R}^3$ acted on by a force $F \in \mathbb{R}^3$. The Euler equations are

$$J\dot{\omega} + \omega \times J\omega = r \times F.$$

In the body-fixed frame, $r = \sum_{i=1}^{3} a_i e_i$ is a constant vector and $F = \sum_{i=1}^{3} f_i R^T e_i$ is the force. These are expressed in terms of scalar constants $a_i,\ f_i,\ i = 1, 2, 3$.

(a) Show that the moment vector is constant in the inertial frame.
(b) What are conditions on the constants $a_i,\ f_i,\ i = 1, 2, 3$ that guarantee that the Euler equations have an equilibrium solution?
(c) What are conditions on the constants $a_i,\ f_i,\ i = 1, 2, 3$ that guarantee that the rigid body dynamical flow on $\mathsf{TSO(3)}$ has an equilibrium solution $(R, \omega) = (I_{3\times 3}, 0) \in \mathsf{TSO(3)}$? Are there other equilibrium solutions in this case? What are they?

6.14. Consider the rotational motion of a rigid body in $\mathbb{R}^3$ acted on by a moment vector that is constant in the body-fixed frame. The Euler equations are

$$J\dot{\omega} + \omega \times J\omega = M,$$

where $M = \sum_{i=1}^{3} a_i e_i$ is the constant moment vector for scalar constants $a_i,\ i = 1, 2, 3$.

(a) Confirm that the moment vector is constant in the body-fixed frame.
(b) Assume the rigid body is asymmetric so that the moment of inertia matrix $J = \mathsf{diag}(J_1, J_2, J_3)$ with distinct entries. Obtain algebraic equations that characterize when the Euler equations have relative equilibrium solutions; that is, the angular velocity vector is constant.

6.15. Consider two concentric rigid spherical shells with common inertially fixed centers. The shells, viewed as rigid bodies, are free to rotate subject to a potential that depends only on the relative attitude of the two spherical shells. The configuration manifold is $(\mathsf{SO(3)})^2$ and the modified Lagrangian function $\tilde{L} : \mathsf{T(SO(3))}^2 \to \mathbb{R}^1$ is given by

$$\tilde{L}(R_1, R_1, \omega_1, \omega_2) = \frac{1}{2}\omega_1^T J_1 \omega_1 + \frac{1}{2}\omega_2^T J_2 \omega_2 - K\mathsf{trace}(R_1^T R_2 - I_{3\times 3}),$$

where $(R_i, \omega_i),\ i = 1, 2$, denote the attitudes and angular velocity vectors of the two spherical shells and J_1, J_2 are 3×3 inertia matrices of the two spherical shells and K is an elastic constant.

(a) What are the Euler–Lagrange equations for the two concentric shells on the tangent bundle $\mathsf{T}\left(\mathsf{SO}(3)\right)^2$?
(b) What are Hamilton's equations for the two concentric shells on the cotangent bundle $\mathsf{T}^*\left(\mathsf{SO}(3)\right)^2$?
(c) What are the conserved quantities of the dynamical flow on $\mathsf{T}\left(\mathsf{SO}(3)\right)^2$?
(d) What are the equilibrium solutions of the dynamical flow on $\mathsf{T}\left(\mathsf{SO}(3)\right)^2$?
(e) Determine the linearization that approximates the dynamical flow in a neighborhood of a selected equilibrium solution.

6.16. It can be shown that the problem of finding the geodesic curves on the Lie group $\mathsf{SO}(3)$ is equivalent to the problem of finding smooth curves on $\mathsf{SO}(3)$ that minimize $\int_0^1 \|\omega\|^2\, dt$ and connect two fixed points in $\mathsf{SO}(3)$.

(a) Using curves described on the interval $[0,1]$ by $t \to R(t) \in \mathsf{SO}(3)$, show that geodesic curves satisfy the variational property $\delta \int_0^1 \|\omega\|^2\, dt = 0$ for all smooth curves $t \to R(t) \in \mathsf{SO}(3)$ that satisfy the boundary conditions $R(0) = R_0 \in \mathsf{SO}(3)$, $R(1) = R_1 \in \mathsf{SO}(3)$.
(b) What are the Euler–Lagrange equations and Hamilton's equations that geodesic curves on $\mathsf{SO}(3)$ must satisfy?
(c) Use the equations and boundary conditions for the geodesic curves to describe the geodesic curves on $\mathsf{SO}(3)$.

6.17. Consider the problem of finding the geodesic curves on the Lie group $\mathsf{SO}(3)$ that minimize $\int_0^1 \omega^T J\omega\, dt$ and connect two fixed points in $\mathsf{SO}(3)$. Here J is a symmetric, positive-definite 3×3 matrix that is not a scalar multiple of the identity $I_{3\times 3}$.

(a) Using curves described on the interval $[0,1]$ by $t \to R(t) \in \mathsf{SO}(3)$, show that geodesic curves satisfy the variational property $\delta \int_0^1 \omega^T J\omega\, dt = 0$ for all smooth curves $t \to R(t) \in \mathsf{SO}(3)$ that satisfy the boundary conditions $R(0) = R_0 \in \mathsf{SO}(3)$, $R(1) = R_1 \in \mathsf{SO}(3)$.
(b) What are the Euler–Lagrange equations and Hamilton's equations that such geodesic curves on $\mathsf{SO}(3)$ must satisfy?
(c) Describe the impediments in obtaining an analytical expression for such geodesics on $\mathsf{SO}(3)$.

6.18. Consider n rotating rigid bodies that are coupled through the potential energy; the configuration manifold is $(\mathsf{SO}(3))^n$. With respect to a common inertial Euclidean frame, the attitudes of the rigid bodies are given by $R_i \in \mathsf{SO}(3)$, $i = 1, \ldots, n$, and we use the notation $R = (R_1, \ldots, R_n) \in (\mathsf{SO}(3))^n$; similarly, $\omega = (\omega_1, \ldots, \omega_n) \in (\mathbb{R}^3)^n$. Suppose the kinetic energy of the rigid bodies is a quadratic function in the angular velocities of the bodies, so that the modified Lagrangian function $\tilde{L} : \mathsf{T}(\mathsf{SO}(3))^n \to \mathbb{R}^1$ is given by

$$\tilde{L}(R,\omega) = \frac{1}{2}\sum_{i=1}^{n} \omega_i^T J_i \omega_i + \sum_{i=1}^{n} a_i^T \omega_i - U(R),$$

where J_i are 3×3 symmetric and positive-definite matrices, $i = 1, \ldots, n$, $a_i \in \mathbb{R}^3$, $i = 1, \ldots, n$, and $U : (\mathsf{SO(3)})^n \to \mathbb{R}^1$ is the potential energy that characterizes the coupling of the rigid bodies.

(a) What are the Euler–Lagrange equations for this modified Lagrangian for n coupled rigid bodies?
(b) What are Hamilton's equations for the modified Hamiltonian associated with this modified Lagrangian for n coupled rigid bodies?