UNIVERSITY OF CALIFORNIA, SAN DIEGO

Geometric Aspects of Discretized Classical Field Theories: Extensions to Finite Element Exterior Calculus, Noether Theorems, and the Geodesic Finite Element Method

A dissertation submitted in partial satisfaction of the
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by

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The dissertation of Joe Salamon is approved, and it is acceptable in quality and form for publication on microfilm:

Co-Chair

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2016

DEDICATION

To all those cast aside, to the downtrodden;
To those out of luck, to those whose time is done.

To the struggling, to the out-of-left-fielders,
To the frustrated, to the blinded seers.

Know this, and hold it close to your hearts:
Keep Life, Time, Love, Change never apart.
For those that separate these Designs
Will never leave the space they left to start.

EPIGRAPH

If you don't know where you're going,
you might not get there.
—Yogi Berra

Se eu te pudesse dizer
O que nunca te direi,
Tu terias que entender
Aquilo que nem eu sei.
— Fernando Pessoa

La bouche douloureuse ou les lèvres inertes,
Jusques à la mort, Vie, emplis mon œnophore;
Et moi, ivre d'amour, les narines ouvertes.
Les seins dressés vers toi, je te crierai: Encore!
—Valentine de Saint-Point

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Life stories, growth, paths, and outcomes: most people like to think that these factors are all purely under their control. In reality, all of these variables are determined by our interactions with others. These pages represent my attempt at giving a little back to all those that have provided the supportive environment and creative grounds for my journey thus far.

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ABSTRACT OF THE DISSERTATION

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In this dissertation, I will discuss and explore the various theoretical pillars required to investigate the world of discretized gauge theories in a purely classical setting, with the long-term aim of achieving a fully-fledged discretization of General Relativity (GR). I will present some results on the geometric framework of finite element exterior calculus (FEEC); in particular, I will elaborate on integrating metric structures within the framework and categorize the dual spaces of the various spaces of polynomial differential forms $\mathcal{P}_r\Lambda^k(\mathbb{R}^n)$. I will also introduce the Rapetti construction, and then demonstrate the general issues with providing geometric interpretations to polynomial order within FEEC. After a brief pedagogical detour through Noether's theorems, I will apply all of the above into discretizations of electromagnetism and linearized GR. I will conclude with an excursion into the geodesic finite element method (GFEM) as a way to work with nonlinear manifolds.

Chapter 1

Introduction

"Everything starts somewhere, though many physicists disagree."
— Terry Pratchett

1.1 Prelude

This thesis is an attempt to lay a modern foundation for a unified theory of discrete classical gauge fields, with the discretization and simulation of General Relativity as the ultimate long-term goal. Einstein's theory of gravity is arguably the most difficult field theory to computationally simulate that is still physically relevant, so I have no delusions about accomplishing such a goal within the confines of this publication. I ask the reader to think of this objective as a holy grail of sorts that is within relatively[1] close grasp.

The first portions of this chapter will be motivational and historical, served with a healthy amount of rumination and dry humor, followed by a structural overview of the thesis. The remaining cuts will consist of brief introductory sprinklings to the conventions and notations of the various physical and mathematical tools I will use in the rest of the thesis. My hope is that this appetizer will leave the reader with a desire to delve into the forthcoming main course, and end with the sense of a well-balanced meal.

[1] This is a pun. Finding the remaining puns is an exercise left to the reader.

1.2 History

The study of continuous gauge theories has a long history, arguably starting with Maxwell's compilation of the equations of motion for electromagnetism over 150 years ago. Within the last 80 years, this direction of research veered strongly into the quantum world due to the descriptive power of such theories within the realm of particle physics. However, there are many aspects of classical gauge theories that are still not well understood, even in the continuous setting.

The modern culmination of all work on classical field theories can be found in the GiMmsy preprints [GMI+12]. This project, founded by Mark Gotay and Jerrold Marsden, started as an attempt to provide a complete description of the Lagrangian and Hamiltonian aspects of classical field theories: this includes theories with constraints, relativistic theories, and couplings to matter. Their work illustrates how many different facets of classical field theories are in fact completely related to each other.

Their treatise shows that first-order classical field theories are structurally well-understood. On the contrary, the analysis of higher-order field theories (such as General Relativity) is fraught with both structural and conceptual difficulties, especially in the context of constraint theory.

In addition, the discretization of such classical theories has only been considered within the last 50 years or so; most notably, Tullio Regge's framework for a simplicial approximation of spacetime manifolds with a linearized action for General Relativity in 1961 [Reg61] marks the first theoretical attempt at analyzing the properties of a discrete classical gauge theory.

Most would argue that considering the discretization of a continuous theory is something left best for once the continuous theory is fully understood, or perhaps even a minor distraction on the path towards the "truly ultimate" goal of quantum gravity. I claim the contrary: I believe there is much to be gained by considering the discrete classical realm first, and studying the myriad ways in which different levels of discretization interact with the dynamics and kinematics of a given field. In fact, the path from the discrete realm to the continuous realm has a homologous relation to the path from the classical to the quantum world; the study of this former transition is usually more transparent and can provide unexpected insights into the latter.

1.3 Why Discretization?

There are four main types of discretization can occur:

- The fields
- The symmetry group
- Spacetime (the underlying manifold)
- All of the above

All four situations are physically interesting, especially when considering the potential connections to the quantum regime. After all, there are natural connections between the discretized classical world and the quantum world, and in fact, there are strong hints from lattice gauge theories (LGTs) that discrete geometry strongly connects the two, as explored in [Oec05] and [VC07].

The most complex scenarios occur when the fields, gauge group, and the underlying manifold are all discretized in potentially different ways, as is the case with a generic numerical simulation of a field theory. Heuristically speaking, when all three types of discretization are different, one expects some set of compatibility conditions between the three discretizations. Requiring consistency in the type of discretization between the fields and the gauge group simplifies the analysis of the dynamics and kinematics of the resulting field theory, i.e. distinguishes physical evolution of the field variables versus gauge components or symmetry constraints. For theories in which the spacetime is a dynamical variable, then all three notions of discretization should coincide to ensure discrete behavior that maps properly into its continuous counterpart. The techniques analyzed in this thesis are geared towards discretizations that connect the fields, gauge groups, and the spacetime together in a coherent fashion. It is my hope that these tools and ideas will eventually lead to proper discretizations of General Relativity, the principal theory of dynamical spacetime.

1.4 A Structural Overview

Currently, the main long term goal of my work is to characterize discretized theories of general relativity, with applications to simulations and numerical relativity kept in mind. The hope is that classifying discretizations of this highly geometric theory would lead to a better understanding of not only continuous second-order gauge theories but also to a better construction of simulations geared towards general relativity.

Chapter 2 will provide an introduction to the framework of finite element exterior calculus (FEEC), which preserves nice properties of exterior calculus on simplicial complexes as applied to finite element methods via the space of polynomial differential forms $\mathcal{P}_r\Lambda^k(\mathbb{R}^n)$. This chapter mostly showcases the geometric interpretation of many aspects of the theory which are usually ignored. More specifically, I will consider the Lie derivative and the Hodge star in this context, and analyze their corresponding dual spaces.

Chapter 3 is an excursion into the higher-order forms in FEEC, and in particular, will focus on the interpretation championed by Francesca Rapetti which associates polynomial order with homotheties of the original simplex. I will discuss the pros and cons of this interpretation, then use the geometric constructions from Chapter 2 to offer my own take on the matter.

Chapter 4 is a pedagogical detour into Emmy Noether's two theorems. This chapter is intended to distinguish the two theorems, as most physicists refer to "Noether's Theorem" as a single entity. I will break down Noether's original paper and apply her methods to well-known examples. The chapter concludes with a discussion of modern terminology, and an ending note on the distinction between the two theorems in a modern context.

Chapter 5 provides an application of the previous chapters into two discretized examples: spacetime electromagnetism and linearized GR via the first-order Fierz-Pauli formalism. Since these theories live on flat background spaces, I will employ a variational integrator-like method on a generic simplicial complex of the underlying spacetime using polynomial differential forms. Both of Noether's theorems appear in this context when considering matter terms and couplings, and both place different but complementary constraints on the ensuing dynamics.

Chapter 6 showcases the geodesic finite element method (GFEM) as a potential path to working with the above notions on non-flat manifolds. This method is a minimization problem over a Riemannian distance function, and thus, the chapter is dedicated towards constructing such metrics over general spaces. The construction of such a metric is provided for a general class of spaces, and particular applications towards the space of Lorentzian metrics and the space of symplectic forms.

Chapter 7 concludes the dissertation. The major results and contributions from each chapter will be summarized here, with a closing outlook towards future work.

Overall, this area of work is an intersection of theoretical physics and applied mathematics. As is evidenced by the subject matter of Chapter 4, much of the requisite work to discretizing GR requires establishing communication between the realms of mathematics and physics. Since I am a physicist, most of the language here will be directed towards physicists, with occasional mathematical digressions as necessary. This thesis presumes some background in variational principles, group theory, relativity, and differential geometry. Before continuing with the rest of this work, I will provide a quick introduction to the general notations and conventions used from this chapter onwards.

1.5 Notations and Conventions

The notations and conventions used in this thesis will generally follow the works of Morita [Mor01], Baez and Muñoz [BM08], and Ryder [Ryd06]. Some chapters may use slightly different notation; in particular, Chapter 4 on Noether's Theorems will stick to the conventions established in Noether's original paper [Noe18], with translations into modern terminology given throughout.

1.5.1 Exterior algebras

If V is an n-dimensional vector space with vector basis $\{v_i\}$, then $\Lambda(V)$ represents the exterior algebra over V. The symbol $\wedge$ will represent the exterior or wedge product of the algebra[2]. Taking all possible wedge products with k total vectors provides a basis for an algebra which we will denote $\Lambda^k(V)$. This space provides a decomposition for the exterior algebra as $\Lambda(V) = \bigoplus_{k=0}^{n} \Lambda^k(V)$.[3] A member ${}^k w \in \Lambda^k(V)$ is a degree k multivector or k-vector; a multivector will normally be written without a superscript if its degree is contextually obvious.

In a general setting, all of the above carries over to the dual vector space V^*. The inner product or metric, denoted by either angle brackets $\langle \cdot, \cdot \rangle$ or $g(\cdot, \cdot)$, provides a way of translating between V and V^*. This is given by the sharp and flat operators, written $\sharp$ and $\flat$ respectively[4]. The interior product $i_X \omega$ contracts a covector ω of degree k with a vector X of degree one, and produces a $(k-1)$-covector. Correspondingly, the contraction of a k-covector ω with a k-vector v is denoted by $\omega(v) = \langle \omega^\sharp, v \rangle = \langle \omega, v^\flat \rangle$. I will take Vol to stand for the volume form, defined as a normalized n-covector such that $\langle Vol, Vol \rangle = 1$.

1.5.2 Relativity and exterior calculus on manifolds

As for relativity, the standard conventions will be adopted; M will typically denote a flat spacetime manifold. The metric will be an inner product over vectors in the tangent bundle TM,[5] which means that it is a symmetric $(0,2)$-tensor. It can also be turned into an equivalent inner product over covectors in the cotangent bundle T^*M. $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ will represent the metric over Minkowski spacetime with volume form $dt \wedge dx \wedge dy \wedge dz$, and $g_{\mu\nu}$ will represent a generic Lorentzian metric over an arbitrary spacetime. $h_{\mu\nu}$ will represent perturbations from a fixed metric over a background spacetime. The indices take on their usual meaning as in tensor calculus, with the Einstein summation convention employed for repeated indices.

A vector field v assigns a vector $v_p \in \Lambda^1(T_pM)$ to each point $p \in M$, and its

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[2] The wedge product functions similarly to the curl operation in $\mathbb{R}^3$: takes two members of the algebra and produces a third member, and is anticommutative ($a \wedge b = -b \wedge a$ where a, b are vectors). The alternating nature of the product strongly constrains the form of the allowed subspaces.

[3] Note that the dimension of $\Lambda^k(V)$ equals $\binom{n}{k}$, the dimension of $\Lambda(V)$ must be 2^n.

[4] Physicists are accustomed to seeing these operators as "raising and lowering indices".

[5] The tangent bundle represents the union of the tangent spaces at each possible point p in the manifold. This means that the metric tensor g on M really is a function of p, as only tangent vectors from the same tangent space TM_p can be compared.

local coordinates will be typically written via the set of basis vectors $\{\left(\frac{\partial}{\partial x_i}\right)_p\}$. A differential k-form ω on M assigns a k-covector $\omega_p \in \Lambda^k(T^*_pM)$ to each point $p \in M$; these will be written via the set of basis covectors $\{dx_{i_1} \wedge dx_{i_2} \cdots \wedge dx_{i_k}\}$[6].

The exterior derivative d takes k-forms to $(k+1)$-forms, with $d^2 = 0$. The integral of a k-form ω over a k-dimensional (sub-)manifold M yields a scalar and is written $\int_M \omega$. If $\omega = d\alpha$, then Stokes' theorem is given by the elegant relation $\int_M \omega = \int_M d\alpha = \int_{\partial M} \alpha$, where ∂ represents the boundary operation.

$\mathcal{L}_X$ denotes the Lie derivative, whose action on differential forms is given by Cartan's magic formula $\mathcal{L}_X = \{i_X, d\} = i_X d + d i_X$.[7] I will use $\star$ to denote the Hodge star, which takes k-forms to $(n-k)$-forms. Its action is defined through the metric via the identification ${}^k\alpha \wedge \star({}^k\beta) = \langle \alpha, \beta \rangle Vol$, or the more direct expression $\star\omega = i_{\omega^\sharp} Vol$. Furthermore, $\star\star\,\omega = (-1)^{n(n-k)} s\omega$, where s represents the signature of the metric, k the degree of the form being acted upon, and n the dimension of the space. The Hodge star also gives rise to the codifferential $\delta = (-1)^{n(k-1)+1} s \star d \star$.[8] Consequently, the Hodge Laplacian operator Δ can also be defined via $\Delta = \{d, \delta\} = d\delta + \delta d$, which is central to Hodge theory and the de Rham cohomology complex.

1.5.3 Variational principles

Throughout this work, S will denote the action, the invariant integral of the Lagrangian density L over a manifold M:

$$S = \int_M L \ .$$

As stated previously, M will usually represent a flat, spacetime manifold since I will only consider relativistic field theories in either their continuous or discrete variations. If the manifold is discretized, then M will also represent a simplicial complex corresponding to the continuous manifold represented by M. Chapter 6 is the main exception to this rule: M can represent an arbitrary spacetime manifold or a symplectic manifold. As a density, L is usually proportional to Vol, as only a volume

[6] It is in this framework that physicists will recognize differential k-forms as totally antisymmetric $(0,k)$-tensors.

[7] I will only use the more general formula applicable to all tensors in a subsection on General Relativity in Chapter 4.

[8] Note that this occurs since the first and second Hodge stars are technically different operators: the degree of the form gets changed as it passes through the sequence of operations.

form can be meaningfully paired with the entire domain of integration M. L will almost always be a 4-form over a Lorentzian spacetime.

Infinitesimal variations on S or L will be represented by the symbol δ; in spite of the notational overlap with the codifferential defined above, it should be clear from context which δ is being applied[9]. The same symbol δ will be used to denote shifts to the fields contained within L as well. This notational overload only occurs in Chapter 4, and I will point out the difference as required for Noether's theorems.

[9]The codifferential of S should vanish, as S is a 0-form; similarly, the codifferential of L is almost never considered outside the context of boundary terms.

Chapter 2

Finite Element Exterior Calculus: Geometry and Dual Operators

"Your pain is the breaking of the shell that encloses your understanding."
— Khalil Gibran

2.1 Introduction

Finite Element Exterior Calculus was developed by Arnold, Falk, and Winther in two seminal papers [AFW10][AFW06]. Their aim was to tie the zoo of existing mixed finite element methods into a unifying framework by considering spaces of r-degree polynomial differential k-forms over $\mathbb{R}^n$, denoted by $\mathcal{P}_r\Lambda^k(\mathbb{R}^n)$ or $\mathcal{P}_r\Lambda^k$ when the dimension of the underlying space is understood from context. In other words, approximating any smooth function on $\mathbb{R}^n$ is a matter of choosing an appropriate order of approximation via r. These spaces are occasionally broken down into spaces of homogeneous polynomials, denoted by $\mathcal{H}_r\Lambda^k$.

The basic underpinnings of the theory lies upon keeping track of polynomial degree on chains and co-chains, thus effectively allowing for a realization of de Rham cohomology on these spaces. By preserving this property from the continuous setting, simulations using FEEC display increased stability and long-term behavior with little to no spurious eigenmodes; [AFW10] provides sample applications to electromagnetism in $\mathbb{R}^3$ with particularly striking improvements in accuracy and stability.

The main ingredients of this framework are:

- the boundary operator, ∂
- the exterior derivative, d
- the Koszul operator, κ_v
- the space of Whitney k-forms, W^k

We will quickly introduce each of these operators and their geometric significance in the remainder of this section, then turn to examining other common operations from standard exterior calculus that are rarely discussed in the context of FEEC. Even if generalizing the scope of the framework is a fruitless endeavor, the exercise still yields interesting insights into its overall structure, and provides groundwork for the analysis in later chapters.

For both this chapter and the next, I will use the following conventions on simplices. $\{v_i\}$ will denote a set of vertex vectors, and $\sigma = [v_0, v_1, \dots, v_n]$ will denote an oriented n-simplex. I will often refer to the standard simplex in $\mathbb{R}^n$, which has the origin and the Cartesian set of unit vectors as its vertex vectors. Since FEEC only works in affine spaces, every simplex in a given simplicial complex is equivalent to the standard simplex up to affine transformations.

2.1.1 Exterior Derivative and the Boundary Operator

The boundary operator ∂ is defined in the usual way over simplicial chains: if $C_l(M)$ represents the group of l-chains over M, then $\partial : C_l(M) \to C_{l-1}(M)$ and $\partial^2 = 0$, as a boundary has no boundary. Given an oriented n-simplex σ, the action of ∂ is given by the following formula

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i [v_0, \dots, \widehat{v}_i, \dots, v_n], \tag{2.1}$$

where the hat represents the omission of that vertex in that term.

As defined in the Introduction, the exterior derivative d is the natural derivative that maps k-forms to $k+1$-forms, with $d^2 = 0$. Specifically, in FEEC, d :

$\mathcal{P}_r\Lambda^k \to \mathcal{P}_{r-1}\Lambda^{k+1}$. With the standard Cartesian basis of 1-forms over $\mathbb{R}^n$ given by $\{dx_i\}$, d acts on a k-form $\omega = f(x_1, \ldots, x_n)\; dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k$ as follows:

$$d\omega = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge dx_1 \wedge \cdots \wedge dx_k.$$

Just as in standard exterior calculus, d is dual to ∂ via integration. This connection is described by Stokes' Theorem, which is most succinctly written as

$$\int_{\partial c} \omega = \int_c d\omega \; . \tag{2.2}$$

Integration computes scalars by pairing chains with cochains, thus it is natural to view d and ∂ as dual operators. Despite acting on different spaces, they provide an equivalent answer under an integral sign. This is a recurring theme for the rest of this chapter: I will analyze the geometric operators dual to standard operations on forms over integration.

2.1.2 Koszul Operator

The Koszul operator κ_v is "almost" the inverse of the exterior derivative, as $\kappa_v : \mathcal{P}_r\Lambda^k \to \mathcal{P}_{r+1}\Lambda^{k-1}$ with $\kappa_v^2 = 0$. In the context of FEEC, it is the operator that allows us to generate a useful space of differential forms called the Whitney forms, which will be discussed in the next subsection.

The Koszul's action depends on a choice of origin, denoted by v, the vector denoting the origin in question, in the subscript. Since FEEC is generally applied to affine spaces, the choice of origin is irrelevant and the subscript is usually dropped. Within the context of a given simplex, v is restricted to one of the vertices of the complex in question. Usually the vertex denoted v_0 is taken as the local origin, and the operator is written as κ_0 or κ. More concretely, the action of κ_0 on the form ω defined above is

$$\kappa_0\omega = \sum_{i=1}^{n} (-1)^{i+1} x_i dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \cdots \wedge dx_k,$$

where as before, the hat denotes the omission of that term in the sum.

As an example, let us say we are analyzing $\mathbb{R}^2$ with standard Cartesian coordinates. Take the standard simplex with vertices $\{v_0 \to (0,0), v_1 \to (1,0), v_2 \to (0,1)\}$, and consider a 2-form $\omega = dx \wedge dy$. Then the action of the Koszul on ω with respect to v_0 is

$$\kappa\omega = xdy - ydx \ .$$

In this Euclidean space, its action is equivalent to the interior product of ω with respect to the radial vector $r = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, i.e. $\kappa_0 = i_r$. Indeed, this implies that given $\omega \in \mathcal{H}_r\Lambda^k$, $(\kappa d + d\kappa)\omega = (r+k)\omega$, as is shown in [AFW06].

The exterior derivative and the Koszul together form a de Rham complex over the space of polynomial differential forms, which is a subcomplex of the usual one over all differential forms. This discrete analogue of the continuous space ensures that all operations in FEEC respect this structure, and thus preserves most aspects of exterior calculus.

2.1.3 Whitney forms

The space of Whitney forms W^k was first used by Hassler Whitney in his classic text [Whi57], and were introduced as a coordinate-free way of working with forms that was useful in a topological setting. In his work, the set of barycentric coordinates λ_i over an n-simplex σ comprised the space W^0. Whitney forms of degree k were given by

$${}^k w_\rho = k! \sum_{j=0}^{k} (-1)^j \lambda_{i_j} d\lambda_{i_0} \wedge d\lambda_{i_1} \wedge \cdots \wedge \widehat{d\lambda_{i_j}} \cdots \wedge d\lambda_{i_k},$$

where ρ is a k-subsimplex of σ, the barycentric coordinates $\lambda_{i_0}, \lambda_{i_1}, \ldots \lambda_{i_k}$ represent the vertices in ρ, and the hat represents the omission of that factor within the wedge product[1].

These forms were eventually found to be useful in the field of finite elements, and are in fact key members of the family of polynomial differential form spaces. In FEEC, the space of Whitney forms W^k is given by the symbol $\mathcal{P}_1^-\Lambda^k$. Their most useful property, in fact, is that they form an orthonormal basis over simplices: each Whitney k-form has a natural pairing with a k-(sub)simplex via integration.

[1] For the remainder of this chapter, we will use the notation ${}^j w_\rho$ to denote a Whitney j-form over the oriented set of $j+1$ vertices of a j-simplex ρ.

Spoken differently, given a top-level n-simplex σ, the set of Whitney k-forms forms an orthonormal basis when integrated over the k-subsimplices of σ.

Essentially, this provides a direct geometric link between the discretization of the underlying space and the discretization of the differential forms used to describe functions on that space. As shown in the next section, Whitney forms additionally provide a natural stepping stone between polynomial differential form spaces of different degree. There are equivalents of these forms in higher degree polynomials (written $\mathcal{P}_r^-\Lambda^k$), but only the "linear" space is directly connected to the underlying simplicial complex in an unambiguous fashion[2]. The higher order spaces are closed under the action of the action of the wedge product:

$$\mathcal{P}_r^-\Lambda^k \wedge \mathcal{P}_s^-\Lambda^j \in \mathcal{P}_{r+s}^-\Lambda^{j+k}. \tag{2.3}$$

The linear order forms also satisfy a useful closure relation: $dW^k \subset W^{k+1}$. This implies they form their own subcomplex of polynomial differential forms.

2.2 FEEC Spaces and Characterizations

The ladder of polynomial differential form spaces can be described as follows

$$\mathcal{P}_r\Lambda^k = d\mathcal{P}_{r+1}\Lambda^{k-1} \oplus \kappa\mathcal{P}_{r-1}\Lambda^{k+1}, \tag{2.4}$$

with an equivalent equation applying to the spaces of homogeneous polynomial differential forms:

$$\mathcal{H}_r\Lambda^k = d\mathcal{H}_{r+1}\Lambda^{k-1} \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1}. \tag{2.5}$$

In the above decompositions, the Koszul is taken with respect to a fixed origin.

Definitionally, the relation between spaces of consecutive degree is described via

$$\mathcal{P}_r\Lambda^k = \mathcal{P}_{r-1}\Lambda^k \oplus \mathcal{H}_r\Lambda^k.$$

Combining this with equation (2.4) yields

$$\mathcal{P}_r\Lambda^k = \mathcal{P}_{r-1}\Lambda^k \oplus d\mathcal{H}_{r+1}\Lambda^{k-1} \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1},$$

[2]These higher-order analogues will be discussed in more depth in Chapter 3.

which leads us to the following reorganization of the components of $\mathcal{P}_r\Lambda^k$ spaces:

$$\mathcal{P}_r^-\Lambda^k = \mathcal{P}_{r-1}\Lambda^k \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1}, \tag{2.6}$$

$$\mathcal{P}_r^+\Lambda^k = d\mathcal{H}_{r+1}\Lambda^{k-1}. \tag{2.7}$$

This implies that the space of Whitney forms can be written as

$$W^k = \mathcal{P}_1^-\Lambda^k = \mathcal{H}_0\Lambda^k \oplus \kappa\mathcal{H}_0\Lambda^{k+1}. \tag{2.8}$$

The Koszul term provides the intermediate step between adjacent polynomial order spaces by constraining the lower order with higher order information. Note that the equation 2.8 can also be written as $W^k = \bigoplus_v \kappa_v\mathcal{H}_0\Lambda^{k+1}$, where the direct sum is taken over all possible vertices v within a given simplex[3].

The dimensions of these spaces are given by

$$\dim \mathcal{H}_r\Lambda^k(\mathbb{R}^n) = \binom{n}{k}\binom{n+r-1}{n-1}, \tag{2.9}$$

$$\dim \mathcal{P}_r\Lambda^k(\mathbb{R}^n) = \binom{n}{k}\binom{n+r}{n}, \tag{2.10}$$

$$\dim \mathcal{P}_r^-\Lambda^k(\mathbb{R}^n) = \binom{n+r}{n-k}\binom{r+k-1}{k}, \tag{2.11}$$

$$\dim \mathcal{P}_r^+\Lambda^k(\mathbb{R}^n) = \binom{n+r}{n-k}\binom{r+k-1}{r}, \tag{2.12}$$

as shown in [AFW06]. As a special case, $\dim W^k = \dim \mathcal{P}_1^-\Lambda^k(\mathbb{R}^n) = \binom{n+1}{k+1}$. Additionally, note that $\dim \mathcal{P}_r^-\Lambda^k(\mathbb{R}^n) = \frac{r}{r+k}\dim \mathcal{P}_r\Lambda^k(\mathbb{R}^n)$ and $\dim \mathcal{P}_r^+\Lambda^k(\mathbb{R}^n) = \frac{k}{r+k}\dim \mathcal{P}_r\Lambda^k(\mathbb{R}^n)$, so the proportion of elements from each space depends on the ratio of r to k.

2.3 Homotopy: the cone and Koszul operators

The cone operator C_v is a natural operator on the space of chains: it props up a given simplex σ to a simplex σ' which includes the vertices of σ with the given v. In effect, C_v takes an l-chain to an $(l+1)$-chain with $C_v^2 = 0$. As we are dealing with a given simplicial complex, the point v is restricted to the vertices within the

[3] This equality holds due to the affine nature of the underlying space.

complex, effectively creating a convex hull out of each element of the original chain with v. The cone and boundary operator must satisfy the relation

$$\partial C_v + C_v \partial = \mathrm{id}$$

much like d and κ_v: $(d\kappa_v + \kappa_v d)\omega = (k+r)\omega$, where $\omega \in \mathcal{H}_r\Lambda^k$. There is one difference, however: for a general $\omega \in \mathcal{P}_r\Lambda^k$, the operation $d\kappa_v + \kappa_v d$ is no longer proportional to the identity. In fact, computing the above in $\mathbb{R}^n$ with v as the origin and the same ω, yields

$$(\kappa d + d\kappa)\omega = k\omega + \left(\sum_{i=1}^{k} x_i \frac{\partial f}{\partial x_i}\right) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k. \tag{2.13}$$

Thus, the cone operation cannot be the exact dual to κ_v.

The homotopy operator $\mathbf{H}_v$ is the usual operation such that $d\mathbf{H}_v + \mathbf{H}_v d = \mathrm{id}$. Following the convention in [Ede85] as applied to the the context of FEEC, it acts on $\omega \in \mathcal{P}_r\Lambda^k$ by

$$\mathbf{H}_v\omega(x) = \int_0^1 \kappa_v\omega(v + \lambda(x-v))\lambda^{k-1}\, d\lambda\ . \tag{2.14}$$

where the ω in the integrand is a function of the flow parameter λ representing the interpolated flow from the origin vector v to the current position. This operation acts much like the Koszul: it reduces a k-form to a $(k-1)$-form and increases the polynomial order by 1. The main difference is that it is closer to being the appropriate "inverse" of d.

This begs the question: what is the proper dual to the Koszul then? It should clearly act like the cone operator C_v, except perhaps for an extra mass being added to the chain it acts upon. The supposed dual to the Koszul must carry a weight that accounts for the dimension of the simplex being acted on. We can tentatively define such an operation C'_v such that

$$(C'_v\partial + \partial C'_v)\sigma = (k+r)\sigma \tag{2.15}$$

with σ a k-simplex; this formulas holds since the only forms dual to the simplicial elements are the Whitney forms W^k, all of which are in $\mathcal{H}_r\Lambda^k$. The dual to κ_v with respect to general polynomial differential forms is an issue we will defer to Chapter 3, in which we will discuss higher-order elements. For the remainder of this chapter, we will assume that such a C'_v exists, and satisfies a Stokes'-like theorem with the Koszul operation:

$$\int_{C'_v\sigma} \omega = \int_\sigma \kappa_v \omega. \tag{2.16}$$

The Koszul operator can then be interpreted as the "discrete" analogue of the homotopy operator on the space of polynomial differential forms.

2.4 Interior Product and Extrusion

The interior product of a differential form with a vector field X is a standard operation in differential geometry, usually denoted by the operator $i_X : \Lambda^k \to \Lambda^{k-1}$. If the vector field is of polynomial degree s, then the interior product takes the form from $\mathcal{P}_r\Lambda^k$ to $\mathcal{P}_{r+s}\Lambda^{k-1}$. Geometrically, this operation acts exactly like contraction in tensor calculus; one can think of the interior product as computing the remnant of the differential form that does not coincide with the vector field X. Within the context of FEEC, the natural vector fields to consider can be constructed from the edge vectors over the entire simplicial complex.

The dual operator to the interior product is rarely discussed, and is related to the extrusion of the domain of integration. The extrusion is defined as the manifold obtained by allowing a vector field to sweep it for a set amount of time. As this discussion uses notions of weak-form problems and variational integrators, it would then make sense to consider the possible geometric dual operation on the space of chains. Essentially, by letting the vector field X 'flow' via a dummy variable then differentiating over that variable, one can extract the complementary effect of the interior product on the underlying chains [Bos06][Hir03].

Let $M_X(t)$ represent the images of the manifold flowed out at specific times t. Then the extrusion is the union of all such images. Taking $E_X(M,t)$ as the extrusion of the manifold M by X as a function of t,

$$\int_M i_X \omega = \frac{d}{dt}\bigg|_{t=0} \int_{E_X(M,t)} \omega \ . \tag{2.17}$$

The extrusion over the flow inherits a natural orientation from the field and its original manifold, as shown in [Bos06]. The procedure is illustrated in the figures on the next page, which differentiate $M_X(t)$ from $E_X(M,t)$. This process leads to a general duality for any vector field, which only becomes meaningless when the flow is completely parallel to the manifold in question. In other words, extrusion only bumps up the dimension of the manifold by 1 when the flow has some components perpendicular to the manifold in question.

In FEEC, M will be restricted to (sub)simplices, which implies that the extrusion will generically not fall within the original complex. However, since the dual operation to the interior product measures the flow of the extrusion, this does not present any issue, especially if only vector fields given by the edge vectors are used in this process. As we will see in Chapter 3, the extrusion operation will also come in handy when analyzing the higher order polynomial spaces.

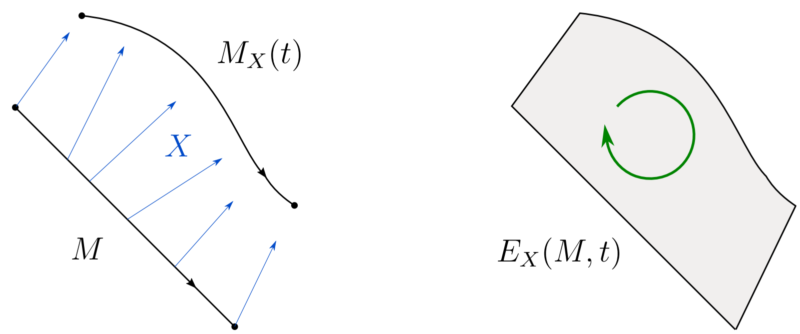


Figure 2.1: Comparison of $M_X(t)$ to $E_X(M,t)$ in $\mathbb{R}^2$; note the induced orientation.

2.5 Lie Derivative and the Flow operator

In essence, the Lie Derivative $\mathcal{L}_X$ is the advective derivative and hence does not require a metric structure. It is colorfully referred to as the "fisherman's derivative" by Arnol'd [Arn95], and on differential forms, its action is typically given by Cartan's magic formula [Mor01]:

$$\mathcal{L}_X = i_X d + d i_X. \tag{2.18}$$

In other words, it is the anticommutator of the interior product with the exterior derivative. Clearly, $\mathcal{L}_X : \Lambda^k \to \Lambda^k$, and if X is a polynomial vector field of s-degree, then $\mathcal{L}_X : \mathcal{P}_r\Lambda^k \to \mathcal{P}_{r+s-1}\Lambda^k$. As with the interior product, X will typically be taken from the set of edge vectors in the complex.

The dual operation to the Lie derivative over integration does not have a standard name in the literature. From the definitions given above,

$$\int_M \mathcal{L}_X\omega = \int_{\partial M} i_X\omega + \frac{d}{dt}\bigg|_{t=0} \int_{E_X(M,t)} d\omega = \frac{d}{dt}\bigg|_{t=0} \int_{\Phi_X(M,t)} \omega \ . \tag{2.19}$$

where the operation on the manifold $\Phi_X(M,t) = E_X(\partial M, t) + \partial E_X(M,t)$ is the dual operator in question. In this work, this operation will be called the flow operation. As with the dual to the interior product, the final t-derivative computes the velocity of this flow, and hence the geometric significance to the Lie derivative.

Although the expression for $\Phi_X(M,t)$ seems unwieldy, it simplifies significantly if one considers the following: the effects of the extrusion of a boundary and the boundary of an extrusion should cancel to some extent. In fact, the remnants of this operation should only involve M and X. Letting $M_X(t)$ denote the flowed-out manifold as above, then

$$\Phi_X(M,t) = M_X(t) - M. \tag{2.20}$$

Plugging this back into the integral above yields the simplified relation:

$$\int_M \mathcal{L}_X\omega = \frac{d}{dt}\bigg|_{t=0} \int_{\Phi_X(M,t)} \omega = \frac{d}{dt}\bigg|_{t=0} \int_{M_X(t)} \omega \ . \tag{2.21}$$

Thus, the dual to the Lie derivative computes the velocity of the extruding manifold as it flows away from its original location. The figure shown below demonstrates the identity in equation (2.20) as applied to the manifold M and vector field X as previously illustrated in Figure 2.1.

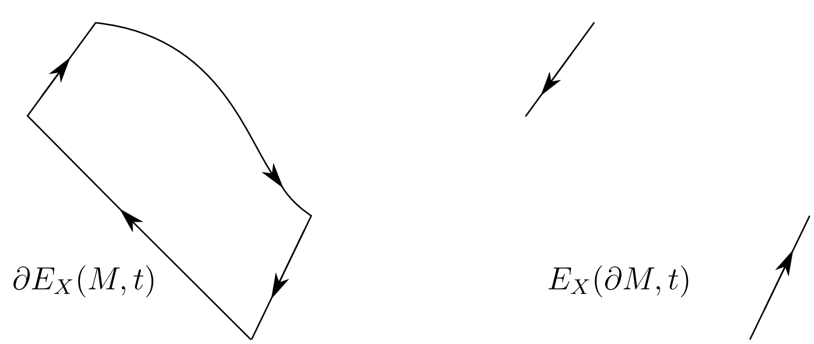


Figure 2.2: The two terms of $\Phi_X(M,t)$. Their sum yields $M_X(t) - M$.

2.6 Hodge star, its geometric dual, and other metric operators

Harrison's work on chainlets ([Har05], [Har06a], [Har06b], [HP12]) provides a very general background on which differential forms can be constructed, especially in the weak sense. Chainlets are a rigorous way to work with spaces as "pathological" as fractal spaces (e.g. soap films), and thus provide a broad context with which to analyze the spaces of all simplicial complexes. Her work has included the development of a geometric dual over integration to the Hodge star on differential forms on chainlets, which yields a Stokes'-like theorem,

$$\int_M \star\omega = (-1)^{n(n-k)} s \int_{\star M} \omega,$$

where ω is a k-form, M is an n-dimensional space, s is the signature of the underlying metric, and $\star$ represents both the analytic and geometric Hodge dual. Thus, these generic spaces allow for a description of the geometric Hodge star on simplices which I will apply to FEEC. Let us first turn to a brief excursion into Harrison's work; the presentation here will follow the ideas in [Har06b]. The paper works with Euclidean spaces, but suitable generalizations to (flat) Lorentzian spaces are straightforward.

Chainlets represent a fractal generalization of the space of polyhedral chains, which are slightly more general than simplicial chains. The fractal nature of chainlets is built up via the use of m-order difference k-cell chains; the order in this case represents the level of iteration. For example, consider the chain in question to be a 2-simplex σ in $\mathbb{R}^2$. An order 1 difference 2-simplex σ^1 would be given by

$$\sigma^1 := \sigma - T_v\sigma,$$

where T_v represents the translation operation by some vector $v \in \mathbb{R}^2$. Higher order difference chains would be given by iterating this process; for example, the order 2 difference 2-simplex σ^2 takes the form

$$\sigma^2 := \sigma^1 - T_v\sigma^1 = \sigma - 2T_v\sigma + T_v^2\sigma.$$

The geometric Hodge star acts naturally over the space of chainlets, in a similar way to the limiting process described above. To find the geometric Hodge dual of a k-simplex σ, take many slices of the simplex and orthogonalize[4] these individual slices into their corresponding $(n-k)$-simplicial slices. Adding these slices up as a chain and taking the limit of infinitesimally small slices yields the Hodge dual.

Consider the figure below as an example. Take σ to represent a 1-simplex in Euclidean $\mathbb{R}^2$ and slice it up. These 1-simplex slices should become orthogonal 1-simplex slices under the action of the Hodge star. Summing the limit of infinitely thin slices yields infinitely small simplices orthogonal to σ as a chain, stacked in the original orientation of σ; $\star\sigma$ is thus an edge whose orientation is perpendicular to the original orientation.

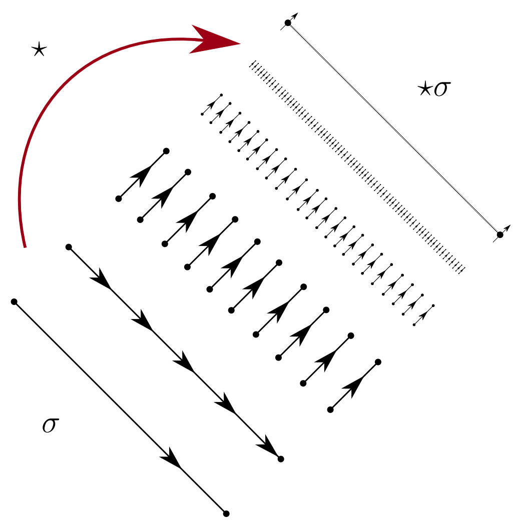


Figure 2.3: The red arrow denotes the limiting process from σ to $\star\sigma$ in Euclidean $\mathbb{R}^2$. The arrows on $\star\sigma$ denote the direction of the infinitesimally small simplices.

Harrison's work addresses the technicalities of providing appropriate measures over these spaces and the corresponding natural norms that give a rigorous meaning to the fractal iteration. In addition, geometric invariants such as volume and area must be preserved upon this transformation, and this is accomplished through the use of an appropriate measure.

[4]This is meant in the sense of the local metric.

Eventually, the following identity is shown within this framework:

$$\int_M \omega = \int_{\star M} \star\omega \ .$$

This equation must be true if the geometric Hodge star is to remain consistent with its analytic dual. Since the the Hodge star acts on both the form and the space equally on a point-by-point basis, the identity holds water, intuitively speaking. In turn, this implies that

$$\int_M \star\omega = \int_{\star M} \star\star\,\omega = (-1)^{k(n-k)} s \int_{\star M} \omega \ . \tag{2.22}$$

In addition, just as with the analytic Hodge dual, the geometric star satisfies $\star\star M = (-1)^{k(n-k)} sM$.

Since FEEC is framed over affine spaces with no metric requirement, the introduction of a Hodge star restricts us to either a Euclidean or Lorentzian metric on the base space, which is entirely compatible with the construction above. Of course, this opens the door to a the usual set of operations that accompany a metric structure; for instance, the $\flat$ and $\sharp$ operations now allow a simultaneous discretization of the space of polynomial vector fields via $(\mathcal{P}_r\Lambda^1)^\sharp$. In addition, we can now consider a slew of Hodge dualized operators and their corresponding duals over integration. The next few subsections will briefly discuss and illustrate these operations on a 1-simplex σ in Euclidean $\mathbb{R}^2$.

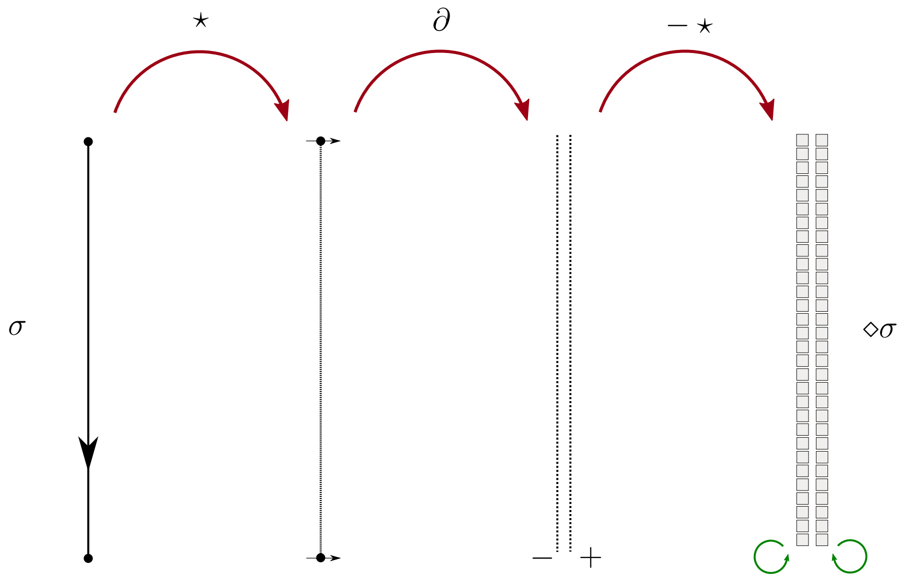


Figure 2.4: The co-boundary acts as $\diamond\sigma = -\star\partial\star\sigma$ for a 1-simplex in Euclidean $\mathbb{R}^2$. The vanishing distance between the transverse "edges" is exaggerated for clarity.

2.6.1 Codifferential and the geometric co-boundary

The codifferential can be defined in the usual way: $\delta = (-1)^{n(k-1)+1}s \star d\star$ as mentioned in the introduction. It takes a form $\omega \in \mathcal{P}_r\Lambda^k$ to the space $\mathcal{P}_{r-1}\Lambda^{k-1}$, and as usual, $\delta^2 = 0$. Following [Har06b], the geometric co-boundary operator is given by $\diamond = (-1)^{n(k-1)+1}s \star \partial\star$, which takes k-chains to $(k+1)$-chains with $\diamond^2 = 0$. Its effect on a 1-simplex σ in Euclidean $\mathbb{R}^2$ is shown in the figure above.

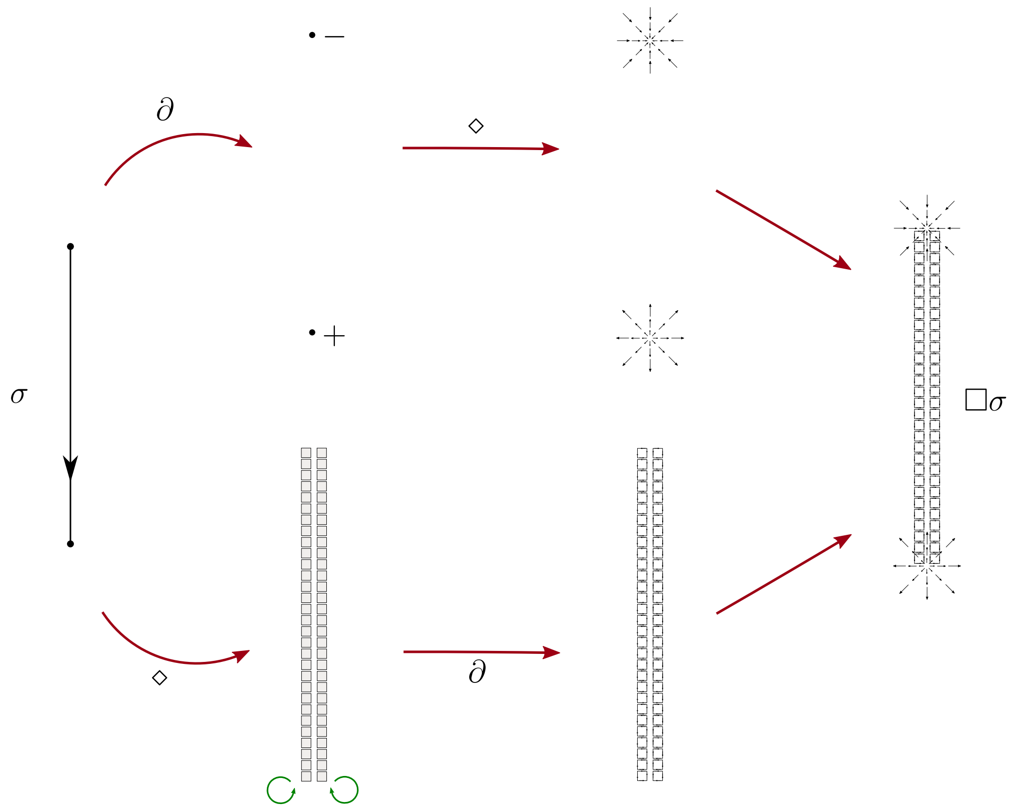


Figure 2.5: The geometric Laplacian $\square$ operates as shown on a 1-simplex in Euclidean $\mathbb{R}^2$. Note the hedgehog-likesink and source at each of the former boundary points, and the rotating edges where σ originally stood.

2.6.2 Hodge Laplacian and the geometric Laplacian

The Laplacian $\Delta = d\delta + \delta d$ as in the usual theory of differential forms. $\Delta : \mathcal{P}_r\Lambda^k \to \mathcal{P}_{r-2}\Lambda^k$, as expected from a second-order differential operator. This completes the extension of the original de Rham complex to FEEC.

The dual over integration to Δ is the geometric Laplacian, which is denoted by $\square = \partial \diamond + \diamond \partial$ in Harrison's work; this symbol should not be confused for the d'Alembertian or wave operator. It transforms a k-chain into a complementary k-chain. The outcome of $\square\sigma$ is depicted above. Interestingly, this can also be used to define a geometric dual to the Dirac operator, a useful notion for lattice gauge theories, but that is beyond the scope of this work.

2.6.3 Co-Koszul and co-cone

The co-Koszul operator, $\xi_v = (-1)^{n(k-1)+1} s \star \kappa_v \star$, takes forms in $\mathcal{P}_r\Lambda^k$ to $\mathcal{P}_{r+1}\Lambda^{k+1}$, with again $\xi_v^2 = 0$. The co-Koszul and codifferential form an equivalent de Rham complex over FEEC with analogous mappings through the Hodge star. A similar dual operator can be defined for the homotopy operation, but it does not provide additional insight beyond the information in ξ_v.

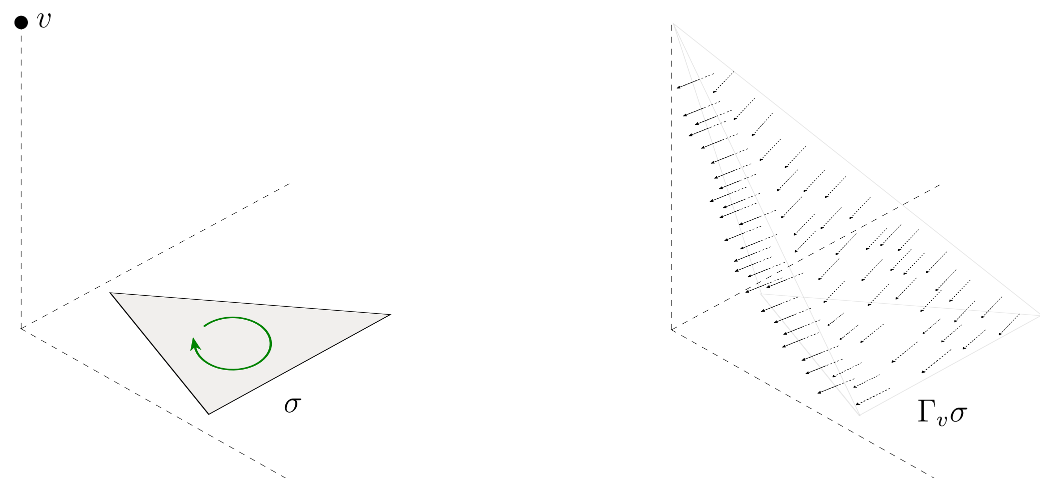


Figure 2.6: Γ_v, the co-cone, operates on a 2-simplex in Euclidean $\mathbb{R}^3$ as shown. The transformed simplex $\Gamma_v\sigma$ is the collection of oriented 1-simplices on the right. $C_v\sigma$ is depicted by the faded gray lines to provide perspective.

However, the co-cone $\Gamma_v = (-1)^{n(k-1)+1} s \star C_v \star$ is more interesting and also difficult to visualize. It reduces the dimension of an k-chain to a $(k-1)$-chain; additionally, $\Gamma_v^2 = 0$, just as with the cone operation. The figure above depicts its action on a 2-simplex in Euclidean $\mathbb{R}^3$ as it is simpler to represent than its effects in $\mathbb{R}^2$. Compare the shape given below to the usual action of C_v on this simplex[5].

Interestingly, the co-cone in this space acts on σ in much the same way the Koszul acts on n-forms to create Whitney forms: the chain of edges 'flows' in the sense of a rotation from one face of the cone to the other, along the orientation of the

[5]We will not consider the dual operator proportional to $\star C'_v \star$ in this chapter, as its action and existence depend entirely on the details of the operation, a discussion deferred until Chapter 3.

original simplex with v as the pivoting point. In Lorentzian $\mathbb{R}^3$, this rotation could instead appear as a shearing motion, depending on the extent to which σ is oriented along the time axis; thus $\Gamma_v\sigma$ can be seen as analogous to the operator $(\kappa_v\cdot)^\sharp$ on a form.

2.6.4 Other operators: odds and ends

The co-interior product, which would be proportional to $\star i_X\star$ is not typically considered in the continuous setting as its own distinct operation. However, it does show up in physical settings when considering relativistic systems, where identities of the form

$$i_X \star \omega = \star(\omega \wedge X^\flat)$$

abound. Given the simplicity of the above expression in terms of previously defined operations, it should be clear why this is never considered as independent operator. In the context of FEEC, this operation would unsurprisingly act much like the co-Koszul operation, bumping up both the dimension of the form and its polynomial degree. Similarly, the co-extrusion operation of the form $\star E_X(M,t)\star$ does not yield much additional insight; the vector field X only properly extrudes the space if it flows in a non-parallel direction to the infinitesimal slices of geometric dual of the given space.

The co-Lie derivative, proportional to $\star\mathcal{L}_X\star$, and the corresponding co-flow are also uninteresting in a flat setting, as they are identical to their original operations up to a sign depending on the signature of the metric. Conceptually speaking, this makes sense, as the Lie derivative merely computes the derivative of a form along the flow of X; the Hodge star simply dualizes the input forms, so the dual of the resulting differential form cannot fundamentally change its character.

Last but not least, the Koszul analogue of the Laplacian, given by $\kappa_v\xi_v + \xi_v\kappa_v$, takes forms from $\mathcal{P}_r\Lambda^k$ to $\mathcal{P}_{r+2}\Lambda^k$. It is not particularly interesting, but its geometric dual has some interesting effects on simplicial meshes. In particular, for the situation depicted in Figure 2.6, the geometric Koszul-Laplacian turns the 2-simplex into a mesh of orthogonal planes filling the interior of the cone $C_v\sigma$. This cubic honeycomb-like structure follows the flow depicted for Γ_v, as well.

2.6.5 Hodge dual of FEEC Spaces

Let us now turn to applying this notion of the Hodge star and its geometric dual to the framework of FEEC. It is clear that by extension from the continuous case and Harrison's construction,

$$\star\,\mathcal{H}_r\Lambda^k = \mathcal{H}_r\Lambda^{n-k} \implies \star\mathcal{P}_r\Lambda^k = \mathcal{P}_r\Lambda^{n-k}. \tag{2.23}$$

Plugging in this identity into the decompositions of section 2.2 leads to a number of interesting formulae. To start, it is clear that the dual of equation (2.4) gives:

$$\mathcal{P}_r\Lambda^k = \delta\mathcal{P}_{r+1}\Lambda^{k+1} \oplus \xi\mathcal{P}_{r-1}\Lambda^{k-1}. \tag{2.24}$$

Furthermore, we find the following relations for the spaces $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r^+\Lambda^k$:

$$\star\,\mathcal{P}_r^-\Lambda^k = \mathcal{P}_{r-1}\Lambda^{n-k} \oplus \xi\mathcal{H}_{r-1}\Lambda^{n-k-1}, \tag{2.25}$$

$$\star\,\mathcal{P}_r^+\Lambda^k = \delta\mathcal{H}_{r+1}\Lambda^{n-k+1}. \tag{2.26}$$

When the Hodge star is specifically applied to the Whitney forms $W^k = \mathcal{P}_1^-\Lambda^k$, we find two interesting properties: 1) the Hodge star of a Whitney form is itself not a Whitney form, and 2) the Hodge star of a Whitney form naturally lives in the Hodge dual of its original simplex, which never coincides with anything in the original simplicial complex. We can also generically state that the codifferential of a Whitney form always vanishes, i.e. the forms in W^k are co-closed. The co-closure of Whitney forms is one of the keys necessary to determining the geometric nature of the $\mathcal{P}_r^-\Lambda^k$ spaces and the corresponding $\mathcal{P}_r^+\Lambda^k$ spaces. The proof of this statement is provided in the subsequent section.

2.7 Metric Reformulation of Whitney Forms

The above pieces let us re-formulate Whitney forms in terms of a metric space. This exercise might seem futile since the effects of these components must conspire to cancel such that the forms remain free of coordinates (as they were originally formulated by Whitney himself). However, there are physical situations in which metrics do affect the nature of interactions. A prominent example is that of matter and sources in electromagnetism; the Hodge star can change its action on the electric displacement and magnetic intensity fields depending on the presence and properties of materials in the problem. Overall then, it is important to know the delineation of metric dependencies of the Whitney form components, at least from a theoretical standpoint.

2.7.1 Whitney Forms: the Covector Approach

The basic properties of Whitney forms lead to a natural and geometrically intuitive construction in terms of the coordinate vertex vectors of a given simplex. We will first showcase this covector representation, then turn to a discussion of its derivation.

Theorem 1 (Metric Decomposition of Whitney Forms in a Metric Space)**.** *Let* $\sigma := [v_0, v_1, ...v_n]$*, an ordered set of vertex vectors, represent an oriented* n*-simplex on a flat* n*-dimensional manifold, and correspondingly, let* $\rho := [v_i, v_{i+1}...v_{i+j}]$ *represent a subsimplex of* σ*, with* $i + j \leq n$*. Taking* $\tau = \sigma \setminus \rho = [v_0, ...v_{i-1}, v_{i+j+1}, ...v_n]$*, the ordered complement of* ρ *in* σ*, the Whitney* j*-form over* ρ *can be written as*

$$^{j}w_{\rho}(x) = \frac{sgn(\rho \cup \tau)}{\star vol(\sigma)} \frac{j!}{n!} \Big(\star \bigwedge_{v_k \in \tau} (v_k - x)^{\flat} \Big), \tag{2.27}$$

where $vol(\sigma)$ *is the volume form of* σ*, defined by* $vol(\sigma) = \frac{1}{n!} \bigwedge_{i=1}^{n} (v_i - v_0)^{\flat}$*,* $sgn(\rho \cup \tau)$ *is the sign of the permutation of the ordered vertex set* $\rho \cup \tau$ *relative to* σ*, and* x *is the position vector. The terms in the wedge product are ordered as in* τ*.*

Note the interesting structure: the Whitney form, a metric-independent object, depends on the *Hodge dual of the form representing its complementary simplex*, up to metric-dependent details. Thus, we can see that the metric dependence of the volume form cancels with the action of the Hodge star over the form on the complementary subsimplex. In addition, $^{j}w_{\rho}(x) = 0, \forall x \in \tau$. The validity of this formula can be proven through the use of a vector proxy representation of the Whitney forms; a self-contained proof by induction is presented in Appendix A.

2.7.2 Consequences

We will now explore a few consequences of (2.27). Expanding the wedge product in the formula, we obtain:

$$ {}^{j}w_{\rho} = \frac{sgn(\rho \cup \tau)}{\star vol(\sigma)} \frac{j!}{n!} \star \Big(\bigwedge_{v_k \in \tau} {v_k}^{\flat} - x^{\flat} \wedge \sum_{v_k \in \tau} (-1)^{\alpha_k} \bigwedge_{\substack{v_l \in \tau \\ l \neq k}} {v_l}^{\flat} \Big), \tag{2.28} $$

where α_k is the number of transpositions required to bring $x^{\flat}$ to the front of the wedge product. If we take $\rho = [v_0, v_1, ...v_j]$ as a standard vertex ordering for the subsimplex of interest, the above expression expands to:

$$ \frac{1}{\star vol(\sigma)} \frac{j!}{n!} \star \Big(v_{j+1}^{\flat} \wedge ... \wedge v_n^{\flat} - x^{\flat} \wedge \sum_{k=j+1}^{n} (-1)^{k-j-1} v_{j+1}^{\flat} \wedge v_{j+2}^{\flat} \wedge ... \hat{v}_k^{\flat} \wedge ... v_n^{\flat} \Big), \tag{2.29} $$

where, as usual, the hat indicates that the term is omitted. With respect to some origin, the first term is akin to the volume form of the complementary simplex τ, and the second term represents the sum of volume forms enclosed by the position 1-form $x^{\flat}$ and each of τ's subsimplices. Whitney forms can thus be thought of as the Hodge dual to the difference between these two volume forms, up to a scaling factor. This is easier to picture if one realizes that the product $\bigwedge_{v_k \in \tau} (v_k - x)^{\flat}$ is the $(n-j)$-volume form of the simplex formed by τ's vertices with the position x as the origin.

As mentioned previously, this representation leads us to an interesting conclusion: the Hodge dual of a Whitney form is not a Whitney form on a simplex. Indeed, the dual Whitney forms only form an orthonormal basis on the space of the dual simplex $\star\sigma$ and its dual subsimplices:

$$ \int_{\rho} {}^{j}w_{\rho} = \int_{\star\rho} \star({}^{j}w_{\rho}) = 1, $$

where the dual simplex is defined in terms of the geometric Hodge dual introduced in Harrison's work [Har06b], as explored in the section before. This implies that the Hodge dual Whitney forms also form a basis, but only on the geometric Hodge dual of the original simplicial complex. Oddly enough, the formula for the Hodge dual Whitney form is slightly easier to interpret in this representation:

$$ \star\, {}^{j}w_{\rho} = (\star\star) \frac{sgn(\rho \cup \tau)}{\star vol(\sigma)} \frac{j!}{n!} \Big(\bigwedge_{v_m \in \tau} (v_m - x)^{\flat} \Big), \tag{2.30} $$

since $\star\star$ is the identity map, up to a sign that depends on the index of the metric, the dimension of the manifold, and the degree of the differential form.

We can thus picture the Hodge dual Whitney forms to be a difference of the complement's volume form from the sum of the complement's subsimplicial volume forms. Indeed, this formula is also more amenable to algebraic manipulation due to the Hodge star's partial cancellation on the RHS of the equation. For example, taking the exterior derivative of a dual Whitney form yields:

$$d(\star^j w_\rho) = 0,$$

as $d(x^\flat) = 0$. This implies that the $\delta^j w_\rho = 0$ as well, where δ is the codifferential. In terms of the breakdown of W^k given by (2.8), this means that:

$$\delta W^k = \delta\kappa\mathcal{H}_0\Lambda^{k+1} = 0 \tag{2.31}$$

as all of the elements of $\mathcal{H}_0\Lambda^{k+1}$ are co-closed. Intuitively, this comes back to the fact that the Koszul operator is a contraction with the "radial" vector with respect to a given origin, along with the vanishing of the exterior derivative of said vector.

2.8 Acknowledgements

Chapter 2, in part, contains material from the following arXiv preprint: Joe Salamon, John Moody, and Melvin Leok. "Geometric Representations of Whitney Forms and their Generalization to Minkowski Spacetime", arXiv preprint 1402.7109, 2014. The dissertation/thesis author was the primary investigator and author of this paper.

Chapter 3

Higher Order Finite Element Exterior Calculus: Rapetti Construction and Beyond

"In the space between chaos and shape there was another chance."
— Jeanette Winterson

3.1 Introduction

In the previous chapter, the interpretation of how the polynomial order of the differential forms in FEEC relates to a natural basis for such forms was largely ignored. From both an aesthetic and theoretical viewpoint, however, a complete understanding of this aspect of FEEC is somewhat lacking. In the original FEEC papers by Arnold, Falk, and Winther ([AFW10], [AFW06]), the basis dual to higher order forms is left mostly untouched; not only is there ambiguity in the construction of such a space, but the main procedure presented in [AFW06] provides a method of pairing k-simplices to high-order l-elements. Needless to say, the interpretation of such a method is unclear, and furthermore, conflicts with the natural pairing of k-forms over k-dimensional space upon integration.

This chapter will delve into the details of such a construction and provide insights into the structure of both higher order Whitney forms and the non-Whitney components of the $\mathcal{P}_r\Lambda^k$ spaces in FEEC. After all, the geometric interpretation of the space of Whitney k-forms W^k as a basis over each k-simplex is very compelling; why should the other spaces not have similarly interesting connections? I will demonstrate that this question ties into the nature of affine spaces and the combinatorics of simplicial data, then delineate how a construction championed by Francesca Rapetti in a variety of papers ([Rap07], [RB09], [CR13]) attempts to tie these notions together.

3.2 Dual spaces of $\mathcal{P}_r^{\pm}\Lambda^k$: hints and connections

In this section, I will show how the relation between higher-order Whitney forms $\mathcal{P}_r^-\Lambda^k$ and the gap space $\mathcal{P}_r^+\Lambda^k$ ties into the underlying simplicial complex, with the hope that this line of reasoning will elucidate the rest of the geometric structure of FEEC.

Let us first consider $\mathcal{P}_r^+\Lambda^k$, as its definition from equation (2.7) is quite simple:

$$\mathcal{P}_r^+\Lambda^k = d\mathcal{H}_{r+1}\Lambda^{k-1}.$$

Let ${}^k w^+ \in \mathcal{P}_r^+\Lambda^k$ such that ${}^k w^+ = dh$, where $h \in \mathcal{H}_{r+1}\Lambda^{k-1}$. The above breakdown tells us that the forms in this space are exact and thus closed; therefore, their integration over a boundary should vanish via Stokes' theorem:

$$\int_{\partial M} {}^k w^+ = \int_M d^k w^+ = 0.$$

In addition, their integration over a k-simplex should be equivalent to the integration of the corresponding homogeneous form over the boundary of that simplex:

$$\int_M {}^k w^+ = \int_M dh = \int_{\partial M} h.$$

This last constraint in particular demonstrates that there should be a connection between the polynomial order of the form and the dual space over which it is being integrated. Consequently, this geometric interpretation of polynomial order must be consistent across the whole space of polynomial differential forms and their dual spaces in FEEC.

The space of higher-order Whitney forms $\mathcal{P}_r^-\Lambda^k$ has a slightly more complicated definition as given by equation (2.6):

$$\mathcal{P}_r^-\Lambda^k = \mathcal{P}_{r-1}\Lambda^k \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1}.$$

When $r = 1$, this definition yields the space of Whitney k-forms which, as previously mentioned, are dual over integration to each k-simplex in the simplicial complex. The space of higher order Whitney forms should likewise have a connection to each simplex, which must include some additional structure.

For example, consider the standard simplex in Euclidean $\mathbb{R}^2$, and consider the form $\omega = \lambda_0 w_{12} \in \mathcal{P}_2^-\Lambda^1$ where λ_0 is the barycentric coordinate for v_0 and w_{12} is the Whitney 1-form corresponding to the edge $[v_1, v_2]$. The geometric nature of this form is difficult to see as

$$\int_{[v_1,v_2]} \lambda_0 w_{12} = \int_{[v_2,v_0]} \lambda_0 w_{12} = \int_{[v_0,v_1]} \lambda_0 w_{12} = 0$$

since, by definition, λ_0 vanishes on $[v_1, v_2]$, and w_{12} integrates over the other two edges. Thus, higher-order forms do not generically 'live' on the original simplex.

For forms of higher degree, this constraint is given by the closure of the $\mathcal{P}_r^-$ spaces under the wedge product, as mentioned in equation (2.3):

$$\mathcal{P}_r^-\Lambda^k \wedge \mathcal{P}_s^-\Lambda^j = \mathcal{P}_{r+s}^-\Lambda^{j+k}.$$

This innocuous equation harks back to an interesting property of Whitney form spaces: the wedge product of any two Whitney forms with a common vertex yields a higher-rank Whitney form scaled by the Whitney 0-form of that vertex (i.e. its barycentric coordinate function). In other words,

$$W^k \wedge W^l = W^0 \cdot W^{k+l}, \tag{3.1}$$

which implies that we can bootstrap to higher order forms just with the space of Whitney forms; the space $\mathcal{P}_r^-\Lambda^k$ can be decomposed as the product $(W^0)^{r-1} \cdot W^k$. Overall, this hints that there should be some deeper significance to the wedge product of these forms, and in particular, this significance should tie into the structure of the simplicial complex that the Whitney forms represent along with the order of the polynomial spaces.

Now, instead, assume we are in $\mathbb{R}^n$ with $n \geq 3$ and

$$w_{ij} \wedge w_{jkl} = \lambda_j w_{ijkl},$$

where j represents the vertex in common. Integrating both sides over a 3-simplex σ,

$$\int_\sigma w_{ij} \wedge w_{jkl} = \int_\sigma \lambda_j w_{ijkl}.$$

If σ does not contain the vertex dual to λ_j or does not overlap with the simplex dual to w_{ijkl}, then the integral on the right-hand side automatically vanishes. This immediately suggests an appealing notion of a wedge product over simplices, however, the notion of polynomial order does not factor into the simplest version of this idea.

Furthermore, the two spaces $\mathcal{P}_r^{\pm}\Lambda^k$ are quite different in terms of their relation to simplicial data: the basic forms in $\mathcal{P}_1^-\Lambda^k$ are completely antisymmetric with respect to the vertices they are formed from, whereas those in $\mathcal{P}_1^+\Lambda^k$ contain the remaining pieces of total and mixed symmetry. This is most simply illustrated again in Euclidean $\mathbb{R}^2$: compare the Whitney 1-form $w_{12} = \lambda_1 d\lambda_2 - \lambda_2 d\lambda_1$ to $s_{12} = d(\lambda_1\lambda_2) = \lambda_1 d\lambda_2 + \lambda_2 d\lambda_1$. A consistent interpretation of polynomial order must account for these differences in a fashion consistent with the standard operations in FEEC.

3.3 Rapetti Construction: an outline

The Rapetti construction is an attempt to provide a geometric interpretation to the spaces of higher-order Whitney forms $\mathcal{P}_r^-\Lambda^k$. Rapetti has developed this notion since 2007 [Rap07], and has worked with Bossavit and Christiansen in expanding the scope of these ideas ([RB09], [CR13]). As discussed in [RB09], this formalism sets out to achieve three goals in the interpretation of higher-order Whitney forms: 1) a sense of "partition of unity" for such forms, 2) pair each such form with an integration domain of the appropriate dimension, and 3) preserve the exact sequence property that is critical in the construction of FEEC. Let us first turn briefly to the notation of the framework before examining its details.

Consider a simplex σ living in $\mathbb{R}^n$. Let $\mathbf{d}$ represent a multi-index $\{d_0, d_1, \ldots, d_n\}$ and $\lambda^{\mathbf{d}}$ stand for a product of $|\mathbf{d}| = \sum_{i=0}^n d_i$ factors of barycentric coordinates over

σ. In other words, $\lambda^{\mathbf{d}}$ is a polynomial of order $|\mathbf{d}|$ given by:

$$\lambda^{\mathbf{d}} = \prod_{i=0}^{n} \lambda_i^{d_i}. \tag{3.2}$$

Take the associated map $\bar{\mathbf{d}}$ to represent the following affine transformation on the set of barycentric coordinates of the simplex:

$$\bar{\mathbf{d}} : \{\lambda_i\} \to \left\{ \frac{\lambda_i + d_i}{|\mathbf{d}| + 1} \right\}. \tag{3.3}$$

Lastly, we will need to define the notion of small k-simplices: a small k-simplex is a scaled-down copy of the original k-simplex in σ. The set of small k-simplices lives within the original volume enclosed by σ, and the set of small n-simplices covers most of the volume of σ, in a sense to be made precise below.

Now we can combine these notational components to paint the big picture. As mentioned previously, the space $\mathcal{P}_r^- \Lambda^k(\mathbb{R}^n)$ can be decomposed into a product of $(r-1)$ copies of barycentric coordinates $W^0 = \mathcal{P}_1^- \Lambda^0$ and $W^k = \mathcal{P}_1^- \Lambda^k$, thanks to the closure of the space of Whitney forms under the wedge product.

Take $\omega \in \mathcal{P}_r^- \Lambda^k$ and decompose it into such a product $\omega = \lambda^{\mathbf{d}} \cdot {}^k w_\rho$, where $|\mathbf{d}| = r - 1$. Let the small n-simplex be a scaled-down copy of σ with barycentric coordinates given by the map $\bar{\mathbf{d}}$. Then, ω can be paired with the corresponding scaled down copy of ρ on this small simplex. The transformation defined by $\bar{\mathbf{d}}$ is a homothety: a scaling down of the original space with respect to an origin. We can then think of $\lambda^{\mathbf{d}}$ as a representation of where to find the appropriate degree of freedom. In fact, these homotheties can be seen as geometric representations of the wedge products of Whitney forms. Thus, a higher order form is associated with a smaller version of the original simplex with which it is usually paired. This association of higher polynomial order with the localization of degrees of freedom provides a natural relation between subdivision and order refinement, an idea reminiscent of wavelets and shearlets.

The figure on the next page demonstrates how this transformation works in $\mathbb{R}^2$. The depicted transformation shows the effect of moving from $\mathcal{P}_1^-$ to $\mathcal{P}_2^-$; in going to quadratic order, each vertex is now associated to a quadratic polynomial of the barycentric coordinates, and each of the small 2-simplices moves up to linear order.

Since these small simplices are scaled-down images of the original simplex, the edges transform similarly: they are associated with differential forms of the form $\lambda_i w_{jk}$. In $\mathbb{R}^2$, the holes are inverted images of the original simplex, but this is not true in $\mathbb{R}^n$.

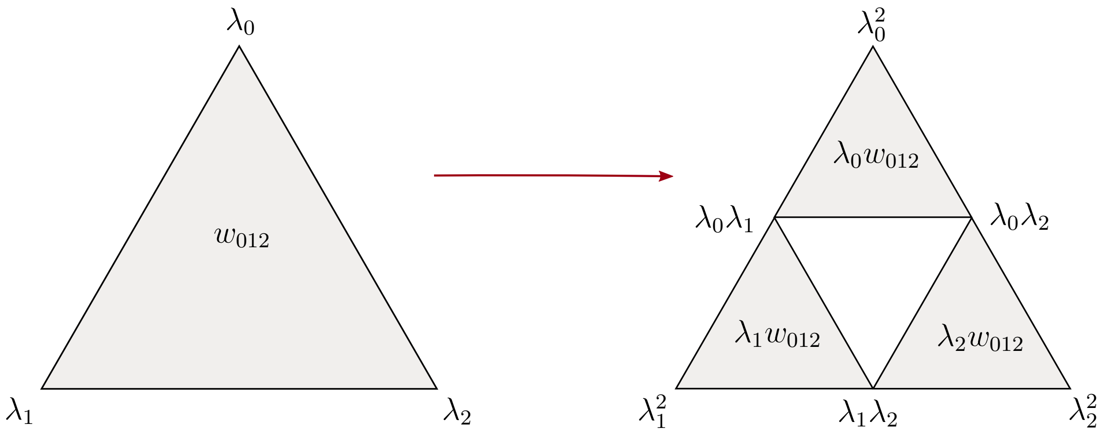


Figure 3.1: The Rapetti construction, as applied to a 2-simplex in $\mathbb{R}^2$. The Whitney 1-forms are not shown to reduce clutter. Note the hole formed in the center of the collection of small 2-simplices on the right.

Geometrically then, integrals over higher-order Whitney forms measure how much the associated small simplices overlap within the domain of integration. The construction thus provides a concise visualization of these integrations as nested volume integrals over the appropriate simplices.

3.4 Consequences of the Rapetti construction

3.4.1 Holes and dimension counting

The dimension of the Rapetti space $\mathcal{R}_r^k(\mathbb{R}^n) = (W^0)^{r-1} \cdot W^k(\mathbb{R}^n)$ should match up exactly with that of $\mathcal{P}_r^- \Lambda^k$. However, a naïve counting of distinct elements in $\mathcal{R}_r^k$ yields

$$\dim \mathcal{R}_r^k \leq (n+1)^{(r-1)} \cdot \binom{n+1}{k+1}, \tag{3.4}$$

whereas the dimension of $\mathcal{P}_r^-\Lambda^k$ is given by equation (2.11):

$$\dim \mathcal{P}_r^-\Lambda^k(\mathbb{R}^n) = \binom{n+r}{n-k}\binom{r+k-1}{k} = \frac{r}{r+k}\binom{n+r}{n}\binom{n}{k}.$$

The excess elements counted in $\mathcal{R}_r^k$ can be accounted for by the partition of unity satsified by the barycentric coordinates

$$\sum_{i=0}^{n} \lambda_i = 1 \implies \sum_{i=0}^{n} d\lambda_i = 0,$$

which reduces the $(r-1)$ copies of W^0 to the space $\mathcal{P}_{r-1}\Lambda^0$. Furthermore, the remaining elements can be reduced by accounting for the commutation of polynomial terms when multiplied by the corresponding element of W^k.

Nonetheless, the naïve count given by equation (3.4) has important consequences. Homotheties associated with the set of polynomials $\lambda^{\mathbf{d}}$ generically introduces holes of different sizes and shapes in between the small simplices. Since these scalings relate a polynomial with a geometric transformation, the commutativity of scalar multiplication creates a degeneracy in the small simplices; hence, the number of simplices bounding these holes relates exactly to the excess in degrees of freedom from the difference between equations (3.4) and (2.11).

As shown in [RB09], the relative cohomology of the complex of the small simplices is identical to that of the original simplex, up to the hole boundaries. Although this might seem a little surprising, the holes essentially do not contribute anything to the construction topologically; furthermore, in the limit of large r, the number of small simplices outweighs the amount of holes, so the continuum limit properly covers the original simplex.

3.4.2 Frames versus bases

The main trade-off with the Rapetti construction is that the space of higher-order Whitney forms does not form a basis with respect to the small simplices over integration. However, the elements of $\mathcal{R}_r^k$ still form a frame, as the presumed “basis” elements are in fact linearly dependent. As described in [RB09], this implies that many of the usual computations over finite elements would need to make use of the pseudoinverse in this framework.

In practice, the notion of a frame is useful, as it provides robustness to certain algorithms. For example, in the realm of signal processing, a frame allows certain signals to be processed more cleanly in terms of frame elements that are easier to compute. In this context, however, the utility of a frame is somewhat unclear; perhaps higher-order polynomial equations of motion become simpler to solve in certain frames. This trade-off could be useful when viewed through the lens of shearlets, allowing the encoding of such localizations to be more robust as well. In Chapter 5, we will see that gauge degrees of freedom for field theories are interlinked with localization in the Rapetti construction; the efficiency of a frame could be exploited when computing complicated gauge conditions throughout a simplicial complex.

3.4.3 Ambiguities with non-Whitney forms in $\mathcal{P}_r^+\Lambda^k$

The Rapetti construction gives a robust lens with which to view the higher order Whitney forms, which furnish the building blocks for FEEC. However, care must be taken when applying Rapetti's interpretation to the elements of $\mathcal{P}_r^+\Lambda^k$, which comprise the rest of the space of polynomial differential forms.

Let us analyze a simple example. Take the standard simplex σ in Euclidean $\mathbb{R}^2$. Consider the form $s_{12} = d(\lambda_1\lambda_2) = \lambda_1 d\lambda_2 + \lambda_2 d\lambda_1$. According to the Rapetti construction, the natural vertex to associate with $\lambda_1\lambda_2$ is the midpoint of the edge connecting v_1 and v_2. There are two possible routes we can now take to formulate a dual space for s_{12}; ideally, these routes will yield compatible answers.

First route. Denoting σ_0 as this midpoint, the 'integral' of $\lambda_1\lambda_2$ evaluates to:

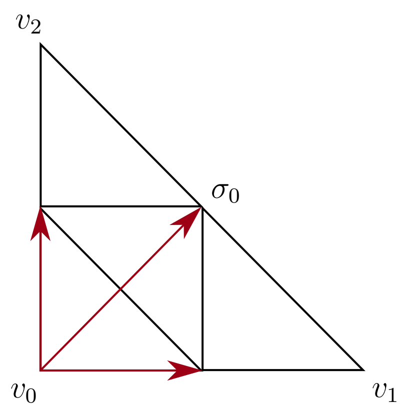
$$\int_{\sigma_0} \lambda_1\lambda_2 = \lambda_1\lambda_2|_{\sigma_0} = \frac{1}{4}.$$

Now, since $\mathbf{H}_{v_0} s_{12} = \lambda_1\lambda_2$, we have the following relation

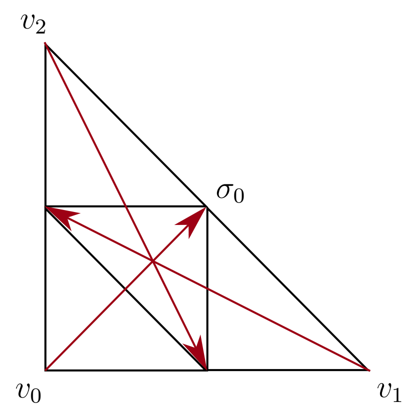
$$\int_{\sigma_0} \lambda_1\lambda_2 = \int_{\sigma_0} \mathbf{H}_{\mathbf{v_0}} s_{12} = \int_{C_{v_0}\sigma_0} s_{12} = \frac{1}{4},$$

where $C_{v_0}\sigma_0$ is the 1-chain which connects the origin v_0 to σ_0. Integrating s_{12} over the cones of the other midpoints yields a value of 0. Alternating the vertices to integrate the other s_{ij} yields identical results: these chains form a dual basis for $\mathcal{P}_1^+\Lambda^1$. Indeed, this basis can be normalized if we add a weight factor of 4 to these small chains.

The trade-off is that these 1-chains will not live in the simplex nor in any of its homothetic images (e.g. σ_0). Additionally, due to the affine nature of this space, this procedure works with any of the vertices v_i taken as the origin; thus, there are many other bases for these forms. In fact, if we fix the homotopy operator to only act from the natural origin v_0, then we find that two of the three chains in the basis line up identically with the small simplices defined by the Rapetti construction. The possible types of bases are compared in Figure 3.2 below.



(a) The first basis for s_{ij} considered. The origin is marked as a special point of reference.



(b) A second possibility for an s_{ij} basis, which is agnostic to the choice of origin.

Figure 3.2: Two sets of s_{ij} bases that are natural when using the first route.

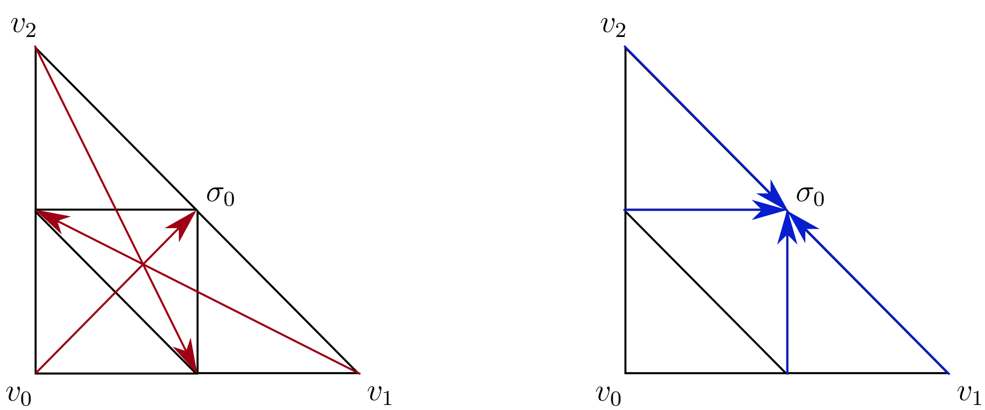
Second route. If we want to find the natural chain to associate with s_{12}, we can instead integrate it over an arbitrary 1-chain C. Since s_{12} is an exact form, this implies that

$$\int_C s_{12} = \int_C d(\lambda_1\lambda_2) = \int_{\partial C} \lambda_1\lambda_2 \ .$$

As discussed in the previous route, the natural element to associate with the quadratic 0-form $\lambda_1\lambda_2$ would be σ_0, the midpoint of the edge $[v_1, v_2]$. ∂C should in principle contain this midpoint. It is impossible, however, for ∂C to contain a single point; nonetheless, we have the freedom to choose a boundary that contains any points for which λ_1 or λ_2 vanish.

Here is where the ambiguity arises: what points do we consider when attempting to build the set of chains C? In principle, if we are to keep consistent with the Rapetti construction, then the only points allowed would be the vertices of the original simplex v_i and the midpoints of the edges. The natural candidate would be v_0, as $\lambda_1 = \lambda_2 = 0$ when evaluated at the origin, which would yield a basis consistent with the first route. However, any set of points along $\lambda_1 = 0$ or $\lambda_2 = 0$ would work as well. The rest of the basis can be built in the same way with respect to each λ_i. As shown in Figure 3.3(a), this is consistent with the possibility shown in Figure 3.2(b).

On the other hand, exact forms $d\lambda_i$ can also be written as a difference of Whitney 1-forms, e.g. $d\lambda_1 = w_{01} - w_{12}$. This means that $s_{12} = \lambda_1(w_{02} - w_{21}) + \lambda_2(w_{01} - w_{12})$. Keeping in line with the Rapetti interpretation here would suggest that the natural chain C would consist of 4 small elements: 2 small simplices connecting $(\frac{1}{2}, \frac{1}{2})$ to the other 2 midpoints and 2 small simplices connecting $(\frac{1}{2}, \frac{1}{2})$ to the vertices v_1 and v_2. This is shown in the figure below. As in the previous route, the appropriate weight factor must be determined by agreement with the evaluation of $\lambda_1\lambda_2$. Furthermore, applying this procedure to the rest of the s_{ij} does not yield a basis for these forms, but still provides a frame, which is consistent with the rest of the Rapetti construction. This chain is depicted in Figure 3.3b; the equivalent chains to the other s_{ij} are not shown for clarity.



(a) This basis for $\{s_{ij}\}$ is in line with the first route.

(b) The blue chain is another potential dual to s_{12}.

Figure 3.3: Two possibilities for s_{ij} bases when using the second route.

Both routes are equally valid in principle and are consistent with the spirit of the Rapetti construction; furthermore, they can be made to agree with each other. However, the decomposition of s_{12} into constituent higher order Whitney forms provides a theoretically compelling application of the Rapetti interpretation, with a frame that is conceptually in line with what is originally presented. Given the affine nature of the spaces in FEEC, however, there is no unambiguously correct answer.

3.5 Beyond the Rapetti construction

Although the Rapetti construction leaves much to be desired, it is entirely possible that it is the cleanest possible interpretation of the localization of higher order forms in FEEC. Nonetheless, there are a few hints of further structure within the spaces of polynomial differential forms that come out in the context of the extensions to FEEC outlined in Chapter 2. This section will discuss some of these interesting avenues accompanied by some speculation for future directions. For the rest of this section, any explicit calculations will refer to the standard simplex in Euclidean $\mathbb{R}^2$; that being said, the formulae involved can be easily extended to other affine spaces.

3.5.1 Interior product and extrusion

The interior product i_X can provide an alternate means to explore whether spaces of polynomial differential forms create a suitable basis over a specific space. Restricting the set of vector fields X to only consist of the edge vectors formed by the simplex in question yields especially enlightening results about the structure of FEEC.

For example, taking v_{ij} to be the vector corresponding to the 1-simplex $[v_i, v_j]$, consider the following interior products on Whitney 1-forms w_{ij}:

$$
\begin{aligned}
i_{v_{ij}} w_{ji} &= \lambda_i + \lambda_j, \\
i_{v_{ij}} w_{jk} &= -\lambda_j, \\
i_{v_{ij}} w_{kl} &= 0.
\end{aligned}
$$

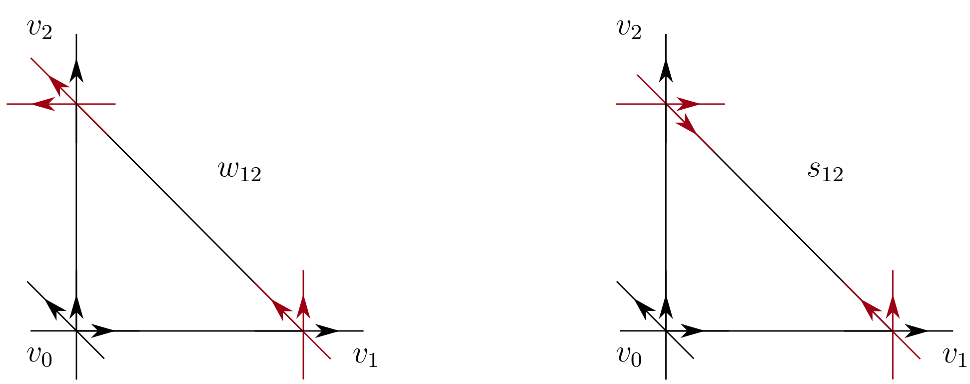
In particular, the first and third equations hint at some notion of a basis; the first expression evaluates to 1 along the entirety of the simplex $[v_i, v_j]$. Let s_{ij} represent the forms in $\mathcal{P}_1^+\Lambda^1$ as before, i.e. $s_{ij} = d\lambda_i\lambda_j = \lambda_i d\lambda_j + \lambda_j d\lambda_i$, then:

$$\begin{aligned} i_{v_{ij}} s_{ji} &= \lambda_i - \lambda_j, \\ i_{v_{ij}} s_{jk} &= \lambda_k, \\ i_{v_{ij}} s_{kl} &= 0. \end{aligned}$$

For these forms, the first expression evaluates to 1 along a reflected image of the simplex $[v_i, v_j]$. The second expression puts a damper on our hopes for a consistent basis: it implies that the form s_{ij} carries information about the opposite vertex from which it is evaluated[1].

Nonetheless, the interior product and extrusion can be used to determine which spaces provide some notion of the degrees of freedom for specific forms. The duality over integration is not quite exact, however. As shown in Chapter 2, the link between the interior product and extrusion accounts for the velocity or flow of the integral as well; more precisely, as given in equation (2.17),

$$\int_M i_X \omega = \frac{d}{dt}\Big|_{t=0} \int_{E_X(M,t)} \omega \ .$$



(a) Note the 'rotation' implied by the red simplices.

(b) s_{12} implies a 'shearing' motion instead.

Figure 3.4: Extrusions dual to the interior products discussed above. Red arrows depict integrations which equal unity; black denotes a vanishing integrand.

[1]These expressions can be recast in terms of the Lie derivative; this implies that the flow of these quadratic functions by the edge vectors relates to the barycentric coordinates.

In $\mathbb{R}^2$, there are 9 possible extrusions: 3 vectors with any 3 of the vertices as possible points of origin. The figure on the previous page shows these possibilities for both the inner products of the Whitney form w_{12} and the form s_{12} over these possible extrusions. In both cases, 5 of the 9 possibilities vanish upon integration, and the remaining 4 integrate to unity. The directions of the non-vanishing extrusions for w_{12} match up with the direction of rotation with v_0 as the pivot. In particular, two of these line up with 1-simplex $[v_1, v_2]$, the edge associated with w_{12}. The directions of the extrusions of s_{12} align with that of a shear transformation along the line $y = x$. Interestingly, two of these directions point to the midpoint of $[v_1, v_2]$, the small vertex predicted by the Rapetti construction.

3.5.2 Dual to the Koszul

As was discussed in Chapter 2, the appropriate dual to the homotopy operator $\mathbf{H}_v$ is the cone operator C_v, such that $\mathbf{H}_v d + d\mathbf{H}_v$ and $\partial C_v + C_v \partial$ equate the identity operator on forms and chains respectively. The dual to the Koszul operator κ_v was tentatively dubbed C'_v, as it had to act similarly to C_v, but had to account for the order and degree of the form by its action on the simplex. As shown in equation (2.13), the action of $d\kappa + \kappa d$ on a general form $\omega \in \mathcal{P}_r\Lambda^k$ is not proportional to ω; therefore, finding a precise dual to κ_v is not possible.

In fact, the action of κ_v is equivalent to that of the interior product with respect to the radial vector field r from the point v. As was shown in Chapter 2, the dual to the interior product is the velocity of the integral of the extrusion; applying this logic to the Koszul gives

$$\int_M \kappa_v \omega = \frac{d}{dt}\bigg|_{t=0} \int_{E_{r_v}(M,t)} \omega \ . \tag{3.5}$$

Thus, we can associate with the Koszul some notion of velocity of a radial flow.

In conclusion, the supposed dual operator C'_v discussed in Chapter 2 cannot exist over integration, and the intuition discussed in the previous subsection is somewhat vindicated. Moreover, the seeds of the Rapetti construction are visible here: homotheties can be construed as extrusions as well. The 'dynamical' aspect of this dual operator suggests that perhaps the notion of polynomial order localization on a simplex is not a well-defined question within the framework of FEEC, as its entire structure relies on the homotopy-like relation (2.13) between these spaces. Nonetheless, the Rapetti construction provides a useful method for framing certain aspects of discrete equations of motion, as we will see in Chapter 5.

Chapter 4

Noether's Theorems

"Meine Methoden sind Arbeits- und Auffassungsmethoden, und daher anonym überall eingedrungen."

— Emmy Noether

4.1 Noether: A Brief Biography

This section is intended as a quick dive into the history and background of Emmy Noether, as the impact of her academic work is broad and deep, all in spite of the severe discrimination she faced as a German Jewish woman attempting to enter academia in the early twentieth century[1].

Dr. Emmy Noether, née Amalie Emmy Noether, was born in Erlangen, Germany, in 1882 to Max and Ida Amalia Noether. Her father Max was a mathematician at Heidelberg and Erlangen, and Ida Amalia was the daughter of a merchant. She was required to request permission from her professors to take their courses, and learned much under the wings of Gordon, Klein, Hilbert, and Minkowski as both friends and collaborators. She completed her PhD in Mathematics, summa cum laude, in 1907 with Gordon as her advisor. Her main contributions were to the theory of invariants, which led to her celebrated theorems in Physics, abstract algebra via her pioneering work on commutative rings and ideals, and additionally contributed work to the

[1] The biographical details in this chapter have been compiled from a number of sources, chiefly [Neu11] and [KS11].

fields of noncommutative algebra, hypercomplex numbers, and group representation theory.

The discrimination she faced coming into academia has been well-documented by a variety of sources. She was required to obtain permission to enroll from every professor when completing her undergraduate work. After obtaining her doctorate, she was barred from lecturing at Göttingen under her own name as women were not allowed to hold faculty positions, and had to teach under Hilbert's name[2]. In fact, she was not given the right to lecture without salary under her own name until 1919. Her appointment as an honorary faculty member only arrived in 1922, only after much protest by the likes of Hilbert and Einstein. Her first official faculty position started in 1933 at Bryn Mawr College, when she had to escape Germany due to the rise of the Nazi party. She continued to teach both there and at the Institute for Advanced Study in Princeton until 1935, when she passed away from post-surgical complications for a tumor removal.

The remainder of this chapter will focus on clarifying the differences and similarities between Dr. Noether's first and second theorems, and their results as applied to two classical field theories: electromagnetism and general relativity. In particular, I plan to contrast the two theorems in a way that is not usually explored in the standard textbook literature. My hope is that the discussion and equations below will serve as a pedagogical note to the broad conception of "symmetry implies conservation". Furthermore, the equations from these sections consequently affect the discretization of the field theories considered in the next chapter, and provide an interesting method to quantify the amount of error within a given discretization and its simulation by analyzing deviations of conserved quantities.

4.2 The Two Theorems: A Prelude

Noether's two theorems often get confused and misused within the physics literature, as has been discussed in [Bra02]. Our first step will be to state the two theorems precisely, and highlight how their similarities and differences work to give a

[2] When other faculty members protested against this practice, Hilbert replied that "I do not see that the sex of the candidate is an argument against her admission as a [faculty member]. After all, the [faculty] senate is not a bathhouse."

perspective on the role of symmetries within least-action principles. In the proceeding subsections, we will first provide the background established in Noether's paper [Noe18], the statements of Noether's original theorems[3] and provide the context with which the theorems were presented, with translations into modern mathematical and physical parlance as necessary. A discussion of how the two statements interrelate will follow afterwards. The classical Klein-Gordon Lagrangian will be used in this section as a reference example.

4.2.1 Background and Notation of "Invariante Variationsprobleme"

Noether's paper starts off with a brief, generic background on standard functionals, then delves immediately into invariance under transformations. The main functional considered in the paper I represents the integration of a function

$$f\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots\right)$$

over the coordinate space x_i, where the u_j are functions over the coordinates[4]. Physically, I can be thought of as the action S, f as some functional such as the Lagrangian density L, x as representing spacetime coordinates, and the various u_j as stand-ins for a physical field.

The transformations discussed in the paper are also fairly general. In Noether's words, only groups of transformation $\mathfrak{G}$ that act on the coordinates x, the functions u, and their derivatives are considered. The difference between two distinct types of transformations is what leads to the first and second theorems, so those details will be explored in the relevant sections.

The invariant variational problem is then defined shortly thereafter. The specific meaning of invariance in this context is any transformation that leaves the value of I unchanged. The explicit example given in her paper is the following: as above, let us say a certain $\mathfrak{G}$ transformation takes the coordinates x_i into a new set of coordinates y_i, with an accompanying transformation of the functions $u_j(x)$ into

[3] Translated into English by myself, then cross-verified with [Bra02], [KS11], and [Neu11].
[4] Note that the subscripts in the derivatives are suppressed to avoid clunkiness.

$v_j(y)$ and the corresponding derivatives. Explicitly,

$$y_i = A_i\left(x, u, \frac{\partial u}{\partial x}\dots\right),$$

$$v_j = B_j\left(x, u, \frac{\partial u}{\partial x}\dots\right).$$

Then I is invariant if:

$$I = \int \dots \int f\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots\right) dx = \int \dots \int f\left(y, v, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial y^2}, \dots\right) dy,$$

where the dx, dy are shorthand for the appropriate volume element of the space. This captures the notion of a symmetry in this paper, and this usage is still in line with modern physics terminology.

Noether then turns to "Lagrange expressions," in reference to the standard variational problem denoted by the equation $\delta I = 0$[5]. The variation $\delta I = 0$ is written as:

$$\delta I = \int \dots \int \delta f dx = \int \dots \int \left(\sum \psi_i \delta u_i\right) dx = 0. \tag{4.1}$$

Here, δu_i represents the infinitesimal change in u_i as a result from the transformation in question, and is assumed to vanish at the boundary as is standard. The ψ_i denote these so-called Lagrange expressions. In modern terminology, the equations $\psi_i = 0$ are typically called the Euler-Lagrange equations and result from setting the variation $\delta I = 0$. However, as it turns out, these equations do not need to be satisfied for either of Noether's results to hold true; in standard physics jargon, we would phrase this as Noether equations holding both "off-shell and on-shell". The equation $\psi_i = 0$ is used to specify throughout the paper that the Euler-Lagrange equations are indeed satisfied, i.e. that we are on-shell.

The last piece of information required to decipher the original context of the theorems is the meaning of the term "divergences." This word refers to the boundary terms left over by a transformation, such that the integration is effectively over a total derivative[6]. In general, the variation δf leaves some boundary terms that are

[5]Here δ denotes not the codifferential, but merely the standard variation of a functional as used in classical physics in the sense of functional differentiation. The notational overload is standard and can occasionally lead to confusion, but clarification will be provided if the distinction is not clear from context.

[6]This could be a simple Euclidean vector divergence, a covariant derivative, or something else entirely depending on the context.

linear in the quantities δu_j and its derivatives, so the variational problem above can be generically represented by:

$$\sum \psi_i \delta u_i = \delta f + \operatorname{Div} A, \tag{4.2}$$

where $\operatorname{Div} A$ represents the total derivative over the term A.

We are now ready to proceed to the basic equations required to tackle Noether's theorems. First, let us make the form of the transformed quantities y_i and v_j more explicit:

$$y_i = A_i \left(x, u, \frac{\partial u}{\partial x} \dots \right) = x_i + \Delta x_i + \dots,$$
$$v_j = B_j \left(x, u, \frac{\partial u}{\partial x} \dots \right) = u_i + \Delta u_j + \dots,$$

with the Δ terms represent the lowest-order terms of the transformations $A_i, B_j \in \mathfrak{G}$. For now, we will assume they are linear; this assumption does not restrict the generality of the arguments to come. If we further assume Δx_i and Δu_j represent infinitesimal transformations, then the invariance of I under $\mathfrak{G}$ gives the following constraint:

$$\int f \left(y, v, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial y^2}, \dots \right) dy = \int f \left(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots \right) dx + \int \operatorname{Div}(f \cdot \Delta x)\ dx,$$

where the multiple integral signs from before are implied, and $\operatorname{Div}(f \cdot \Delta x)$ represents a total derivative over the f multiplied over the set of transformations Δx_i. We can massage the expression a little further if we modify Δu_j: as it stands, the form of this transformation does not reflect the fact u_j is a function of x_i, which in turn provides a little extra information about Δu_j. Noether here introduces the variation:

$$\bar{\delta} u_j = v_j(x) - u_j(x) = \Delta u_j - \sum \frac{\partial u_j}{\partial x_\alpha} \Delta x_\alpha. \tag{4.3}$$

The variation $\bar{\delta} u_j$ leads to an associated variation $\bar{\delta} f$ over the function. Plugging these results into the invariance of I leads to the final condition:

$$\int [\bar{\delta} f + \operatorname{Div}(f \cdot \Delta x)]\ dx = 0.$$

Since this is identical to the condition required for the Lagrange expressions, this leaves us with:

$$\sum \psi_j \bar{\delta} u_j = \operatorname{Div} B, \tag{4.4}$$

where $B = A - f \cdot \Delta x$ represents the boundary terms resulting from the variation $\bar{\delta} f$, and A represents the boundary terms from δf. Equation (4.4) is the most general expression of Noether's objectives in this paper without imposing additional constraints on the transformation group $\mathfrak{G}$.

4.2.2 Example: Classical Klein-Gordon Lagrangian

Let us synthesize the above by applying it to the classical Klein-Gordon equation, a relativistic physical field theory over a pair of complex scalar field ϕ and ϕ^*. The invariant integral in question is the action S is given by

$$S = \int_M L = \int_M (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi) \; d^4 x \; . \tag{4.5}$$

M represents $(3+1)$-spacetime with the volume element d^4x given by the Minkowski metric $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ for raising and lowering indices over the coordinates x^μ. L is a 4-form called the Lagrangian density. The symbol ∂_i is used as shorthand for $\frac{\partial}{\partial x^i}$ and the Einstein summation convention is employed. Physically, the first term represents the kinetic energy of these fields, while the second term represents the rest-mass of the fields. In terms of Noether's notation, it is easy to see that $I \to S$, $f \to L$, and $u_j \to (\phi, \phi^*)$.

The Euler-Lagrange equations for a classical relativistic field theory are given by

$$\psi_i = \partial_\mu \left(\frac{\partial I}{\partial(\partial_\mu u_i)} \right) - \frac{\partial I}{\partial u_i},$$

which yields the equations of motion for ϕ and ϕ^*:

$$\psi_\phi = \partial_\mu \partial^\mu \phi^* + m^2 \phi^* = 0, \tag{4.6}$$

$$\psi_{\phi^*} = \partial_\mu \partial^\mu \phi + m^2 \phi = 0. \tag{4.7}$$

The variation δL can be calculated by perturbing the fields by $\phi \to \phi + \delta\phi$ and $\phi^* \to \phi^* + \delta\phi^*$, giving

$$S \to S' = \int_M L - \delta\phi(\partial_\mu \partial^\mu \phi^* + m^2 \phi^*) - \delta\phi^*(\partial_\mu \partial^\mu \phi + m^2 \phi) + \partial_\mu(\delta\phi^* \partial^\mu \phi + \delta\phi \partial^\mu \phi^*) \; .$$

We can clearly identify the equations of motion from this variation, and furthermore: we have found an expression for $A = \delta\phi^* \partial^\mu \phi + \delta\phi \partial^\mu \phi^*$.

4.3 Noether's First Theorem

Below is Noether's first theorem, translated from her original paper:

Theorem 2 (Noether's First Theorem)**.** *If the integral I is invariant with respect to a $\mathfrak{G}_\rho$, then ρ linearly independent combinations of the Lagrange expressions become divergences — the converse follows from the invariance of I with respect to a $\mathfrak{G}_\rho$. This statement is still valid even in the limit of infinitely many parameters.*

The symbol $\mathfrak{G}_\rho$ represents a global symmetry group. It is a restriction of the group $\mathfrak{G}$ introduced previously: it is a finite continuous group of transformations such that the most general transformations within that group depend on precisely ρ unique parameters, which are denoted by the set of symbols ϵ_i. Since the infinitesimal terms $\Delta x, \Delta u$ are linear in the transformation, that implies they must be linearly dependent on the ρ parameters $\epsilon_1, \dots \epsilon_\rho$. This also constrains $\bar{\delta} u$ and its derivatives along with A and B to be linear in the parameters as well. Explicitly writing this constraint gives

$$B = \sum B^{(i)} \epsilon_i,$$

$$\bar{\delta} u = \sum \bar{\delta} u^{(i)} \epsilon_i.$$

The consequences of Noether's first theorem then result from plugging in these restrictions into (4.4), yielding the formal statement of the theorem in ρ separate equations:

$$\sum \psi_i \bar{\delta} u_i^{(1)} = \mathrm{Div}\, B^{(1)}; \quad \dots \quad \sum \psi_i \bar{\delta} u_i^{(\rho)} = \mathrm{Div}\, B^{(\rho)}. \tag{4.8}$$

In other words, the ρ Euler-Lagrange expressions become total derivatives; in modern parlance, this means that whether or not the equations $\psi_i = 0$ are satisfied, there is a current associated with the each of the symmetries in question. Here, the B terms represent those currents. If indeed the Euler-Lagrange equations are satisfied, then the B currents are conserved as $\mathrm{Div}\, B = 0$. It is common in quantum field theory to integrate both sides of the expression to obtain an associated conserved "charge", but whether or not this is possible depends on the details of the integration (e.g. boundary conditions). This still holds even if there are countably infinitely many parameters of transformation. Noether's proof of the converse of this theorem does not yield any additional physical insights.

4.3.1 Example: First Theorem in Klein-Gordon

The obvious symmetry group that is of the form $\mathfrak{G}_\rho$ for the Klein-Gordon Lagrangian is the pair of transformations:

$$\phi \to \phi' = e^{i\theta}\phi,$$

$$\phi^* \to \phi'^* = e^{-i\theta}\phi^*,$$

where θ is a constant real number representing a phase shift. Clearly, the Lagrangian in equation (4.5) is completely unaffected by this shift. $\rho = 1$ for this group, so we expect one divergence expression. If we assume that θ is infinitesimal, then

$$\phi' = \phi + i\theta\phi - \ldots,$$

$$\phi'^* = \phi^* - i\theta\phi^* + \ldots \ .$$

This implies that the shifts take the form $\Delta x = 0$ and $\Delta u_\phi = i\theta\phi, \Delta u_{\phi^*} = -i\theta\phi^*$. Plugging in for each of the variables in equation (4.8) yields

$$i\theta[\phi(\partial_\mu\partial^\mu\phi^* + m^2\phi^*) - \phi^*(\partial_\mu\partial^\mu\phi + m^2\phi)] = i\theta\partial_\mu[\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi],$$

which upon simplification yields the identity,

$$\phi\partial_\mu\partial^\mu\phi^* - \phi^*\partial_\mu\partial^\mu\phi = \partial_\mu[\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi].$$

If the Euler-Lagrange equations are satisfied, this tells us that the current $j^\mu = \phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi$ is conserved for the Klein-Gordon Lagrangian. This is indeed the "probability current" that indicated there were conceptual issues with the quantization of this relativistic system, as this current is not generically positive-definite.

We will see in later sections that this current represents a useful notion of continuity for matter fields; in particular, when considering electromagnetism, this current can be naturally identified with electric current. It was this key insight that led to the development of the first 'complete' gauge theories; this connection allowed for the first meaningful quantization of an electrodynamic Lagrangian.

4.4 Noether's Second Theorem

Below is Noether's second theorem, as translated from the original paper:

Theorem 3 (Noether's Second Theorem)**.** *If the integral I is invariant with respect to a $\mathfrak{G}_{\infty\rho}$ in which the arbitrary functions occur up to the σ-th derivative, then there subsist ρ identity relationships between the Lagrange expressions and their derivatives up to the σ-th order. The converse holds here as well*[7]*.*

The group $\mathfrak{G}_{\infty\rho}$ under consideration represents a local symmetry group. It is defined as a continuous group $\mathfrak{G}$ such that the transformations depend on ρ independent, arbitrary functions $p_i(x)$ and their derivatives (up to the specified order σ). As in the first theorem, this means that B and $\bar{\delta}u$ along with its derivatives will depend linearly on the functions $p_i(x)$ and their derivatives. Explicitly, let us write the transformation as

$$\bar{\delta}u_i = \sum_j a_{ij}p_j(x) + b_{ij}\frac{\partial p_j}{\partial x} + \cdots + s_{ij}\frac{\partial^\sigma p_j}{\partial x^\sigma},$$

where the coefficients $a_{ij}, b_{ij}, \ldots, s_{ij}$ are actually functions of x, u, and the derivatives of u. We can plug in the above expression back into equation (4.4) to obtain

$$\sum_{ij} \psi_i \left(a_{ij}p_j(x) + b_{ij}\frac{\partial p_j}{\partial x} + \cdots + s_{ij}\frac{\partial^\sigma p_j}{\partial x^\sigma} \right) = \operatorname{Div} B.$$

Since the above expressions are technically under integrals, we can employ integration by parts to remove the derivatives off the $p_i(x)$ and onto the coefficients and the ψ_i. If we lump the boundary terms into a term Γ, then we are left with:

$$\sum_{ij} p_j(x) \left(a_{ij}\psi_i - b_{ij}\frac{\partial \psi_i}{\partial x} + \cdots + (-1)^\sigma s_{ij}\frac{\partial^\sigma \psi_i}{\partial x^\sigma} \right) = \operatorname{Div}(B - \Gamma). \tag{4.9}$$

If we now take $p_i(x)$ and its derivatives to vanish at the boundary, along with $B - \Gamma$, then the parenthetical expressions must vanish as the functions $p_i(x)$ are arbitrary. Thus, the above equation reduces to Noether's final form for her second theorem

$$\sum_{i} \left(a_{ij}\psi_i - b_{ij}\frac{\partial \psi_i}{\partial x} + \cdots + (-1)^\sigma s_{ij}\frac{\partial^\sigma \psi_i}{\partial x^\sigma} \right) = 0, \tag{4.10}$$

[7]Original footnote: "For certain trivial exceptions, compare Section 2, Note 13." These exceptions refer to the case when the divergences mentioned previously automatically vanish, leaving the equations trivially satisfied.

with one such equation holding for each value of $j = 1, 2, \ldots, \rho$. The second theorem thus implies that the Euler-Lagrange expressions satisfy additional constraints under such symmetries, whether or not the equations of motion are satisfied.

4.4.1 Example: Second Theorem in Klein-Gordon

Noether's Second Theorem is trivial if applied to the Klein-Gordon Lagrangian as is, as there are no arbitrary functions that can transform the fields and leave the action unchanged. In fact, if one takes as the transformation the function $p(x) = e^{i\theta}$ as used previously, then the second theorem reduces to the equation (4.4).

However, it is here that we can find the seed of gauge symmetries: physical intuition suggests that if the value of θ arbitrarily changed as a function of position, then the underlying physics should not really change. So let θ become a local function $\theta(x)$[8], thus promoting $p(x) = e^{i\theta(x)}$ to a local symmetry, and the Klein-Gordon action becomes:

$$
\begin{aligned}
\delta S &= \int_M i\left(\phi\partial_\mu\phi^*\partial^\mu\theta - \phi^*\partial_\mu\theta\partial^\mu\phi\right) + \partial_\mu\theta\partial^\mu\theta\phi^*\phi \; d^4x \\
&= \int_M \partial_\mu\left[-i\theta\left(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right)\right] + \partial_\mu\theta\partial^\mu\theta\phi^*\phi \; d^4x \\
&= \int_M \partial_\mu(-i\theta j^\mu) + \partial_\mu\theta\partial^\mu\theta(\phi^*\phi) \; d^4x \; .
\end{aligned}
$$

The appearance of the conserved current as a boundary term is strongly suggestive of a deeper physical significance to this theory and this tentative gauge symmetry. We will not elaborate any further on this idea in this section, but instead leave the connection between gauge invariance and Noether's second theorem for the end of the next section.

4.5 First vs. Second: Scope and Overlap

In a nutshell, we can state that the first theorem relates symmetries at the global level to how they lead to local conservation laws of currents and associated charges on spacelike surfaces. The second theorem relates symmetries at the local

[8] For now, we are implicitly assuming that this function vanishes on the boundary of the manifold in question as to keep in line with Noether's assumptions.

level to how they constrain the global dynamics by specifying the distinction between the kinematic and dynamic variables of motion. In fact, the kinematic variables of a field theory are the components of the field that are entirely determined by the symmetries of the action, whereas the dynamic variables contain information regarding the physical evolution of the system. The restrictions from the second theorem essentially constrain the equations of motion, providing explicit ways to identify the kinematic field variables. Generally speaking, we can see these as a more global property of a given action principle, in direct contrast to the first theorem's identities.

Noether's theorems, of course, do not apply to discrete symmetries (e.g. reflections), but these symmetries are generally easier to identify and analyze than their continuous counterparts.

The two theorems become far more interesting when applied to gauge symmetries, both classically and quantum mechanically. In the quantum field theory literature, gauge symmetries are global symmetries that are promoted to local symmetries; in the classical realm, typically only local symmetries are called gauge symmetries. In modern parlance, the term is used for both global and local symmetries in both classical and quantum contexts.

In most relevant physical contexts, theories in which local gauge symmetries originate from global gauge symmetries therefore are restricted by both of Noether's theorems: local and global conservation laws strongly restrict the dynamics, and any resulting discretization must respect these conserved quantities. Applications of both theorems to specific classical field theories will be discussed in the next subsections to clarify the interplay of the two theorems.

4.5.1 As applied to spacetime electromagnetism

The action for Maxwell's equations over a manifold M is given by

$$S = \int_M \frac{1}{2} F \wedge \star F - A \wedge J \tag{4.11}$$

where A is the 1-form representing the electromagnetic potential, $F = dA$ is the electromagnetic 2-form, and J is the current 3-form satisfying $dJ = 0$ as a constraint (conservation of electric charge)[9]. We will assume M represents a flat spacetime with some Lorentzian metric.

Let us vary the action by $A+\chi$. This gives

$$\delta S = \int_M d(\chi \wedge \star dA) + \chi \wedge (d \star dA - J),$$

which identifies $\chi \wedge \star dA$ as the boundary term "A" as defined in Noether's notation. The equation of motion ψ results from the rightmost term, so $\psi = d \star dA + J$. Along with the geometric identity $dF = 0$, this yields all of Maxwell's equations (when boundary terms vanish).

Electromagnetism is invariant under shifts of the potential A by exact forms,

$$A \to A' = A + d\omega \implies F' = F$$

where ω is a scalar function. This symmetry group technically falls under the type $\mathfrak{G}_{\infty\rho}$, so this action must obey Noether's second theorem. Plugging the above transformation into the action gives,

$$S \to S' = S - \int_M d\omega \wedge J = S - \int_M d(\omega J) + \omega dJ = S - \int_M d(\omega J)\ .$$

The term ωJ represents the term Γ in Noether's notation, and $dJ = 0$ by our initial constraint of conservation of charge. If we assume the boundary terms to be irrelevant, as in Noether's work, then we immediately find that the action is invariant *only because charge is conserved*, and not vice-versa as is commonly touted in the physics literature. In fact, Noether's second theorem in this context amounts to the statement $d\psi = 0$, which is trivially satisfied as $d(d \star dA - J) = 0$.

If we want to enlarge the gauge group to include shifts by closed forms, i.e. shifts by ω such that $d\omega = 0$, then Noether's theorems imply that conservation of charge must be modified to account for the presence of boundary effects. In other words, let ω be a closed 1-form, with a Hodge decomposition $\omega = d\alpha + \delta\beta + \gamma$ and $\Delta\gamma = 0$. Shifting the electromagnetic action by ω yields

$$S' = S - \int_M \omega \wedge J = S - \int_M d\alpha \wedge J + \delta\beta \wedge J + \gamma \wedge J\ .$$

[9] As we will see in the next few examples, charge conservation can not come about from Noether's theorem, as J is not a variational field term. In other words, a matter action with symmetry needs to be defined for an associated conservation law to follow. J in this example acts as an auxiliary field, and thus must satisfy a constraint.

If we are to force this symmetry group upon electromagnetism, then it is clear that both of Noether's theorems would come into play, and require the topology of the underlying manifold to contribute to charge conservation via boundary terms.

4.5.2 As applied to General Relativity

In General Relativity, the Einstein-Hilbert action yields the dynamics of pure gravity[10]

$$S_{EH} = \alpha \int_M R\sqrt{-g}\; d^4x, \tag{4.12}$$

where $\alpha = \frac{c^4}{16\pi G}$. Since the domain of integration is a general spacetime manifold M, the volume form is explicitly written, with the notation $g = \det g_{\mu\nu}$ and R standing for the Ricci scalar. The Einstein Field Equations, obtained by taking variations with respect to the metric tensor, are given by (as demonstrated in [Wal84])

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \implies G_{\mu\nu} = 0, \tag{4.13}$$

where $R_{\mu\nu}$ represents the Ricci tensor, and $G_{\mu\nu}$ is the Einstein tensor.

The symmetry group of this action is the group of diffeomorphisms of the manifold, denoted $\mathrm{Diff}\, M$. The operation which generates infinitesimal diffeomorphisms is the Lie derivative $\mathcal{L}_X$; its action on a general tensor field Y is (following [Nak03])

$$\mathcal{L}_X Y = \lim_{\epsilon\to 0} \frac{1}{\epsilon}[(\sigma_{-\epsilon})_* Y|_{\sigma_\epsilon(x)} - Y|_x], \tag{4.14}$$

where σ is a parametrization of the flow generated by X, $\sigma_\epsilon(x)$ is a point close to x, and $(\sigma_{-\epsilon})_*$ represents the pushforward. Only Noether's Second Theorem is valid in the context of pure gravity, as $\mathrm{Diff}(M)$ shifts the metric tensor $g_{\mu\nu}$ and its derivatives by arbitrary functions. We can thus expect a conservation law to occur from this theorem.

Let us determine the action of the infinitesimal diffeomorphisms on the action. Acting on the Lagrangian by the Lie derivative $\mathcal{L}_X$, with X an arbitrary small displacement vector field,

$$\delta S = \int_M \mathcal{L}_X(R\sqrt{-g})\; d^4x = \int_M (X^\mu \nabla_\mu R + R\nabla_\mu X^\mu)\sqrt{-g})\; d^4x, \tag{4.15}$$

[10]Matter will be added to the system later in this subsection.

where ∇ represents the covariant derivative. The above equation represents a boundary term, thus if X vanishes on the defined boundaries, δS automatically vanishes. Assuming this constraint is satisfied, we find that upon applying Noether's Second Theorem,

$$\nabla_\mu G^{\mu\nu} = 0. \tag{4.16}$$

In other words, we are left with an identity: the Einstein tensor is a conserved quantity, both definitionally and by invariance of the action. This identity is a consequence of the Bianchi identities on the curvature tensors, thus Noether's Second Theorem demonstrates that the symmetry is consistent with the underlying geometry. This should not be very surprising: after all, first-order variations of the metric take on the general form

$$\delta g_{\mu\nu} = -\mathcal{L}_X g_{\mu\nu},$$

which implies that in General Relativity, the kinematic constraint $\nabla_\mu G^{\mu\nu} = 0$ is an inherent part of the motion[11].

Non-trivial outcomes of Noether's Second Theorem result from coupling a matter Lagrangian to the action, which then results in a generic statement of local energy-momentum conservation via the stress-energy tensor. Let us add such a coupling via a (non-fermionic) matter Lagrangian L_{matter}:

$$S = S_{EH} + S_{matter} = \int_M (\alpha R + L_{matter})\sqrt{-g}\ d^4xl, \tag{4.17}$$

with the stress-energy tensor given by the Euler-Lagrange equations,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}}.$$

Varying the action by $\delta g_{\mu\nu}$ yields a nearly identical constraint to equation (4.15) containing L_{matter}. Applying Noether's Second Theorem, we find that

$$\nabla^\mu (G_{\mu\nu} - \frac{1}{2\alpha}T_{\mu\nu}) = \psi_{matter}, \tag{4.18}$$

where ψ_{matter} represents the Euler-Lagrange expression for the matter terms. Thus, if the Euler-Lagrange equations for the matter terms are satisfied, we find that not only is the stress-energy tensor $T_{\mu\nu}$ is conserved, but also that $\nabla^\mu G_{\mu\nu}$ holds true.

[11] In Einstein-Cartan theory, this constraint is not satisfied due to non-vanishing torsion; however, a similar constraint involving a general analogue to angular momentum is satisfied instead.

4.5.3 As applied to the Maxwell Klein-Gordon System

Coupling electromagnetism to the Klein-Gordon Lagrangian leads to a theory of electromagnetism in which the current J is now dynamically represented by the Klein-Gordon matter fields ϕ. It is often in this context that the conclusions from Noether's theorems are applied and misused. As we will see, both theorems are in play which tends to be the main source of confusion.

First, consider writing the two Lagrangians together under the same action:

$$S = \int_M d\phi^* \wedge \star d\phi - m^2 \phi^* \phi + \frac{1}{2} F \wedge \star F - A \wedge J \ . \tag{4.19}$$

Clearly, the term J must connect to the field ϕ for any interesting interactions to occur between the two systems. Our only constraint on the current is that it must be closed, $dJ = 0$. The natural candidate that satisfies this constraint is the symmetry current obtained from the application of the first theorem to the Klein-Gordon system; in other words,

$$J \to \star j = \phi \star d\phi^* - \phi^* \star d\phi, \tag{4.20}$$

which promotes J to a dynamical quantity.

However, we must recall that $\partial_\mu j^\mu = 0$ as defined earlier only holds when the Euler-Lagrange equations are satisfied. This seemingly innocuous difference is what leads to a modification of the standard gauge symmetry in the coupled action (4.19). Indeed, shifting the potential A by an exact 1-form is only a symmetry of this system when on-shell. In addition, the Klein-Gordon global phase symmetry trivially holds in this context, and does not provide any apparent avenue to resolving this issue.

As it turns out, there is a symmetry transformation of this system that requires combining the two symmetries of the Klein-Gordon and Maxwell Lagrangians. By promoting the trivial Klein-Gordon symmetry to a local symmetry, then coupling the gauge symmetry to differentiation, we find a neat and compact physical system which provides a way to intimately tie charge (a material property) into the dynamics of the potential field.

Consider result for δS found in section 4.4.1, with θ representing a local function of the spacetime:

$$\delta S = \int_M \partial_\mu (-i\theta j^\mu) + \partial_\mu \theta \partial^\mu \theta (\phi^* \phi) \ . \tag{4.21}$$

Canceling these terms requires a series of well-known steps: first, allow A to transform under this symmetry, then allow the potential to serve as a connection on this manifold. The final Lagrangian invariant under this gauge symmetry is

$$S = \int_M \nabla\phi^* \wedge \star\nabla\phi - m^2\phi^*\phi + \frac{1}{2}F \wedge \star F - A \wedge J, \tag{4.22}$$

with $\nabla = d + igA$, $J = ig\star j = ig(\phi\star d\phi^* - \phi^*\star d\phi)$, and g is a constant. Furthermore, when ϕ is phase-shifted by $\theta(x)$, A is assumed to simultaneously transform as

$$A \to A' = A + \frac{1}{g}d\theta(x),$$

which still preserves the form of the purely electromagnetic part of the action. Overall, this guarantees that the above action remains invariant under the gauge symmetry, up to the usual boundary considerations.

The constant g is an electric coupling constant which we can interpret as the fundamental charge e; it is a parameter that must connect how the matter fields ϕ relate to the potential field A. Indeed, this latter fact is typically used as justification for the following statement: gauge symmetry explains the origin of charge, and furthermore, proves that it must be conserved. Clearly, as the above exposition of Noether's theorems shows, this physicists' shorthand erases the subtleties and distinctions involved in the process, especially in the presence of non-trivial boundary terms.

4.6 Modern Terminology: updates to Noether's Theorems

The general modern setting for discussing variational problems is provided by the language of exterior calculus. This section will discuss conventions and notations for Noether's theorems, as presented in a variety of modern texts on the intersection of variational principles, differential geometry, and physics; in particular, I will follow [Ede85], [GS95], and [GMI$^+$12].

If only symmetries on the underlying manifold are considered, then the Lie derivative provides a means of characterizing Noether's theorems. A vector field X

is called a *Noetherian vector field* if the flow of the Lagrangian density L by this field vanishes; in other words, $\mathcal{L}_X L = 0$.[12] Since a Lagrangian density is generally proportional to the volume form of the underlying manifold $L = \lambda vol$, we can apply Cartan's magic formula to the above:

$$\mathcal{L}_X L = (i_X d + d i_X)(\lambda vol) = d(\lambda i_X vol) = d\lambda \wedge i_X vol + \lambda d i_X vol = 0.$$

Clearly, the above constraint is satisfied if the manifold in question is compact. Assuming this trivial case is not true, then it is also true if L or X vanish on the boundary of the manifold in question; alternatively, any X tangent to the boundary would satisfy the above constraint.

As it turns out, this constraint is too restrictive, as the invariance of the action is more physically relevant in a field theory, and furthermore, is not representative of the symmetries typically encountered in physical systems. Allowing the $\mathcal{L}_X L$ to vanish up to a closed form provides an equation defining *Noetherian vector fields of the first kind*. For *Noetherian vectors fields of the second kind*, the flow can vanish if the above condition corresponds to an exact form. Generalizations of these ideas to include boundary data and constraints are detailed in [Ede85] and [GMI$^+$12].

For more general symmetries, the full machinery used in the GiMmsy preprints is necessary. Since this involves much more advanced topics beyond what is presented in the rest of this thesis, I will provide no introduction to the terminology and instead refer the interested reader to look at the GiMmsy preprints themselves.

In short, Noether's First Theorem is referred to as the *Noether Conservation Law*, which states that the divergence of the multisymplectic analogue of the Noether current vanishes when the equations of motion are satisfied and the Lagrangian is equivariant with respect to the symmetry group; the contributions from each of these terms is computed explicitly in the proof of Theorem 4D.3 of [GMI$^+$12].

Noether's Second Theorem is referred to as the *Vanishing Theorem*: the multisymplectic Noether current must vanish when the pullback of its inclusion is integrated over a hypersurface of the full spacetime for any solution of the equations of motion. This can also be stated as the vanishing of the energy-momentum map on

[12] Note that this is distinct from what was shown for General Relativity, as the manifold in question was not flat.

the same hypersurface. The converse of this theorem holds as well: if the energy-momentum map vanishes over all hyperspaces, then the equation of motion in the Noether current is satisfied. The proof of the theorem and its converse are provided under Theorem 9B.1 of [GMI+12].

4.7 Noether Currents and Conservation Laws as related to Errors

Both Noether's first and second theorems capture the restrictions imposed on a variational field theory by its symmetries. Thus, in the realm of computation, the Noether quantities resulting from these theorems can provide a method of computing the extent to which a symmetry is preserved within a given simulation. Spoken differently, these constraints from the continuous realm should translate to discrete constraints that can yield bounds on the extent to which the symmetries are violated. However, such applications typically involve computing violations of energy or momentum conservation; generalizing this idea to any symmetry is rarely discussed. Specific examples will be analyzed in the subsequent chapter in the context of an FEEC discretization of spacetime electromagnetism and linearized relativity.

4.8 Generalizations and Conclusions

Theories in which local gauge symmetries originate from global gauge symmetries therefore are restricted by both of Noether's theorems: local and global conservation laws strongly restrict the action. Morally speaking, any discretization of such field theories must respect these conserved quantities to ensure qualitatively correct behavior. I suggest the following convention: the singular term "Noether's theorem" should refer specifically to equation (4.4),

$$\sum \psi_j \bar{\bar{\delta}} u_j = \operatorname{Div} B,$$

as it is the most general result that can be obtained without restricting the form of the continuous symmetry group $\mathfrak{G}$ or the form of the invariant integral in question.

In fact, (4.4) fully displays the connection between the allowed group transformations in situations for which global considerations come into play. For theories in which the boundary data play a dynamical role, (4.4) shows that the resulting equations of motion must include the effects of such boundary terms as well. Spoken differently, a gauge transformation is only valid and meaningful if it reflects the symmetries of the underlying space in addition to that of the action in question.

Chapter 5

Electromagnetism and Linearized General Relativity in FEEC

"Tout ce qu'on invente est vrai, sois-en sûre. La poésie est une chose aussi précise que la géométrie. L'induction vaut la déduction, et puis, arrivé à un certain point, on ne se trompe plus quant à tout ce qui est de l'âme."

— Gustave Flaubert

5.1 Introduction

This chapter will synthesize the previous three chapters' work into an FEEC discretization of two relativistic Lagrangians with the conceptual machinery of variational integrators: electromagnetism with current sources in a Minkowski spacetime and a corresponding discretization of linearized General Relativity via the Fierz-Pauli formalism. I will analyze some of the geometric aspects of these discretizations using insights from the Rapetti construction, and then turn to examining the outcome of Noether's theorems in both scenarios. In future work, generalizing these expressions to include the dynamics of coupled matter will enable the Noether currents to quantify the numerical error in such simulations. As shown in Chapter 4, this notion of error should correlate directly to topological and boundary quantities in these field theories, especially in the context of electromagnetism with material effects.

5.2 Electromagnetism in spacetime

The action for Maxwell electromagnetism in Minkowski spacetime M is given by

$$S = \int_M \frac{1}{2} dA \wedge \star dA - A \wedge J, \tag{5.1}$$

where A represents the electromagnetic potential 1-form and J is a closed 3-form ($dJ = 0$) denoting the conserved current and charge due to matter sources. As shown in Chapter 4, a variation $A + \chi$ gives for the variation δS:

$$\delta S = \int_M d(\chi \wedge \star dA) + \chi \wedge (d \star dA - J) \ .$$

The first term represents a boundary contribution. For a general simulation, we cannot assume that M is compact, so we must impose the additional constraint that dA vanishes at the boundary, i.e. the electromagnetic fields vanish[1]. The resulting equation of motion is:

$$d \star dA = J \implies \delta dA = \star J. \tag{5.2}$$

Thus, for a consistent discretization of Maxwell's equations of order r, $J \in \mathcal{P}_r\Lambda^3$ and $A \in \mathcal{P}_{r+2}\Lambda^1$. Let us now turn to analyzing different orders of discretization.

As shown in Chapter 4, the gauge symmetry of electromagnetism is typically given as shifts by exact forms, i.e. $A \to A' = A + d\omega \implies S \to S' = S$. This gives the equation:

$$S' = S - \int_M d\omega \wedge J = S - \int_M d(\omega J) - \omega dJ = S - \int_M d(\omega J) \ .$$

This implies that J must vanish at the boundary, much like dA.

5.2.1 0^{th} order FEEC discretization

At $r = 0$, the current J can be represented by constant 3-forms over a given simplex. J must also represent a closed form; this is trivially satisfied at this order of discretization since $\mathcal{P}_0\Lambda^3 = d\mathcal{P}_1\Lambda^2$. Let σ represent the simplex in question. Generically, the current is given by

$$J_\sigma = \sum_{i<j<k}^{4} c_{\sigma_{ijk}} dw_{\sigma_{ijk}}, \tag{5.3}$$

[1] Adding dynamics to the J term, as was discussed in Chapter 4, allows us to circumvent this issue by adding appropriate boundary currents.

where $w_{ijk} \in W^2$ and $i, j, k \neq 0$, as λ_0 has been taken as the superfluous degree of freedom for σ.

The electromagnetic potential A must be quadratic to be consistent with a constant current. Since $\mathcal{P}_2\Lambda^1 = d\mathcal{P}_3\Lambda^0 \oplus \kappa\mathcal{P}_1\Lambda^2$, the potential A can be written as

$$A_\sigma = \sum_{i=0}^{4}\sum_{j<k}^{4} a_{\sigma_{ijk}}\lambda_i w_{jk} + \sum_{i\leq j}^{4}\sum_{k=1}^{4} b_{\sigma_{ijk}}\lambda_i\lambda_j d\lambda_k, \tag{5.4}$$

where in the first term, the $w_{\sigma_{jk}}$ represent the Whitney 1-forms on the edge jk and the $j < k$ summation does not include $j, k = 0$; in the second term, the $i \leq j$ summation includes the zero values.

The electromagnetic action then takes on the form:

$$S_d = \sum_{\sigma\in M}\int_\sigma \frac{1}{2}dA_\sigma \wedge \star dA_\sigma - A_\sigma \wedge J_\sigma \ . \tag{5.5}$$

Adding a perturbation $\chi_\sigma \in \mathcal{P}_2\Lambda^1$ to A_σ such that $A_\sigma \to A'_\sigma = A_\sigma + \epsilon\chi_\sigma$ produces a first-order variation in the action given by

$$\left.\frac{dS_d}{d\epsilon}\right|_{\epsilon=0} = 0 \implies \sum_{\sigma\in M}\int_\sigma d(\chi_\sigma \wedge \star dA_\sigma) + \chi_\sigma \wedge (d\star dA_\sigma - J_\sigma) = 0. \tag{5.6}$$

As χ_σ is arbitrary, this requires that $d\star dA_\sigma + J_\sigma = 0$ on each simplex with $dA_\sigma = 0$ and $J_\sigma = 0\,\forall\,\sigma \in \partial M$. Simplifying the equation of motion in terms of wedge products of Whitney forms yields:

$$\sum_{i,l,m}^{4}\sum_{j<k}^{4} a_{ijk}[2w_{im}\wedge\star w_{jkl} + w_{km}\wedge\star w_{ijl} + w_{jm}\wedge\star w_{ilk}] = \sum_{i<j<k}^{4}\sum_{l=0}^{4} 3c_{ijk}w_{ijkl},$$

where the σ is understood and the triple summation over i, l, m goes from 0 to 4.

This can only be solved once a determination of how the combination $w_{ij} \wedge \star w_{klm}$ maps onto the space W^3. For the standard simplex in $\mathbb{R}^4$, the above equation reduces to

$$\sum_{i=1}^{4}\sum_{j<k}^{4}(3a_{ijk} - 2a_{0jk})d\lambda_i \wedge \star(d\lambda_j \wedge d\lambda_k) = \sum_{i<j<k}^{4} 3c_{ijk}d\lambda_i \wedge d\lambda_j \wedge d\lambda_k.$$

On the standard simplex, $d\lambda_i \wedge \star(d\lambda_j \wedge d\lambda_k)$ is non-zero iff $i = j$ or $i = k$ as each $d\lambda_i$ is identical to the corresponding dx_i. There are 12 such terms (up to the antisymmetry of the wedge product); given the definition of the Hodge star over the standard simplex in Minkowski spacetime and the identification of $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \to \{t, x, y, z\}$, we can simplify the sum above using the correspondence:

$$
\begin{aligned}
d\lambda_1 \wedge \star(d\lambda_1 \wedge d\lambda_2) &= d\lambda_1 \wedge d\lambda_3 \wedge d\lambda_4, & d\lambda_3 \wedge \star(d\lambda_1 \wedge d\lambda_3) &= d\lambda_2 \wedge d\lambda_3 \wedge d\lambda_4,\\
d\lambda_1 \wedge \star(d\lambda_1 \wedge d\lambda_3) &= -d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_4, & d\lambda_3 \wedge \star(d\lambda_2 \wedge d\lambda_3) &= d\lambda_1 \wedge d\lambda_3 \wedge d\lambda_4,\\
d\lambda_1 \wedge \star(d\lambda_1 \wedge d\lambda_4) &= d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_3, & d\lambda_3 \wedge \star(d\lambda_3 \wedge d\lambda_4) &= -d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_3,\\
d\lambda_2 \wedge \star(d\lambda_1 \wedge d\lambda_2) &= d\lambda_2 \wedge d\lambda_3 \wedge d\lambda_4, & d\lambda_4 \wedge \star(d\lambda_1 \wedge d\lambda_4) &= d\lambda_2 \wedge d\lambda_3 \wedge d\lambda_4,\\
d\lambda_2 \wedge \star(d\lambda_2 \wedge d\lambda_3) &= d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_4, & d\lambda_4 \wedge \star(d\lambda_2 \wedge d\lambda_4) &= d\lambda_1 \wedge d\lambda_3 \wedge d\lambda_4,\\
d\lambda_2 \wedge \star(d\lambda_2 \wedge d\lambda_4) &= -d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_3, & d\lambda_4 \wedge \star(d\lambda_3 \wedge d\lambda_4) &= -d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_4.
\end{aligned}
$$

The sum then provides explicit conditions between the coefficients in A and J:

$$
\begin{aligned}
3c_{123} &= 3(a_{114} - a_{224} - a_{334}) - 2(a_{014} - a_{024} - a_{034}),\\
3c_{124} &= 3(-a_{113} + a_{223} - a_{434}) - 2(-a_{013} + a_{023} - a_{034}),\\
3c_{134} &= 3(a_{112} + a_{424} + a_{323}) - 2(a_{012} + a_{024} + a_{023}),\\
3c_{234} &= 3(a_{212} + a_{313} + a_{414}) - 2(a_{012} + a_{013} + a_{014}).
\end{aligned}
$$

The gauge symmetry of the theory can be seen in the disappearance of the coefficients b_σ in A_σ. These coefficients represent the group of shifts by exact quadratic 1-forms, a subgroup of the full group in the continuous theory. Setting all these coefficients to zero represents the simplest possible gauge condition within this context, and corresponds to the axial gauge in the continuous theory as setting $A \wedge \star x^\flat = 0$ is akin to acting on A with the Koszul operator[2]. Other classical gauges, such as the Lorenz gauge $\delta A = 0$, are possible to enforce, but they are not as clean, algebraically speaking. Furthermore, enlarging the symmetry group to the space of closed forms does not add any extra physics, as topologically speaking, all simplices are equivalent to $\mathbb{R}^n$. The addition of a non-trivial manifold with a boundary would change this, however, and the effects could be calculated by finding how well dJ_σ tracks zero.

[2]Following the terminology of [Ede85], this would be called the anti-exact gauge in the framework of exterior calculus. Coincidentally, this is the condition that also defines the Whitney forms in FEEC. The goemetric nature of this connection traces back to the “rotational” aspect of Whitney forms over simplex.

5.2.2 1^{st} order FEEC discretization

At this order of discretization, the behavior of the theory becomes a little more interesting. Since J is now linear, we have:

$$J_\sigma = \sum_{i=0}^{4} \sum_{j<k<l}^{4} c_{\sigma_{ijkl}} \lambda_{\sigma_i} d\lambda_{\sigma_j} \wedge d\lambda_{\sigma_k} \wedge d\lambda_{\sigma_l}. \tag{5.7}$$

As before, the summation over $j < k < l$ does not include the $d\lambda_0$ terms as we have set it as our origin in σ. We must also guarantee that J is a closed form to ensure that the equations of motion behave properly. Imposing this constraint on J_σ gives the equation on the coefficients:

$$\sum_{i=0}^{4} \sum_{j<k<l}^{4} c_{\sigma_{ijkl}} d\lambda_{\sigma_i} \wedge d\lambda_{\sigma_j} \wedge d\lambda_{\sigma_k} \wedge d\lambda_{\sigma_l} = 0.$$

The potential 1-form A must now go to cubic order. $\mathcal{P}_3\Lambda^1 = d\mathcal{P}_4\Lambda^0 + \kappa\mathcal{P}_2\Lambda^2$, which leaves us with the following formula for A_σ:

$$A_\sigma = \sum_{i\leq j}^{4} \sum_{k<l}^{4} a_{\sigma_{ijkl}} \lambda_{\sigma_i} \lambda_{\sigma_j} w_{\sigma_{kl}} + \sum_{i\leq j\leq k}^{4} \sum_{l=1}^{4} b_{\sigma_{ijkl}} \lambda_{\sigma_i} \lambda_{\sigma_j} \lambda_{\sigma_k} d\lambda_{\sigma_l}, \tag{5.8}$$

with similar conditions on the indices as in the 0^{th} order discretization. Plugging into the equation of motion and dropping the σ labels yields

$$\sum_{i\leq j}^{4} \sum_{k<l}^{4} \sum_{m,n=0}^{4} a_{ijkl} \lambda_m (w_{jn} \wedge \star w_{ikl} + w_{in} \wedge \star w_{jkl}) = \sum_{i=0}^{4} \sum_{j<k<l}^{4} c_{ijkl} \lambda_i d\lambda_j \wedge d\lambda_k \wedge d\lambda_l \tag{5.9}$$

where again, the mapping from the forms $w_{ij} \wedge \star w_{klm}$ to the barycentric 3-forms must be done as above; the mapping provided previously allows for an explicit calculation.

It is at this order that a clear division emerges between the gauge portion of A and its dynamical component. Within the Rapetti framework, the gauge variables of A live over chains of edges at the third homothetic order of the simplex; spoken differently, the gauge piece of A is given by

$$b_{ijkl} \lambda_i \lambda_j \lambda_k d\lambda_l = \sum_{m=0}^{4} b_{ijkl} \lambda_i \lambda_j \lambda_k w_{lm},$$

which clearly lives in $(W^0)^3 \cdot W^1$, whereas the dynamical piece

$$a_{ijkl} \lambda_i \lambda_j w_{kl}$$

lives in the space $(W^0)^2 \cdot W^1$. Thus, dual space to the gauge portion in the Rapetti construction is one order higher, so their relative localizations do not align, as illustrated over $\mathbb{R}^2$ in Figure 5.1 below. However, due to the self-similar nature of the construction, the dual space to the gauge condition is in the vicinity of that of the dynamical variable. In this light, gauge symmetry takes on a new meaning, as it would not physically make sense for non-local effects to change the dynamics of the theory. This intuition still lines up in the limit of large polynomial order, as the difference between n^{th} and $(n+1)^{th}$ homothetic order vanishes as $n \to \infty$.

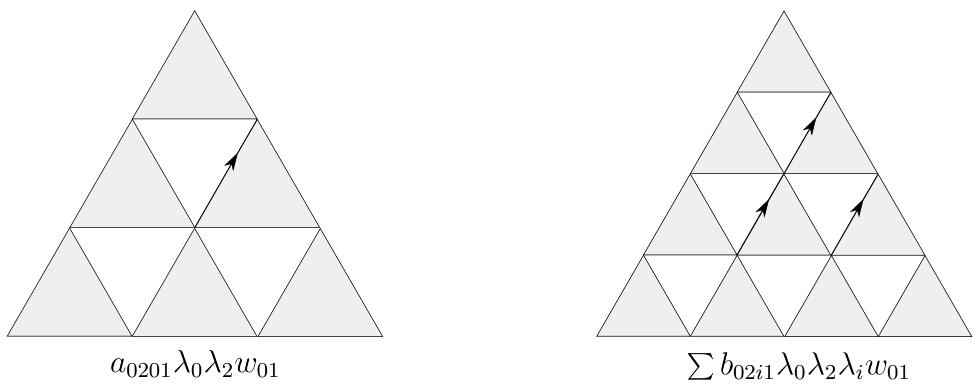


Figure 5.1: The dynamical term is on the left, and the gauge term is on the right.

5.3 Linearized General Relativity

The Fierz-Pauli Lagrangian provides a linearized action of General Relativity, in terms of a perturbation metric $h_{\mu\nu}$ around flat Minkowski spacetime $\eta_{\mu\nu}$; in other words, the metric tensor is given by the expansion $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. This Lagrangian represents the most general Lorentz-invariant field theory over a symmetric tensor of rank 2, up to the addition of boundary terms. The full action for linearized General Relativity is

$$S = \int_M \frac{1}{2}\partial^\rho h^{\mu\nu}\partial_\rho h_{\mu\nu} - \partial_\nu h^{\mu\nu}\partial_\rho h^\rho{}_\mu + \partial_\nu h^{\mu\nu}\partial_\mu h - \frac{1}{2}\partial_\mu h \partial^\mu h - \gamma T^{\mu\nu} h_{\mu\nu} \ . \tag{5.10}$$

Here indices are raised and lowered by $\eta_{\mu\nu}$ and $h = h_{\mu\nu}\eta^{\mu\nu}$, and the invariant volume element over Minkowski spacetime d^4x is dropped for simplicity. $T^{\mu\nu}$ in principle represents the stress-energy tensor, but since we are in the linearized regime, its meaning is not as clear-cut as in the full theory of GR. We will regard it as a conserved, symmetric matter term ($\partial_\mu T^{\mu\nu} = 0$) with the constant γ proprotional to G, the gravitational constant, acting as a coupling constant.

Varying perturbation metric by some $\epsilon_{\mu\nu}$ gives

$$\delta S = \int_M \partial^\rho \epsilon_{\mu\nu} \partial_\rho h^{\mu\nu} + \partial^\nu \epsilon_{\mu\nu} \partial^\mu h + \partial_\rho h^{\alpha\rho} \partial_\alpha \epsilon_{\mu\nu} \eta^{\mu\nu} - \partial_\rho h \partial^\rho \epsilon_{\mu\nu} \eta^{\mu\nu} - \gamma \epsilon_{\mu\nu} T^{\mu\nu}$$

for the variation δS. Using integration by parts yields

$$\delta S = \int_M \epsilon_{\mu\nu}(-\partial^\rho \partial_\rho h^{\mu\nu} - \partial^\mu \partial^\nu h - \eta^{\mu\nu} \partial_\rho \partial_\alpha h^{\rho\alpha} + \eta^{\mu\nu} \partial^\rho \partial_\rho h - \gamma T^{\mu\nu}) + B,$$

where the boundary terms B equal

$$B = \partial^\rho [\epsilon_{\mu\nu}(\partial_\rho h^{\mu\nu} + \eta^\nu{}_\rho \partial^\mu h + \eta^{\mu\nu} \partial^\alpha h_{\rho\alpha} - \eta^{\mu\nu} \partial_\rho h)].$$

As with the electromagnetic case, we will assume that $h_{\mu\nu}$ vanishes on the boundary. The equation of motion is then

$$-\partial^\rho \partial_\rho h^{\mu\nu} - \partial^\mu \partial^\nu h - \eta^{\mu\nu} \partial_\rho \partial_\alpha h^{\rho\alpha} + \eta^{\mu\nu} \partial^\rho \partial_\rho h = \gamma T^{\mu\nu}. \tag{5.11}$$

Thus, just as with electromagnetism, the polynomial order of $h_{\mu\nu}$ must be two higher than that of $T_{\mu\nu}$ to ensure a consistent discretization.

The symmetry of this action is analogous to that of electromagnetism. Any shift in the perturbation metric of the form

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \chi_\nu + \partial_\nu \chi_\mu, \tag{5.12}$$

where χ_μ is a 1-form, leaves the action in the following state:

$$S' = S - \int_M \gamma T^{\mu\nu}(\partial_\mu \chi_\nu + \partial_\nu \chi_\mu) = S - 2\gamma \int_M \partial_\mu (\chi_\nu T^{\mu\nu}) \ .$$

Thus, just as with J, we will assume that $T_{\mu\nu}$ vanishes on the boundary as well. In fact, the Fierz-Pauli Lagrangian encounters difficulties when the matter terms of the theory become dynamical, as the appropriate stress-energy tensor to such a theory does not include the full gravitational self-coupling through the term $h_{\mu\nu} T^{\mu\nu}$. As shown in [Ort04], one can add corrective terms to the action, but this merely pushes the inconsistency to a higher order in the perturbation. Bootstrapping this process, however, builds up to the only completely consistent theory: General Relativity[3].

[3] This insight is useful in the perturbative regime, as we can conceptualize the metric as a nonlinear function of the perturbations.

5.3.1 Discretizing general tensors in FEEC

Since the perturbation metric $h_{\mu\nu}$ is a symmetric $(0, 2)$-tensor, adapting the machinery of FEEC to this situation will require a little work. Polynomial 1-forms can be used to construct the space of symmetric tensors using the tensor product $\otimes$ to build up tensor ranks. In other words, $h_{\mu\nu}$ can be thought of as living in $(\Lambda^1 \otimes \Lambda^1)$.

An order r discretization of $h_{\mu\nu}$ cam be achieved in the space $\mathcal{P}_s\Lambda^1 \otimes \mathcal{P}_t\Lambda^1$ where s, t are any integers such that $r = s + t$. The symmetry of the tensor product makes the exact split of polynomial order irrelevant. $T_{\mu\nu}$ also lives in this space. Writing a general tensor of mixed symmetry in FEEC would involve taking the appropriate tensor and alternating products of the elements in the appropriate spaces.

Keeping in line with the themes of Chapters 2 and 3, this begs the question: is there a natural dual interpretation of the tensor product akin to the wedge product over integration? The answer is no. In short, scalar or vector combinations of tensors (e.g. $h = h_{\mu\nu}\eta^{\mu\nu}$) do take on a specific meaning, as there is no ambiguity in the identification of the simplicial chains attached to these quantities. For chains of higher degree, however, there is no unique interpretation to the pairing of these forms and chains over the tensor product.

The most natural indices in this framework should represent the set of 1-forms given by the basis $\{d\lambda_i\}$. Conventionally, $i \neq 0$ as v_0 is usually taken as the origin of the simplex; as a result, $d\lambda_0 = -\sum_{i=1}^{4} d\lambda_i$ as it is not linearly independent of the other forms. This introduces a complication with respect to the the partial derivative: ∂_μ must be taken with respect to the barycentric coordinate function. This will be indicated by $\tilde{\partial}_\mu$. In addition, the background Minkowski metric $\eta_{\mu\nu}$ must be modified so that the act of raising and lowering indices reflects the affine transformation inherent in using the appropriate metric over the barycentric coordinates instead.

Let us then convert the Minkowski metric $\eta_{\mu\nu}$ into a form suited to working with barycentric indices. Let us label this metric $\tilde{\eta}$ and impose the constraint $\tilde{\eta}_{ij}\tilde{\eta}^{jk} = \delta_i{}^k$, where δ_{ij} represents the Kronecker delta. Barycentric coordinates relate to the coordinates of the underlying space via a matrix affine transformation with respect to the vertex vectors. Let V represent the 4×4 matrix of vertex column vectors except without v_0. Then, as a matrix, $\tilde{\eta}$ can be written as

$$\tilde{\eta} = V^T \eta V. \tag{5.13}$$

These notational differences will be used in the 0^{th} and 1^{st} order discretizations in the subsections to follow. Although these differences might seem to make the variation of the linearized regime unnecessarily complicated, the effects are fairly insignificant. Technically, the Lagrangian must change to reflect the change in the natural volume element at a simplex-by-simplex level; however, since this change is a constant dependent on the volume of the simplex in question, this does not affect the overall dynamics of the system.

5.3.2 0^{th} order FEEC discretization

At the lowest-order of discretization, $T_{\mu\nu}$ is a constant tensor, so we can take it to live in $\mathcal{P}_0\Lambda^1 \otimes \mathcal{P}_0\Lambda^1$; like J, it is trivially conserved at this polynomial order. As previously, let σ represent the simplex in question, and take λ_0 as the origin for σ. Letting the indices on T represent slots over the barycentric coordinates $\{\lambda_i\}|_{i\neq 0}$ yields the discretization

$$T_{ij}|_\sigma = c_{\sigma_{ij}} d\lambda_{\sigma_i} \otimes d\lambda_{\sigma_j}, \tag{5.14}$$

where the $|$ distinguishes tensor indices from simplicial labels. The coefficients can be defined such that $c_{ij} = c_{ji}$; either way, the symmetry of the tensor product ensures that T is symmetric.

The discretization for the dynamical variable h is slightly more complex: $\dim \mathcal{P}_1\Lambda^1 \otimes \mathcal{P}_1\Lambda^1 = 150$, yet $\dim \mathcal{P}_1\Lambda^1 = 20$. The high level of degeneracy makes it difficult to find a representative basis. Since a generic element in $\mathcal{P}_1\Lambda^1$ can be written as a sum of the Whitney 1-forms $\sum a_{ij} w_{ij}$ and the non-Whitney forms $\sum b_{ij} s_{ij}$ with $s_{ij} \in \mathcal{P}_1^+\Lambda^1$, the sum representing h should have terms of the form $w_{ij} \otimes w_{kl}, w_{ij} \otimes s_{kl}, s_{ij} \otimes w_{kl}$, and $s_{ij} \otimes s_{kl}$. For compactness of notation, let the symbol $\{ijkl\}$ denote the tensor $\lambda_i \lambda_j d\lambda_k \otimes d\lambda_l$. Then the entire tensor for h_σ can be written as

$$h_\sigma = \sum_{i<j}^{4}\sum_{k<l}^{4} \{ikjl\}(a_{ij}+b_{ij})(a_{kl}+b_{kl}) + \{iljk\}(a_{ij}+b_{ij})(b_{kl}-a_{kl}) \\ + \{jkil\}(b_{ij}-a_{ij})(a_{kl}+b_{kl}) + \{jlik\}(b_{ij}-a_{ij})(b_{kl}-a_{kl}), \tag{5.15}$$

where the summations on the indices include 0. Writing out the h_σ in terms of barycentric indices,

$$h_{ij}|_\sigma = \sum_{k\leq l}^{4} e_{\sigma_{ijkl}}\ \lambda_{\sigma_k}\lambda_{\sigma_l} d\lambda_{\sigma_i}\otimes d\lambda_{\sigma_j}, \tag{5.16}$$

with the coefficients e_σ symmetric under the exchange of i, j and k, l.

We can finally turn to the discretization of the Lagrangian and the resulting equations of motion. The discrete action takes on the form:

$$S_d = \sum_{\sigma\in M}\int_\sigma \tilde{\partial}_\xi h_{\mu\nu}|_\sigma \tilde{\partial}_\rho h_{\alpha\beta}|_\sigma \Big[\tilde{\eta}^{\xi\nu}(\tilde{\eta}^{\alpha\beta}\tilde{\eta}^{\mu\rho} - \tilde{\eta}^{\alpha\rho}\tilde{\eta}^{\mu\beta}) + \\ \frac{1}{2}\tilde{\eta}^{\xi\rho}(\tilde{\eta}^{\mu\alpha}\tilde{\eta}^{\nu\beta} - \tilde{\eta}^{\mu\nu}\tilde{\eta}^{\alpha\beta})\Big] - \tilde{\eta}^{\alpha\mu}\tilde{\eta}^{\beta\nu}\gamma T_{\mu\nu}|_\sigma h_{\alpha\beta}|_\sigma, \tag{5.17}$$

with a volume element over the new metric $\tilde{\eta}$. Dropping the σ labels and tildes, the equation of motion associated to this action is given by

$$\eta_{\mu\nu}(\eta^{\xi\rho}\eta^{\alpha\beta} - \eta^{\xi\alpha}\eta^{\rho\beta})\partial_\xi\partial_\rho h_{\alpha\beta} - \eta^{\alpha\beta}(\partial_\alpha\partial_\beta h_{\mu\nu} + \partial_\mu\partial_\nu h_{\alpha\beta}) = \gamma T_{\mu\nu}, \tag{5.18}$$

along with the further constraints $h_\sigma = 0$ and $T_\sigma = 0\ \forall\ \sigma \in \partial M$. Plugging in the discretizations above yields the equation

$$\sum_{\alpha\leq\beta}^{4}\sum_{i\leq j}^{4}\Big(e_{\alpha\beta ij}[\eta^{\alpha\beta}\delta_{i\mu}\delta_{j\nu} + \eta_{\mu\nu}(\eta^{ij}\eta^{\alpha\beta} - \eta^{i\alpha}\eta^{j\beta})]d\lambda_\alpha\otimes d\lambda_\beta \\ + e_{\mu\nu ij}\ \eta^{ij} d\lambda_\mu\otimes d\lambda_\nu\Big) = \frac{\gamma}{2}c_{\mu\nu}d\lambda_\mu\otimes d\lambda_\nu, \tag{5.19}$$

where the Einstein summation convention has been dropped as to clarify the index gymnastics required between the summed over bases and the tensor contractions from the equation of motion.

Analyzing the gauge aspects of the linearized Fierz-Pauli formalism in this context is difficult, as the Rapetti construction does not generalize straightforwardly to tensor product spaces. In addition, the gauge portions of h at this level of discretization are not explicitly separated from the dynamical pieces, as is evident from

(5.16). Nonetheless, it is clear that the gauge condition lives at least at the same polynomial order as h, as the symmetry shifts h by $\partial_\mu \chi_\nu + \partial_\nu \chi_\mu$. If a conceptually similar analogue to the Rapetti identification holds over polynomial tensor fields, then the gauge portion should behave almost identically to that of electromagnetism.

5.3.3 1^{st} order FEEC discretization

Even at first-order, the above theory becomes algebraically cumbersome. Therefore, this section will only delineate the pieces required to assemble the equation of motion. I hope to investigate this in more depth in future work.

The stress-energy tensor is now linear in this framework, so we expect $\dim \mathcal{P}_1\Lambda^1 \otimes \mathcal{P}_0\Lambda^1 = 50$ degrees of freedom for this tensor. This gives a discretization of the form

$$T_{ij}|_\sigma = \sum_{k=0}^{4} c_{\sigma_{ijk}} \lambda_{\sigma_k} d\lambda_{\sigma_i} \otimes d\lambda_{\sigma_j}. \tag{5.20}$$

However, T must now satisfy the constraint $\partial^\mu T_{\mu\nu} = 0$ to ensure proper energy-momentum conservation. Using the same notation as the previous section, this translates to the equation:

$$\tilde{\eta}^{\rho\mu} \tilde{\partial}_\rho T_{\mu\nu} = \tilde{\eta}^{\rho\mu} \sum_{i=0}^{4} c_{\mu\nu i} \delta_{i\rho} d\lambda_\mu \otimes d\lambda_\nu = 0. \tag{5.21}$$

The corresponding equation for the perturbation metric h is further complicated as well, as there are 350 possible terms for cubic symmetric rank 2 tensors. The most general form for h is given by

$$h_{ij}|_\sigma = \sum_{k \le l \le m}^{4} e_{\sigma_{ijklm}} \; \lambda_{\sigma_k} \lambda_{\sigma_l} \lambda_{\sigma_m} d\lambda_{\sigma_i} \otimes d\lambda_{\sigma_j}, \tag{5.22}$$

with the additional condition that $h_{ij}|_\sigma = 0 \; \forall \; \sigma \in \partial M$.

In the next chapter, I will describe a potential computational avenue for considering discretizations of classical actions over nonlinear manifolds. Although this does not quite answer the questions posed above, it is entirely possible that a novel approach is required to make sense of discrete General Relativity.

Chapter 6

Geodesic Finite Element Method on Lorentzian metrics and Symplectic forms

"Donde venho, ninguém sabe, nem eu sei...Para onde vou, diz a lei tatuada no meu corpo..."

— Alda Lara

6.1 Introduction

So far in this work, I have considered spaces which are inherently flat. Of course, one must consider methods for non-flat spaces in discussing General Relativity. The work in this chapter describes an attempt at achieving this goal via the geodesic finite element method (abbreviated GFEM for short).

The GFEM is a conceptual framework independently developed by both Philipp Grohs and Oliver Sander. Overall, this method works over any manifold, and guarantees that the resulting finite element computations lie within the original manifold. This guarantee is especially useful within the context of general relativity, as landing back on the original manifold is a requirement for the energy-momentum to be locally conserved and for the equivalence principle to make sense.

In more technical terms, say we are given Lorentzian metric data at the nodes of a simplicial complex. We would then need to interpolate the data over each of the simplices such that the interpolant remains in the space of Lorentzian metrics. However, the space of Lorentzian metrics, when embedded into the space of $n \times n$ matrices, is not a convex space; this means that linear interpolation will generally result in an interpolant that takes does not take values in the space of Lorentzian metrics. More generally, polynomial interpolation does not respect the geometric properties of the manifold and is dependent on the choice of embedding – even when the interpolation does not leave the manifold.

The approach developed below resolves both of these issues by constructing a Riemannian metric on the space of interest, and then using the geodesic finite element method to interpolate that space over a given simplex. In the rest of this chapter, we will first take a generic look at this technique as applied to the quotients of $GL(n)$[1] by the classical Lie groups viewed as symmetric spaces, then hone in on specific applications to general relativity via the space of Lorentzian metrics, and Hamiltonian mechanics via the space of even-dimensional skew-symmetric matrices (i.e. symplectic forms). In the former case, we will consider the quotient of $GL(4)/O(1,3)$, and in the latter case we will explore the quotient $GL(2n)/Sp(2n)$. Before we begin, let us turn to a brief overview of the notation and framework involved in discussing geodesic finite elements. We will follow the conventions established in Sander's introductory papers to the method [San12]. In particular, pay close attention to Lemma 1 on the next page, as it is critical to the method: it guarantees that geodesic interpolation transforms appropriately under the isometry group of the manifold. This tells us that this method provides a robust representation of the manifold.

Definition 1. *The d-**dimensional simplex** is the set:*

$$\Delta^d = \left\{ w \in \mathbb{R}^{d+1} \mid w_i \geq 0, i = 0, \ldots, d, \textstyle\sum_{i=0}^d w_i = 1 \right\}.$$

Definition 2. *Let M be a Riemannian manifold. Let $w \in \Delta^d$ and $v = (v_0, \ldots, v_d) \in$*

[1]All Lie groups here will be considered over the real numbers. Of course, all the relevant machinery in this chapter carries over to the complex numbers by the usual substitutions (e.g. changing ^{T} to †). However, the symmetric spaces in these situations are of dubious physical relevance. For example, consider the complex symmetric space $GL(n)/U(n)$; this is the space of positive-definite Hermitian matrices. Although the correspondence to finite-dimensional quantum mechanical operators might seem physically appealing, the positive-definiteness strongly restricts the allowed results of physical measurements by these operators.

M^{d+1}. *Let dist* $: M \times M \to R$ *be a metric on* M*. Then we refer to*

$$\Upsilon : M^{d+1} \times \Delta \to M,$$

$$\Upsilon(v; w) = \underset{q \in \mathbb{R}}{\arg\min} \sum\nolimits_{i=0}^{d} w_i \operatorname{dist}(v_i, q)^2$$

as simplicial geodesic interpolation on M*.*

Lemma 1. *Let* G *be the isometry group of* M*, and let* G *act on* M^k *diagonally. Then for all* $v \in M^{d+1}$ *that is in a neighborhood of the diagonal,* $w \in \Delta$*, and* $g \in G$*, we have*

$$g \circ \Upsilon(v; w) = \Upsilon(g \circ v; w).$$

This implies that simplicial geodesic interpolation on M *is equivariant with respect to the action of the isometry group* G*.*

6.2 General features of symmetric spaces

Most of the definitions and notation in this section follows Wallner et. al [WYW11] and Helgason's classic text [Hel78] for working with homogeneous and symmetric spaces. A homogeneous space is a geometric object characterized by the action of group, as explored in Felix Klein's Erlangen program[2]. A symmetric space is a homogeneous space with extra requirements; intuitively, these requirements amount to having some notion of a reflection symmetry. For what follows below, we will limit ourselves to considering these constructions over Lie (sub)groups, their corresponding Lie algebras, and smooth manifolds, but the overarching theory is more general.

We can define a homogeneous space X in the following manner: let us say that you have a group G with identity element e that acts on on the set X. G acts as a transformation group in the expected way. We must then choose a base point b and identify it with some $x \in X$. This leads us to a description of X as a quotient G/K, where K is a symmetry of the space. More formally,

[2]It should come as no surprise that Emmy Noether's influence on the developments of this program was monumental, especially with respect to her second theorem. In fact, she even sent a postcard to Klein with a sketch of a proof for her theorem from Erlangen! [KS11]

Definition 3. *Let X denote a set and G denote a transformation group acting on X. For any $g \in G$, the mapping $x \to g \circ x$ maps $X \to X$ such that*

1. $e \circ x = x$
2. $(gh) \circ x = g \circ (h \circ x))$
3. $\forall x$ *and* y*, there exists a* g *such that* $g \circ x = y$.

Choose a base point b and identify a point $x \in X$ with the set of transformations which map the base b to x; in other words, if $\pi(g) = g \circ b$, identify x with $\pi^{-1}x$. $\pi^{-1}x$ is a set of type $g \cdot K$, where g is any element of $\pi^{-1}x$ and K consists of those $g \in G$ with $g \circ b = b$. Then, the space X can be called a ***homogeneous space*** *with $X = G/K = \{gK|g \in G\}$. If a point of X is written as "gK", then $g \circ hK = (gh)K$.*

The type of symmetric space we will consider here has been called the *infinitesimal version* of a symmetric space by [WYW11], as the global properties of the space are not relevant. In what follows, X is a smooth manifold, G, K are Lie groups, and $\mathfrak{g}$, $\mathfrak{k}$ denote their respective Lie algebras.

Definition 4. *A homogeneous space X is a* ***symmetric space*** *if and only if there exists a reflection $s : \mathfrak{g} \to \mathfrak{g}$ such that*

1. $s^2 = \mathrm{id}$
2. $s([u, v]) = [s(v), s(w)]\ \forall v, w \in \mathfrak{g}$
3. $\mathfrak{k}$ *is the* $+1$ *eigenspace of* s*, with* $\mathfrak{s}$ *denoting the* -1 *eigenspace.*

The first two conditions in the definition above correspond to the requirement of s being a Lie algebra automorphism. The existence of this reflection symmetry can be thought of as an identification of "poles" in G with respect to K. In addition, the complementary eigenspace $\mathfrak{s}$ is the tangent space of the base point $b \in X$ lifted to G. In other words, the $\circ$ mapping also maps tangent vectors via the differential: for a curve $x(t)$ with tangent vector $v = \frac{d}{dt}x(t)|_{t=0}$, the mapping $g \circ x$ maps v to $g \odot v = \frac{d}{dt}(g \circ x(t))|_{t=0}$. Furthermore, the tangent vector attached to the base point b can be uniquely identified with a vector $\tilde{v} \in \mathfrak{s}$.

As it turns out, it is this property that makes the definition of the induced exponential function over X meaningful. Let exp denote the exponential over the group G. This leads to the following definition:

Definition 5. *In a symmetric space $X = G/K$, let v represent the vector attached to the base point b, and $w = g \odot v$ be the vector attached to $x = g \circ b$. Then the* ***induced exponential function*** Exp *is given by* $\mathrm{Exp}_x(w) = \mathrm{Exp}_{g\circ b}(g \odot v) := g \exp(\tilde{v}) \circ b$, *where $\tilde{v} \in \mathfrak{s}$ represents v.*

The induced function Exp then lets us move within the symmetric space X, due to the structure from the overarching space G. In a sense, this exponential can be thought of as a distance function on the symmetric space: as mentioned above, the identification of the “poles” makes this notion work. Even though the choice of $g \circ b = x$ is not unique, it can be shown that $\mathrm{Exp}_x(w)$ does not depend on this choice, as Exp is an invariant with respect to transformations in G.[3] The existence of the induced exponential function also means that there is a corresponding logarithm, as induced by G. For most applications, the value of the matrix is not globally well-defined as it is generically non-unique, but this is not in general a major obstacle.

6.3 Symmetric spaces from classical Lie groups

Although it is not entirely obvious from the above construction of symmetric spaces, the choice of the base point b almost entirely determines the nature of the resulting space X. This statement is especially true in the context of classical Lie groups, as the $\circ$ mapping is taken as a symmetrized product of the form gxg^{α}, where α is some involution[4]. In fact, when considering a quotient space $X = G/K$ where $G = GL(n)$ and K is a classical Lie group[5], a few interesting simplifications occur. We will call these spaces classical symmetric spaces.

As it turns out, the above formula for the exponential function simplifies to the form $\mathrm{Exp}_x(w) = x \exp(x^{-1}w)$ for classical symmetric spaces.

[3]See [WYW11] for a brief proof.

[4]For spaces of direct physical interest, the involution in question is the transpose; considering all the classical Lie groups for K leads to a complete description of the space of quadratic forms.

[5]More specifically, we are considering the classical Lie groups to be the orthogonal, symplectic, and unitary groups. The quotient of $GL(n)/SL(n)$ is not very interesting.

Theorem 4 (Simplified Exponential Function for Classical Symmetric Spaces)**.** *Let $x, b \in GL(n)$. Assume an SVD for x such that $x = U\Sigma V^T$ with $U, V \in O(n)$ and Σ diagonal. If the additional conditions $Ub^T = V$ and $\Sigma b = b\Sigma$ hold, then $\mathrm{Exp}_x(w) = x\exp(x^{-1}w)$.*

Before proceeding with the proof, let us discuss the given conditions on the SVD. First, consider $\Sigma b = b\Sigma$. This commutativity constraint implies that either b is diagonal or that Σ is proportional to the identity matrix. This first scenario holds true for the orthogonal and unitary groups, and the second holds true for the symplectic group as the eigenvalues of symplectic matrices come in block-diagonal pairs. The condition $V = Ub^T$ ties back into the form of the $\circ$ mapping mentioned previously for classical Lie groups; essentially, we are ensuring that the second matrix V leads to some form of spectral decomposition for x. These two combined constraints most likely apply to other types of symmetric spaces, but again: we are only interested in cases of physical interest and thus focus on the classical symmetric spaces.

We now turn to a proof of the simplification. The details for the specific cases of the Lorentzian metrics and the skew-symmetric forms will be demonstrated in their respective sections.

Proof. Let $g = U\Sigma^{\frac{1}{2}}$, where $\Sigma^{\frac{1}{2}}$ is any matrix square-root of Σ. This implies that $\pi(g) = g \circ b = gbg^T = U\Sigma^{\frac{1}{2}}b\Sigma^{\frac{1}{2}}U^T = U\Sigma bU^T = U\Sigma V^T = x$, thus connecting $GL(n)$ to the symmetric space implicit in this construction. Now, plugging in the above into the induced exponential function:

$$\begin{aligned}
\mathrm{Exp}_x(w) &:= g\exp(\tilde{v}) \circ b \\
&= g\exp(\tilde{v})b\exp(\tilde{v})^T g^T \\
&= g\exp(\tilde{v})b\exp(\tilde{v}^T)g^T \\
&= gb\exp(b^{-1}(\tilde{v}b + b\tilde{v}^T))g^T \\
&= gb\exp(b^{-1}v)g^T \\
&= gb\exp(b^{-1}g^{-1}wg^{-T})g^T \\
&= gbg^T\exp(g^{-T}b^{-1}g^{-1}w) \\
&= x\exp(x^{-1}w).
\end{aligned}$$

$\square$

6.4 Lorentzian metrics and General Relativity

The space of all Lorentzian metrics, which we will denote by $LM(1,3)$, can be thought of as the quotient of $GL(4)$ to the Lorentz group $O(1,3)$. Translated into the notation given in the exposition above, we have $X = LM(1,3)$, $G = GL(4)$, and $K = O(1,3)$. The mapping $\circ$ will act as $g \circ x = gxg^T$, and the base point b will be taken to be the Minkowski metric, $b = \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. This can be trivially shown to be a homogeneous space. The intuition for this comes from the fact that the metric is a bilinear form which can be represented as a matrix; the subset of such matrices which is invariant to the Lorentz transformations should provide us with all allowed Lorentzian metrics. In equation form, for any $x \in LM(1,3)$, there exists a non-unique $g \in GL(4)$ such that $x = g \circ b = gbg^T$, with the non-uniqueness given by transformations in $O(1,3)$.

The appropriate s reflection that proves that $LM(1,3)$ is indeed a symmetric space is given by the mapping $s(v) = -bv^Tb$. It is easy to show that s is a Lie algebra automorphism over $\mathfrak{gl}_4$, the Lie algebra of $GL(4)$. To prove that the given s satisfies the third property in definition 4, we set up the eigenvalue equation, with $v \in GL(4)$:

$$s(v) = -bv^Tb = v.$$

Using the form $v = \begin{bmatrix} a & b^T \\ c & D \end{bmatrix}$, where a is a scalar, b and c are 3×1 vectors, and D is a 3×3 matrix, we have:

$$\begin{bmatrix} -a & c^T \\ b & -D^T \end{bmatrix} = \begin{bmatrix} a & b^T \\ c & D \end{bmatrix}.$$

The above conditions on v are exactly those required for $\mathfrak{so}_{1,3}$, the Lie algebra of $O(1,3)$, and thus $LM(1,3)$ must be a symmetric space. Repeating the above calculation for the -1 eigenspace yields the conditions:

$$\begin{bmatrix} a & -c^T \\ -b & D^T \end{bmatrix} = \begin{bmatrix} a & b^T \\ c & D \end{bmatrix}.$$

This space $\mathfrak{s}$ has a symmetric 3×3 block, an antisymmetric 3×1 row, along with an unconstrained scalar. This space does not have a standard name in the literature as it is rarely considered in its own right. In this work, we will simply denote it as $\mathfrak{lm}_{1,3}$. The relation to the space $\mathfrak{k} = \mathfrak{so}_{1,3}$ and the structure of $LM(1,3)$ is fairly clear, so we will not worry about the risk of considering $\mathfrak{lm}_{1,3}$ as the "Lie algebra" of $LM(1,3)$.

The final piece required before turning to the exponential is the calculation of the vector $\tilde{v} \in \mathfrak{s}$, which represents the tangent vector v attached to the base point. Setting $v = \frac{d}{dt}|_{t=0}\pi(e + t\tilde{v})$, we find that $\tilde{v}b + b\tilde{v}^T = v \implies \tilde{v} = \frac{1}{2}vb \in \mathfrak{s}$.

The exponential function $\mathrm{Exp}_x(w)$ simplifies to the form $x\exp(x^{-1}w)$ as shown above, since the basepoint b is diagonal, and the decomposition $x = U\Sigma bU^T$ simplifies to $U\Lambda U^T$, where Λ is the matrix of eigenvalues of x.

6.5 Skew-symmetric matrices and Hamiltonian Mechanics

Similarly, the space of non-degenerate, even-dimensional, skew-symmetric forms $Skew(2n)$ can be written as a quotient of $GL(2n)$ by $Sp(2n)$. $Skew(2n)$ is in fact identical to the space of symplectic bilinear forms. Thus, viewing $Skew(2n)$ lets us move between all possible symplectic vector spaces by moving between the associated forms. Again, translating into the notation given in the previous exposition, we have $X = Skew(2n)$, $G = GL(4)$, and $K = Sp(2n)$. The mapping $\circ$ will act as $g \circ x = gxg^T$, and the base point b will be taken to be the base symplectic matrix $\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, where 0 and I_n represent the $n \times n$ zero and identity matrices respectively. As before, this can be trivially shown to be a homogeneous space.

The appropriate s reflection that proves that $Skew(2n)$ is indeed a symmetric space is given by the mapping $s(v) = \Omega v^T \Omega$. As before, it is easy to show that s is a Lie algebra automorphism over $\mathfrak{gl}_4$, the Lie algebra of $GL(4)$. To prove that the given s creates the appropriate eigenspaces (i.e. that it satisfies the third property in definition 4), we set up the eigenvalue equation, with $v \in GL(4)$:

$$s(v) = \Omega v^T \Omega = v.$$

Using the form $v = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A, B, C, and D are $n \times n$ matrices, we have:

$$\begin{bmatrix} -D^T & B^T \\ C^T & -A^T \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

The above conditions on v are exactly those required for $\mathfrak{sp}_{2n}$, commonly called the space of Hamiltonian matrices and the Lie algebra of $Sp(2n)$, and thus $Sp(2n)$ must be a symmetric space. Repeating the above calculation for the -1 eigenspace yields the conditions

$$\begin{bmatrix} D^T & -B^T \\ -C^T & A^T \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Unlike in the Lorentzian case, this space $\mathfrak{s}$ does in fact have a standard name: these conditions define what are typically called the skew-Hamiltonian matrices [FMMX99], which we will denote in this work as $\mathfrak{sh}_{2n}$. As it turns out, there is a symmetry between the Hamiltonian and skew-Hamiltonian spaces, as the square of a Hamiltonian matrix is skew-Hamiltonian. Furthermore, as detailed in [Wat05], there is nearly no distinction between the two spaces. Indeed, the only proper subspace they contain is that of the alternating Hamiltonian matrices: $2n \times 2n$ matrices of the form $\begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix}$, where A is an $n \times n$ matrix.

The final piece required before turning to the exponential is the calculation of the vector $\tilde{v} \in \mathfrak{s}$, which represents the tangent vector v attached to the base point. Once more, setting $v = \frac{d}{dt}|_{t=0}\pi(e+t\tilde{v})$, we find that $\tilde{v}\Omega + \Omega\tilde{v}^T = v \implies \tilde{v} = \frac{1}{2}v\Omega \in \mathfrak{s}$.

At first glance, the applications of GFEM to the space of symplectic forms might seem a little less obvious than that of the Lorentzian metrics. However, the interpolation of a symplectic form could have interesting applications in complicated Hamiltonian systems. In fact, since a phase space requires only a Hamiltonian function H and a symplectic form ω, the nonlinearities and complexity of a given H could be transformed into that of ω as needed throughout phase space. Essentially, this method allows for a straightforward way to stitch symplectomorphisms as needed within a space, and thus provide a numerical approach to finding canonical transformations over a discretized phase space.

6.6 Generalized Polar Decomposition

The standard polar decomposition is generally given for a square complex matrix A by

$$A = UP.$$

where U represents a unitary matrix and P is a positive-semidefinite Hermitian matrix. This decomposition mirrors the famous representation of complex numbers in the form $z = re^{i\theta}$; here, U represents a complex rotation and P the "radial" component of the matrix A. Of course, the above can be used for real matrices, making U orthogonal and P automatically Hermitian.

The most natural way to generalize this notion to a variety of Lie groups is through the general Cartan decomposition, and has been extensively explored in the work of Munthe-Kaas and others in [MKQZ01] and [KMKZ09]. Here we will briefly outline the intuition behind this method, then apply it to our the context of GFEM.

Symmetric spaces are guaranteed to have a general Cartan decomposition, as this decomposition is an intrinsic part of their definition. More specifically, the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$,[6], with each space corresponding to the appropriate ± 1 eigenvalues of s, is in fact a general Cartan decomposition for the group's Lie algebra. We then have that

$$\gamma = \kappa + \sigma,$$

where $\gamma \in \mathfrak{g}, \kappa \in \mathfrak{k}$, and $\sigma \in \mathfrak{s}$. Multiplying by a scalar t and exponentiating both sides gives the equality

$$\exp(t\gamma) = \exp(t(\kappa + \sigma)).$$

Call $\exp(t\gamma) = g$, which clearly implies $g \in G$. As proven in Theorem 3.1 of [MKQZ01], for t sufficiently small, g can be taken to have the generalized polar decomposition:

$$g = ks = \exp(\kappa(t)) \exp(\sigma(t)), \tag{6.1}$$

[6] As these two subspaces are complementary, the sum is direct.

where $k \in K, s \in X$, and $\kappa(t) \in \mathfrak{k}, \sigma(t) \in \mathfrak{s}$. In comparison to the standard polar decomposition, it is easy to see that the matrix k acts as the generalized rotational analogue to U, while s is the analogous radial component to P.

This notion then provides some intuition for the SVD described above in Theorem 4 over the quotients of classical Lie groups. The matrix U plays the role of the rotation while $\Sigma b U^T$ acts as the radial portion. More precisely, in the Lorentzian case, elements of $\mathfrak{so}_{1,3}$ takes on the mantle of generalized rotations (physical rotations and boosts), and the elements of $\mathfrak{lm}_{1,3}$ provide the metric "radius" to be transformed. Similarly, in the symplectic case, $\mathfrak{sp}_{2n}$ provides the space of symplectomorphisms as a rotation, and the Hamiltonian matrices behave as a radial component.

6.7 Induced exponential as a metric and geodesic interpolation

In this section, I will provide a rough sketch on how to use the induced exponential function on classical symmetric spaces above as a distance function in the context of geodesic finite elements, and how it provides a nonlinear interpolation that is in some sense 'geodesic.'

Assume we are applying the geodesic finite element method over a classical symmetric space X with basepoint b and multiplication $g \circ x = gxg^T$. Then, as discussed in Theorem (4), the induced exponential function from $GL(n)$ takes on the form

$$y = \mathrm{Exp}_x(w) = xe^{x^{-1}w},$$

with $x, y \in X$ and w a tangent vector attached to x. To create some notion of distance out of this exponential, we must search for a scalar, positive-definite function $\mathrm{dist}(\cdot,\cdot)$ that is symmetric in its arguments, i.e. $\mathrm{dist}(x,y) = \mathrm{dist}(y,x)$ and $\mathrm{dist}(x,y) = 0$ iff $x = y$.

Clearly, in the above expression, w acts as a proxy of some sort for the distance between the two elements. Inverting the exponential gives

$$w = x\log(x^{-1}y). \tag{6.2}$$

Symmetrizing this equation and taking the trace gives a candidate expression

$$\operatorname{dist}(x,y)^2 = \operatorname{Tr}(\log(x^T y^{-T}) y^T x \log(x^{-1}y))^2, \tag{6.3}$$

which indeed satisfies the requirements of a metric function over X. In terms of the Lorentzian example described previously, equation (6.3) represents a metric over $LM(1,3)$; in other words, it tells us how far apart two metric tensors are from each other.

Given a simplicial complex[7] with nodal values in X, the machinery of the geodesic finite element method allows us to smoothly interpolate between the nodes, while still guaranteeing that the interpolated values remain in X and preserve its symmetries [San12]. In short, given a linear interpolation

$$v = \sum_{i=0}^{n} v_i \lambda_i,$$

with λ_i representing the barycentric coordinates[8], the appropriate notion required for a nonlinear interpolation Υ is

$$\Upsilon = \operatorname*{arg\,min}_{q\in M} \sum_{i=0}^{n} \lambda_i \operatorname{dist}(v_i, q)^2. \tag{6.4}$$

which is called simplicial geodesic interpolation on the manifold M, as this equation minimizes over the distance function. Correspondingly, the above equation makes it clear that the isometries of M are preserved in this interpolation, and that the values given by Υ lie within the manifold. Applying this method to variational problems introduces an extra layer of complexity, as the corresponding algorithm must minimize the action over the $\operatorname{arg\,min}_{q\in M}$ in the interpolation.

[7] More general complexes are allowed, but we will look only look at simplicial complexes to remain consistent with the rest of this work.

[8] Again, the method applies to more general interpolations than barycentric coordinates, but we will restrict our attention to the simplest case for consistency.

Chapter 7

Conclusion

"With that disappearance... came the end of Eternity.

—And the beginning of Infinity."

— Isaac Asimov

7.1 Summary

In this work, I demonstrated a variety of results relating to the discretization of classical field theories; in particular, I expanded the structure of Finite Element Exterior Calculus, discussed symmetry aspects of both continuous and discrete classical field theories, developed a variational method for simulating electromagnetism and linearized General Relativity over Minkowski spacetime, and laid the groundwork for working with nonlinear manifolds via the Geodesic Finite Element Method. I will now turn to summarize the major results from each chapter.

7.1.1 Extensions to FEEC

In Chapter 2, I showed how to expand FEEC to include a metric structure through the introduction of the Hodge star $\star$, which forces the underlying affine space to be Euclidean or Lorentzian in nature. This trivially extends FEEC to cover the space of polynomial vector fields, and allows a variety of operators from the continuous theory to be applied to the spaces of polynomial differential forms $\mathcal{P}_r\Lambda^k$, such as the codifferential δ, the Hodge-Laplacian Δ, the raising and lowering operators $\sharp$ and $\flat$, and other Hodge-dualized operations such as the co-Koszul $\xi_v = (-1)^{n(k-1)+1} s \star \kappa_v \star$.

Another major theme of this chapter involved identifying the corresponding dual operators over integration to aforementioned operations in FEEC. This included introducing a sense of 'dynamic' duality with respect to vector fields, as shown by the extrusion $E_X(M,t)$ and flow $\Phi_X(M,t)$ operators. I also fleshed out Harrison's characterization of the geometric Hodge dual to simplicial complexes, and extrapolated the results to other dual operations such as the geometric co-boundary $\diamond = (-1)^{n(k-1)+1} s \star \partial \star$ and the geometric Laplacian $\Box = \partial \diamond + \diamond \partial$.

I extended this theme into Chapter 3, in which I tackled the geometric interpretation of polynomial order over the degrees of freedom in FEEC. I highlighted issues and ambiguities of interpretation with the Rapetti construction, a framework which localizes polynomial refinement with homotheties. Additionally, I suggested that a consistent interpretation of polynomial order might not be possible, as the usual notion of duality over integration must include the 'dynamical' dualities discussed in Chapter 2.

7.1.2 Noether Theorems

My work on Noether's Theorems in Chapter 4 is a pedagogical walkthrough of Noether's original paper on symmetries of invariant functionals. I distinguished the First and Second Theorems in terms of the actions of the symmetries involved in each case, accompanied by examples from classical physics. I showcased some of the modern terminology associated with the two theorems, and provided an alternative equation to accompany the use of the term "Noether's Theorem"; this equation does not impose a specific symmetry on the action, and does not erase the effects of boundary or topological data.

7.1.3 Applications of Spacetime FEEC

Chapter 5 applies the results of the previous three chapters to spacetime electromagnetism and linearized General Relativity. By using FEEC over Minkowski spacetime, I constructed a discrete action, derived the associated equations of motion, and considered Noether's Theorems as applied to the discrete symmetries for each case. Using the Rapetti framework, I showed that the gauge degrees of freedom in electromagnetism worked at higher polynomial order, and hence lived on a different dual space than the dynamical field variables. I found that a proper dual to the symmetric tensor product is necessary to fully make sense of discrete linear GR in this setting.

7.1.4 Geodesic Finite Element Method

In Chapter 6, I discussed the Geodesic Finite Element Method and its application to symmetric spaces. I provided a brief introduction to the theory of symmetric spaces, then found a result for the exponential function induced on classical symmetric spaces, which are quotients of $GL(n)$ by classical Lie groups. I worked out the specifics of this framework as applied to the space of Lorentzian metrics $LM(1,3) = GL(4)/O(1,3)$ and the space of skew-symmetric forms $Skew(2n) = GL(2n)/Sp(2n)$. I showed the intuition behind these results through the generalized polar decomposition, then demonstrated how GFEM allows for nonlinear interpolation through the construction of a distance function over classical symmetric spaces.

7.2 Future work

This section will delineate some aspects of future work that can follow from the material presented in this thesis.

7.2.1 Geometric dual to polynomial order in FEEC

I intend to further analyze the geometric interpretation of polynomial order in FEEC, in the hopes that the answer might yield a better interpretation of the dual elements to the variety of spaces of discrete differential forms in FEEC. Currently, studying the dynamical dual to the Koszul over integration seems to be the best avenue for this approach, as it is a fundamental building block of the ladder of the $\mathcal{P}_r\Lambda^k$ spaces. A complete theory, if it exists, would probably need to connect these ideas to the space of polynomial vector fields via the metric.

7.2.2 Noether Theorems

A pedagogical restatement of Noether's Theorems in more modern language is my next goal for this line of work, as I believe it would assist both physics experts and novices alike in understanding the role of symmetries in physical theories. Part of this endeavor would require rewriting the notation in the GiMmsy preprints in a language and notation more accessible to physicists. I also plan to make the interplay of boundary effects, topology, and symmetries explicit, with well-known, motivating examples from electromagnetism.

7.2.3 Applications of spacetime FEEC

Extensions of the FEEC spacetime electromagnetism formulation would involve the inclusion of dynamical matter and non-flat spatial geometries. Such additions would create nontrivial Noether currents and constraints which could be verified through the lens of FEEC. In fact, even the addition of linear matter interactions (i.e. permeability and permittivity tensors) would extend the practical application of the method. Furthermore, I plan on examining how to best make use of gauge freedom, numerically speaking: for a given simulation, the computations might be optimized by considering patches of specific gauges in certain regions, which would then need to be stitched together appropriately.

Future work for linearized GR would involve extending the framework to perturbations over non-flat background spacetimes. This requires generalizing the notion of "perturbing a metric", as for a general manifold there is no equivalent to the Minkowski metric. Essentially, one must linearize the relevant operations (e.g. the equation of motion) on the metric to find an analogous notion of perturbations as in flat spacetime. This idea is thoroughly fleshed out in [FM79]; Chapter 7.5 of [Wal84] provides a brief summary of the method for the interested reader. Although the technique is technically not variational as presented, the overall method is still compatible with the structure of FEEC and hence the discretization presented above. This would allow for the simulation of gravitational waves over physically interesting spacetimes, e.g. the Kerr black hole.

Additionally, there is much to be explored in the sense of gauge degrees of freedom and their localization in the context of linearized GR. More foundational work is required to make sense of the tensor product in a sense that is appropriate to the dual spaces, while still keeping general tensor operations consistent with the known dualities for forms. In short, a proper generalization of FEEC to the full space of tensors over a simplicial complex is the natural next step. This direction of research would necessarily overlap with the work done in the ever-growing field of discrete differential geometry, which could potentially provide fruitful avenues of approach towards extending FEEC. Ideally, the method presented above could be connected to Regge calculus, in a similar vein to Christiansen's work [Chr11].

7.2.4 Applications of GFEM to General Relativity

In future work, I intend to focus on applications of the Geodesid Finite Element Method to GR. A simple proof of concept of the interpolation scheme over a physical, analytically solvable situation (e.g. the Schwarzschild metric) would provide some sense of the error involved in the numerical approximation. In the long run, I plan to work on an application of the above to the ADM formalism [ADM59], which would evolve a spatial metric over some initial boundary data, and would hopefully provide a field theory which faithfully maps to GR in the continuum limit.

7.3 Conclusion

My hope is that the reader that has made it this far has been left with a sense of the vastness of the topic of discretization: it is difficult to make rigorous in a general fashion for field theories, yet the intuition behind the word is appealing and deceptively simple. Using variational principles to ensure symmetries are preserved provides a foundation for building discretizations that approach the continuous theory and display good, long-term behavior. The frameworks and methods discussed in this work showcase one possible discretization which contains most of the necessary machinery required for a faithful representation of classical field theories. Indeed, the last major obstacle to overcome requires generalizing the above to non-flat manifolds. I look forward to what the future brings on the topic; I hope the reader does too.

Appendix A

Metric Reformulation of Whitney forms: Proof by induction

This is a proof of equivalence of the metric reformulation of Whitney forms to the Whitney's original barycentric representation.

Proof. We will assume a flat n-dimensional manifold with some signature containing an n-simplex σ. Then, a Whitney j-form over a j-subsimplex ρ must satisfy the following set of constraints:

$$^{j}w_{\rho}(v_i) = 0, \forall v_i \in \tau,$$

where $\tau = \sigma \setminus \rho$. Note that this implies that $^{j}w_{\rho}(x) = 0, \forall x$ spanned by τ. Since Whitney forms are linear functions of the position vector x, they must be composed of a product of difference 1-forms $(v_i - x)^{\flat}$. Furthermore, Whitney forms are anti-symmetric under vertex exchange; in other words, an even/odd permutation of ρ's vertices will multiply the form by a ± 1 respectively. Thus, we are lead to a formula of the form

$$^{j}w_{\rho} = C_{\sigma,j} sgn(\rho \cup \tau)(\star \bigwedge_{v_k \in \tau} (v_k - x)^{\flat}). \tag{A.1}$$

The $\star$ is the usual Hodge star required to convert the $(k-j)$-form from the wedge product into a j-form, and $sgn(\rho \cup \tau)$ is required to maintain consistency with the arbitrary ordering of the vertices in ρ and τ. The constant $C_{\sigma,j}$ can be constrained by the conventional normalization requirement:

$$\int_{\rho} {}^{j}w_{\rho} = 1. \tag{A.2}$$

This will be shown by induction on j, the order of the form. Starting with the first base case $j = 0$ and $\rho = [v_0]$, we find that

$$\begin{aligned}\int_{[v_0]} {}^{0}w_{[v_0]} = {}^{0}w_{[v_0]}(v_0) &= \frac{sgn(\sigma)}{\star vol(\sigma)} \frac{0!}{n!} (\star \bigwedge_{m=1}^{n} (v_m - v_0)^{\flat}) \\ &= \frac{1}{n! \star vol(\sigma)} \star ((v_1 - v_0)^{\flat} \wedge ... \wedge (v_n - v_0)^{\flat}) = 1.\end{aligned}$$

Thus, equation 2.27 for $j = 0$ corresponds exactly to barycentric coordinates. Next, we must consider the second base case $j = 1$. Taking $\rho = [v_0, v_1]$, the LHS of the normalization condition becomes:

$$\begin{aligned}&\int_{[v_0,v_1]} {}^{1}w_{[v_0,v_1]} = \frac{1}{n! \star vol(\sigma)} \int_{[v_0,v_1]} (\star \bigwedge_{m=2}^{n} (v_m - x)^{\flat}) \\ &= \frac{1}{n! \star vol(\sigma)} \int_{[v_0,v_1]} \star (v_2^{\flat} \wedge ... \wedge v_n^{\flat} - x^{\flat} \wedge \sum_{m=2}^{n} (-1)^{n-m} v_2^{\flat} \wedge v_3^{\flat} \wedge ... \hat{v}_m^{\flat} \wedge ... v_n^{\flat})) \ .\end{aligned}$$

Now, let's exploit the translation invariance of the representation and set v_2 as our origin. Note that translation invariance is not required to complete the proof, but leads to the most geometrically simple result. The equation above then simplifies to

$$\frac{-1}{n! \star vol(\sigma)} \int_{[v_0 - v_2, v_1 - v_2]} \star ((x - v_2)^{\flat} \wedge (v_3 - v_2)^{\flat} \wedge ... (v_n - v_2)^{\flat})),$$

as only the term without $v_2^{\flat}$ survives. Since the integration is over the 1-simplex $[v_0, v_1]$, we can parametrize our path by $x(t) = (v_1 - v_0)t + v_0$ with $x'(t) = (v_1 - v_0)$, where $t \in [0, 1]$. The integral above then turns into:

$$\begin{aligned}&= \frac{-1}{n! \star vol(\sigma)} \int_0^1 \left\langle \star (((v_1 - v_0)t + (v_0 - v_2))^{\flat} \wedge (v_3 - v_2)^{\flat} \wedge ... (v_n - v_2)^{\flat}), (v_1 - v_0)^{\flat} \right\rangle dt \\ &= \frac{-1}{n! \star vol(\sigma)} \left\langle \star ((\frac{1}{2}(v_1 - v_0) + (v_0 - v_2))^{\flat} \wedge (v_3 - v_2)^{\flat} \wedge ... (v_n - v_2)^{\flat}), (v_1 - v_0)^{\flat} \right\rangle \\ &= \frac{-1}{n! \star vol(\sigma)} \star ((v_1 - v_0)^{\flat} \wedge (\frac{1}{2}(v_1 - v_0) + (v_0 - v_2))^{\flat} \wedge (v_3 - v_2)^{\flat} \wedge ... (v_n - v_2)^{\flat}) \\ &= \frac{1}{n! \star vol(\sigma)} \star ((v_0 - v_2)^{\flat} \wedge (v_1 - v_2)^{\flat} \wedge (v_3 - v_2)^{\flat} \wedge ... (v_n - v_2)^{\flat}) = 1.\end{aligned}$$

Therefore, equation 2.27 for $j = 1$ corresponds exactly to Whitney 1-forms.

Next, as our inductive hypothesis, we will assume that the barycentric Whitney l-forms correspond to the $j = l$ case as shown in 2.27.

Then, for the Whitney $(l+1)$-forms over $\rho = [v_0, ...v_{l+1}]$, we can use the following well-known decomposition from the usual formulation in barycentric coordinates:

$$
\begin{aligned}
{}^{l+1}w_{[v_0,v_1,...v_l,v_{l+1}]} &= (l+1)\frac{{}^{l}w_{[v_0,v_1,...v_l]} \wedge {}^{1}w_{[v_l,v_{l+1}]}}{{}^{0}w_{[v_l]}} \\
&= \frac{1}{\star vol(\sigma)}\frac{(l+1)!}{n!}\frac{(\star \bigwedge_{i=l+1}^{n}(v_i - x)^\flat) \wedge (\star \bigwedge^{n}_{\substack{m=0 \\ m\neq l,l+1}}(v_m - x)^\flat)}{\star \bigwedge_{\substack{k=0 \\ k\neq l}}^{n}(v_k - x)^\flat}.
\end{aligned}
$$

The coefficient matches the normalization as given in 2.27, but now we must answer the question of whether the fraction of wedge products yields the correct behavior. Let's consider the vertex constraints imposed earlier in the proof:

- Taking $x \in [v_{l+2}, ...v_n]$, all three products vanish identically.
- Upon inserting $x \in [v_0, v_1, ...v_{l-1}]$, the fraction converges to a finite answer, as both ${}^{1}w_{[v_l,v_{l+1}]}$ and ${}^{0}w_{[v_l]}$ contain the same identically vanishing term.
- If $x = v_{l+1}$, then both ${}^{l}w_{[v_0,v_1,...v_l]}$ and ${}^{0}w_{[v_l]}$ instead contain the same, identically vanishing term, leaving a finite answer.
- If $x = v_l$, then none of the terms in fraction are zero.

Thus, since ${}^{l+1}w_{[v_0,v_1,...v_{l-1}]}(x) = 0, \forall x \in \tau$, and ${}^{l+1}w_{[v_0,v_1,...v_{l-1}]}(x)$ is finite $\forall x \in \rho$, as required by our construction, we are left to conclude that the equation above must correspond to

$$
{}^{l+1}w_{[v_0,v_1,...v_l,v_{l+1}]} = \frac{1}{\star vol(\sigma)}\frac{(l+1)!}{n!}(\star \bigwedge_{m=l+2}^{n}(v_m - x)^\flat),
$$

as it is the only product of differential forms that can mimic the behavior stipulated at the beginning of the proof. Thus, the vertex representation for Whitney forms given by 2.27 is completely equivalent to the barycentric formulation $\forall j$. $\square$

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