Abstract. This paper develops the theory of discrete Dirac reduction of discrete Lagrange–Dirac systems with an abelian symmetry group acting on the configuration space. We begin with the linear theory and, then, we extend it to the nonlinear setting using retraction compatible charts. We consider the reduction of both the discrete Dirac structure and the discrete Lagrange–Pontryagin principle, and show that they both lead to the same discrete Lagrange–Poincaré–Dirac equations. The coordinatization of the discrete reduced spaces relies on the notion of discrete connections on principal bundles. At last, we demonstrate the method obtained by applying it to a charged particle in a magnetic field, and to the double spherical pendulum.

1. Introduction

Symmetry reduction plays a central role in the field of geometric mechanics [1; 4; 20], and it involves expressing the dynamics of a mechanical system with symmetry in terms of the equivalence classes of group orbits on the space of solutions. This allows one to derive reduced equations of motion on a lower-dimensional reduced space which is obtained by quotienting the phase space by the symmetry action. The modern approach to symmetry reduction was introduced in Arnold [3], Smale [31], Meyer [27], and Marsden and Weinstein [23], but the notion of symmetry reduction arises in earlier work of Lagrange, Poisson, Jacobi, and Noether.

Discrete variational mechanics [11; 15; 24] provides a discrete (in time) notion of Lagrangian dynamics, based on a discrete Hamilton’s principle. This leads to discrete flow maps that are symplectic, and exhibit a discrete Noether’s theorem. In turn, this naturally raises the question of whether one can develop a corresponding theory of symmetry reduction for discrete variational mechanics. Interest in this direction was motivated in part by attempts to understand the integrable discretization of the Euler top due to Moser and Veselov [28]. Prior work on discrete symmetry reduction includes a constrained variational formulation of discrete Euler–Poincaré reduction [7; 25], the associated reduced discrete Poisson structure [26], discrete Euler–Poincaré reduction for field theories [32] and discrete fluids [29], discrete Lie–Poisson integrators [5; 17], discrete higher-order Lagrange–Poincaré reduction [6], and a discrete notion of Routh reduction for abelian groups [10]. The resulting symplectic and Poisson integrators can be viewed as geometric structure-preserving numerical integrators, and this is an active area of study that is surveyed in [9]. In addition, discrete reduction theory can also be expressed in terms of composable groupoid sequences, which was the approach introduced in [35], and explored further in [18], and extended to field theories in [33].

Reduction theory can be addressed either in terms of the reduction of geometric structures like the symplectic or Poisson structures, or in terms of the reduction of variational principles. In this paper, we will start with the discrete Lagrange–Dirac mechanics that was developed in [14], which can be viewed as a discrete analogue of Lagrange–Dirac mechanics, that can be formulated both in terms of Dirac structures [36] and the Hamilton–Pontryagin variational principle [37]. In order to coordinatize the reduced spaces arising in the reduction that we will perform, we rely on the notion of discrete principal connections that was introduced in [10] and further developed in [8].

2020 Mathematics Subject Classification. 37J39, 65P10, 70G65, 70H33.

Key words and phrases. discrete mechanical systems, geometric numerical integration, Lagrange–Poincaré–Dirac equations, reduction by symmetries.

ÁLVARO RODRÍGUEZ ABELLA AND MELVIN LEOK
Overview. We will recall the notion of discrete principal connection and discrete Dirac mechanics in Section 2. Initially, we will consider the case where the configuration space $Q$ is a vector space, and the symmetry group $G$ is a vector subspace acting on $Q$ by addition, and show how the discrete Dirac structure is group-invariant, descends to a discrete Dirac structure on the quotient space in Section 3. In Section 4, we will show how the reduced discrete Dirac structure can be used to derive the reduced discrete equations of motion, after we derive the reduced discrete Dirac differential, and an atlas of retraction compatible charts [14] to develop a global discrete theory, whose local representatives recover the vector space theory that we considered in the earlier part of the paper. This is significant, because that implies that with respect to an atlas of retraction compatible charts, and an equivalent class of $q \in Q$ by $[q] \in \Sigma$. In general, we use a superscript to denote the space where $G$ acts, and brackets $[\cdot]$ to denote the corresponding equivalence classes.

The diagonal action on $Q \times Q$ is given by

$$\Phi^Q : G \times Q \times Q \rightarrow Q \times Q, \quad (g, (q_0, q_1)) \mapsto (g \cdot q_0, g \cdot q_1).$$

Likewise, the lift of the action to the cotangent bundle is given by

$$\Phi^{T^*Q} : G \times T^*Q \rightarrow T^*Q, \quad (g, p_q) \mapsto \left( (d\Phi^Q_{g^{-1}})_{g^{-1}_q} (p_q) \right),$$

where the superscript $*$ denotes the adjoint map. In other words, for each $v_{g,q} \in T_{g,q}Q$ we have

$$\langle g \cdot p_q, v_{g,q} \rangle = \left( p_q, (d\Phi^Q_{g^{-1}})_{g^{-1}_q} (v_{g,q}) \right).$$

The infinitesimal generators (or fundamental vector fields) of the action $\Phi^Q$ are denoted by $\xi_Q \in \mathfrak{X}(Q)$ for each $\xi \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of $G$, and analogously for the actions defined on the other spaces. In the same fashion, the momentum map $J : T^*Q \rightarrow \mathfrak{g}^*$ of the action is implicitly defined by $\langle J(p_q), \xi \rangle = \langle p_q, \xi_Q(q) \rangle$ for each $p_q \in T^*_Q Q$ and $\xi \in \mathfrak{g}$.

2.1. Group actions. Let $Q$ be a smooth manifold and $G$ be a Lie group acting freely and properly on $Q$ on the left, thus yielding a principal bundle $\pi_{Q,\Sigma} : Q \rightarrow \Sigma$, where $\Sigma = Q/G$. We denote the action $\Phi^Q : G \times Q \rightarrow Q$ by

$$\Phi^Q(g, q) = g \cdot q = \Phi^Q_g(q) = \Phi^Q_g(q)$$

and the equivalence class of $q \in Q$ by $[q] \in \Sigma$. In general, we use a superscript to denote the space where $G$ acts, and brackets $[\cdot]$ to denote the corresponding equivalence classes.

The diagonal action on $Q \times Q$ is given by

$$\Phi^{Q \times Q} : G \times Q \times Q \rightarrow Q \times Q, \quad (g, (q_0, q_1)) \mapsto (g \cdot q_0, g \cdot q_1).$$

Then, in Section 6, we show that, in the nonlinear setting, we can use the notion of retractions [2] and an atlas of retraction compatible charts [14] to develop a global discrete theory, whose local representatives recover the vector space theory that we considered in the earlier part of the paper. Finally, we will summarize our contributions in the conclusion, and discuss future research directions.

2. Preliminaries

2.1. Group actions. Now we suppose that $Q$ is a vector space, which allows us to identify

$$TQ = Q \times Q, \quad T^*Q = Q \times Q^*.$$
In addition, assume that $G \subset Q$ is a vector subspace acting by addition, i.e. $g \cdot q = g + q$ for each $g \in G$ and $q \in Q$. In this case, we have $g^{-1} = -g$ and the action on $T^*Q$ reduces to,

$$\Phi_T^* : G \times T^*Q \rightarrow T^*Q, \quad (g, (q_0, p_1)) \rightarrow (g + q_0, p_1).$$

To conclude, note that $\mathfrak{g} = G$ and $\mathfrak{g}^* = G^*$, and that the exponential map is the identity. Hence, $\xi Q(q) = \xi$ for each $\xi \in \mathfrak{g}$ and $q \in Q$. Subsequently, the momentum map is given by $(J(q_0, p_0), \xi) = \langle p_0, \xi \rangle$ for each $(q_0, p_0) \in T^*Q$ and $\xi \in \mathfrak{g}$.

2.2. Discrete principal connections. Discrete principal connections were first introduced in [10] and further studied in [8]. Let $Q$ be a smooth manifold and $G$ be a Lie group acting freely and properly on $Q$.

**Definition 2.1.** A discrete principal connection on the principal bundle $\pi_{Q, \Sigma} : Q \rightarrow \Sigma$ is a (smooth) function $\omega_d : Q \times Q \rightarrow G$ such that

(i) $\omega_d(q_0, q_1) = e$, the identity element, for each $q_0 \in Q$.

(ii) $\omega_d(q_0, q_1, q_1) = g_1 \omega_d(q_0, q_1)g_1^{-1}$ for each $(q_0, q_1) \in Q \times Q$ and $g_0, g_1 \in G$.

Discrete principal connections are not generally defined globally on $Q \times Q$, but only on a $G$-invariant open subset $U \subset Q \times Q$ containing the diagonal, i.e. $(q_0, q_0) \in U$ for each $q_0 \in Q$. Nevertheless, in the following we assume that they are globally defined in order to simplify the notation.

A discrete principal connection $\omega_d$ enables us to define the discrete horizontal bundle as

$$H_d = \{(q_0, q_1) \in Q \times Q \mid \omega_d(q_0, q_1) = 0\} \subset Q \times Q.$$

In addition, $\omega_d$ induces a discrete horizontal lift, i.e. a map $h_d : Q \times \Sigma \rightarrow Q \times Q$ that is the inverse of the diffeomorphism $(id_G \times \pi_{Q, \Sigma})|_{H_d} : H_d \rightarrow Q \times \Sigma$. We denote $\bar{h}_d = pr_2 \circ h_d : Q \times \Sigma \rightarrow Q$, where $pr_2$ denotes the projection onto the second component.

2.2.1. Local expression of discrete connections on vector spaces. Suppose that $Q$ is a vector space and $G \subset Q$ is a vector subspace acting by addition. Therefore, $\Sigma = Q/G$ is a vector space and the projection $\pi_{Q, \Sigma} : Q \rightarrow \Sigma$ is linear. In this case, we assume that the horizontal lift, $h_d : Q \times \Sigma \rightarrow Q \times Q$, is a linear map (where the vector structure on the product is given by the direct sum). This, in turn, ensures that $\bar{h}_d : Q \times \Sigma \rightarrow Q$ is also linear.

Working in a trivialization of $\pi_{Q, \Sigma}$, i.e. supposing that $Q = \Sigma \times G$ with $\pi_{Q, \Sigma} = pr_1$ and the vector structure given by the direct sum, then the horizontal lift of the discrete connection is determined by a map $h_d : Q \times \Sigma \rightarrow G$ defined by $h_d = pr_2 \circ \bar{h}_d$, that is,

$$\bar{h}_d(q_0, x_1) = (x_1, h_d(q_0, x_1)), \quad q_0 = (x_0, g_0) \in Q, \quad x_1 \in \Sigma.$$

Of course, $h_d$ is also linear and, from [8] Equation (4.2)], it satisfies

$$\omega_d(q_0, q_1) = g_1 - h_d(q_0, x_1), \quad q_0, q_1 \in Q.$$

In particular, $h_d(q_0, x_0) = g_0$ and $h_d(g + q_0, x_1) = g + h_d(q_0, x_1)$ for each $g \in G$ (to obtain this second equation we have used the equivariance of $\omega_d$).

On the other hand, the dual of $Q$ is given by $Q^* = \Sigma^* \times G^*$, and the adjoint of $h_d$ can be written as $h_d^* = (h_d^*_{\Sigma, Q}, h_d^*_{\Sigma, \Sigma}) : G^* \rightarrow Q^* \times \Sigma^*$ for some linear maps $h_d^*_{\Sigma, Q} : G^* \rightarrow Q^*$ and $h_d^*_{\Sigma, \Sigma} : G^* \rightarrow \Sigma^*$. They are related to the adjoint of $\bar{h}_d$ as follows

$$\bar{h}_d^*(p_0) = (h_d^*_{\Sigma, Q}(r_0), w_0 + h_d^*_{\Sigma, \Sigma}(r_0)), \quad p_0 = (w_0, r_0) \in Q^*.$$

Similarly, we denote by $h_{d, Q} : \Sigma \rightarrow G$ the adjoint map of $h_d^*_{\Sigma, Q}$, and analogous for $h_{d, \Sigma} : \Sigma \rightarrow G$. In fact, we have $h_{d, Q}(q_0) = h_d(q_0, 0)$ and $h_{d, \Sigma}(x_1) = h_d(0, x_1)$. Hence,

$$h_d(q_0, x_1) = h_{d, Q}(q_0) + h_{d, \Sigma}(x_1).$$

1Recall that any vector space can be regarded as a Lie group with the additive structure.
For the sake of simplicity, we define the map \( h_d^0 : \Sigma \times \Sigma \to G \) as \( h_d^0(x_0, x_1) = h_d((x_0, 0), x_1) \). Lastly, observe that \( h_d : Q \times \Sigma \to G \) may be regarded locally as a map defined on \((\Sigma \times G) \times \Sigma \). For this reason, we denote its partial derivatives by

\[
D_1 h_d(q_0, x_1) = \frac{\partial h_d}{\partial q_0}(q_0, x_1) = \left( \frac{\partial h_d}{\partial x_0}(x_0, q_0, x_1), \frac{\partial h_d}{\partial q_0}(x_0, q_0, x_1) \right), \quad D_2 h_d(q_0, x_1) = \frac{\partial h_d}{\partial x_1}(q_0, x_1)
\]

for each \((q_0, x_1) = ((x_0, q_0), x_1) \in (\Sigma \times G) \times \Sigma \). Due to the linearity of \( h_d \), they are given by

\[
D_1 h_d(q_0, x_1)(q) = h_d(q, 0), \quad D_2 h_d(q_0, x_1)(x) = h_d(0, x), \quad q \in Q, \quad x \in \Sigma.
\]

We conclude with the following straightforward result.

**Lemma 2.1.** Let \( Q = \Sigma \times G \) be a trivialization of \( \pi_Q : \Sigma \). Then for each \( q_0 = (x_0, q_0) \in Q \) and \( p_0 = (x_0, r_0) \in Q^* \) we have \( J(q_0, p_0) = r_0 \).

### 2.3. Discrete Dirac mechanics

Let us recall the formulation of discrete Dirac mechanics introduced in \([13, 14]\). Here, we present only the unconstrained case. Let \( Q \) be a vector space. Making use of generating functions of types 1 and 2, we obtain the so-called \((+)\)-discrete Tulczyjew triple,

\[
\begin{array}{ccc}
\kappa^d \& \quad \gamma^d \& \quad \Omega^d \\\
T^*(Q \times Q) \& \quad T^*Q \times T^*Q \& \quad T^*(Q \times Q^*)
\end{array}
\]

In the same vein, using generating functions of types 1 and 3, we obtain an analogous diagram called the \((-)\)-discrete Tulczyjew triple. Hereafter, we will focus on the former triple, but analogous results hold when using the latter.

Denoting \( z = (q, p) \in T^*Q \simeq Q \times Q^* \), the \((+)\)-discrete induced Dirac structure is defined as

\[
D^d = \{(z_0, z_1, \alpha_{z^+}) \mid z_0, z_1 \in T^*Q, z^+ = (q_0, p_1), \alpha_{z^+} = \Omega^d(z_0, z_1) \} \subset (T^*Q \times T^*Q) \times T^*(Q \times Q^*).
\]

Given a discrete Lagrangian \( L_d : Q \times Q \to \mathbb{R} \), its derivative is the map \( dL_d : Q \times Q \to T^*(Q \times Q) \) given by

\[
dL_d(q_0, q_1) = (q_0, q_1, D_1 L_d(q_0, q_1), D_2 L(q_0, q_1)), \quad (q_0, q_1) \in Q \times Q,
\]

where \( D_i \) denotes the partial derivative with respect to the \( i \)-th component, \( i = 1, 2 \). The \((+)\)-discrete Dirac differential is the map

\[
D^* L_d = \gamma^d \circ dL_d : Q \times Q \to T^*(Q \times Q^*).
\]

At last, a discrete vector field on \( T^*Q \) is a sequence

\[
X_d = \{ X^k_d = ((q_k, p_k), (q_{k+1}, p_{k+1})) \in T^*Q \times T^*Q \mid 0 \leq k \leq N - 1 \}.
\]

**Definition 2.2.** A \((+)\)-discrete implicit Lagrangian system, also called a \((+)\)-discrete Lagrange–Dirac system, is a pair \((L_d, X_d)\), where \( L_d \) is a discrete Lagrangian on \( Q \) and \( X_d \) is a discrete vector field on \( T^*Q \), satisfying the \((+)\)-discrete Lagrange–Dirac equations, i.e.,

\[
(X^k_d, D^* L_d(q_k, q^+_k)) \in D^d, \quad 0 \leq k \leq N - 1.
\]

The equations are locally given by

\[
q^+_k = q_{k+1}, \quad p_{k+1} = D_2 L_d(q_k, q^+_k), \quad p_k = -D_1 L_d(q_k, q^+_k), \quad 0 \leq k \leq N - 1.
\]
2.3.1. Variational structure for Lagrange–Dirac systems. The discrete Lagrange–Dirac equations may also be obtained from a variational principle, as shown in [14]. As above, there exist two possible choices when performing discretization, but we will focus on the (+) case. Recall that the (+)-discrete Pontryagin bundle is the (vector) bundle over \( Q \) given by

\[
(Q \times Q) \oplus (Q \times Q^*) \simeq Q \times Q \times Q^* = \{(q_0, q_0^*, p_1) | q_0, q_0^* \in Q, p_1 \in Q^*\}.
\]

Given a discrete Lagrangian \( L_d : Q \times Q \to \mathbb{R} \), the (+)-discrete Lagrange–Pontryagin action is the discrete augmented action defined as

\[
S_{L_d}[(q_k, q_k^+, p_{k+1})_{k=0}^N] = \sum_{k=0}^{N-1} \left( L_d(q_k, q_k^+) + (p_{k+1}, q_k^+ - q_k^-) \right).
\]

The (+)-discrete Lagrange–Pontryagin principle,

\[
\delta S_{L_d}[(q_k, q_k^+, p_{k+1})_{k=0}^N] = 0,
\]

is obtained by enforcing free variations \( \{(\delta q_k, \delta q_k^+, \delta p_{k+1}) \in Q \times Q \times Q^* | 0 \leq k \leq N \} \) that vanish at the endpoints, i.e., \( \delta q_0 = \delta q_N = 0 \).

**Theorem 2.1.** The (+)-discrete Lagrange–Pontryagin principle is equivalent to the (+)-discrete Lagrange–Dirac equations.

3. Reduction of the discrete Dirac structure

Let \( Q \) be a vector space and \( G \subset Q \) be a vector subspace acting by addition on \( Q \). In this section we will show that the discrete induced Dirac structure on \( Q \) is \( G \)-invariant and we will reduce it to the corresponding quotient.

3.1. Trivializations of the tangent bundle and cotangent bundles. Let \( \omega_d : Q \times Q \to G \) be a discrete connection form on \( \pi_{Q, \Sigma} \). We define a right trivialization of \( TQ = Q \times Q \) as

\[
\lambda_d : Q \times Q \to Q \times (\Sigma \times G), \quad (q_0, q_1) \mapsto (q_0, [q_1], \omega_d(q_0, q_1)).
\]

Observe that it is a linear map with the vector structure of \( TQ \). Using the direct sum [8, Remark 4.3] it is straightforward to check that \( \lambda_d \) is an isomorphism (of bundles over \( T\Sigma = \Sigma \times \Sigma \)) with inverse given by

\[
\lambda_d^{-1} : Q \times (\Sigma \times G) \to Q \times Q, \quad (q_0, x_1, g_1) \mapsto (q_0, g_1 + T_d(q_0, x_1)).
\]

The action of \( G \) on \( Q \times Q \) induces an action on \( Q \times (\Sigma \times G) \) by means of \( \lambda_d \). Using the equivariance of \( \omega_d \), such action is given by

\[
g \cdot (q_0, x_1, g_1) = (g + q_0, x_1, g_1), \quad g \in G, \quad (q_0, x_1, g_1) \in Q \times (\Sigma \times G).
\]

By construction, \( \lambda_d \) is equivariant, i.e., \( \lambda_d \circ \Phi^Q \times Q = \Phi^Q \times (\Sigma \times G) \circ \lambda_d \) for each \( g \in G \). Subsequently, it descends to a left trivialization of \( (Q \times Q) / G \), i.e., an isomorphism of the corresponding quotients,

\[
[\lambda_d] : (Q \times Q) / G \rightarrow (Q \times (\Sigma \times G)) / G.
\]

Furthermore, note that \( (Q \times (\Sigma \times G)) / G \simeq \Sigma \times (\Sigma \times G) \) via the isomorphism

\[
[q_0, x_1, g_1] \mapsto ([q_0], x_1, g_1).
\]

In the same vein, we may define a right trivialization of \( T^*Q \simeq Q \times Q^* \) as

\[
\tilde{\lambda}_d : Q \times Q^* \to Q \times (\Sigma^* \times g^*), \quad (q_0, p_0) \mapsto \left(q_0, pr^2 \circ T_d(q_0), J(q_0, p_0)\right).
\]

Again, this trivialization is a linear map with the vector structure on \( Q \times Q^* \) given by the direct sum.
Remark 3.1. Observe that the adjoint of the projection $\pi_{Q,\Sigma}: Q \to \Sigma$ yields a canonical embedding of $\Sigma^*$ into $Q^*$. However, there is not a canonical linear projection of $Q^*$ onto $\Sigma^*$. The discrete principal connection gives a choice of this projection via the adjoint of the horizontal lift, thus yielding the following split

$$Q^* = \text{im}(\pi_{Q,\Sigma}^*) \oplus \ker \left( pr_2 \circ \tilde{h}_d^* \right).$$

In addition, $\langle \pi_{Q,\Sigma}^*(w), \xi_Q(q_0) \rangle = 0$ for every $w \in \Sigma^*$ and $\xi \in g$, so we have

$$\text{im}(\pi_{Q,\Sigma}^*) = \{ \xi_Q(q_0) \in Q \mid \xi \in g \}^0,$$

where the superscript 0 denotes the annihilator.

Using the previous remark, it can be seen that the inverse of $\hat{\lambda}_d$ is given by

$$\hat{\lambda}_d^{-1}: Q \times (\Sigma^* \times g^*) \to Q \times Q^*, \quad (q_0, w_0, \mu_0) \mapsto (q_0, \pi_{Q,\Sigma}^*(w_0) + (\mu_0)Q(q_0)),$$

where $(\mu_0)Q(q_0) \in Q^*$ is implicitly defined by the relations $( (\mu_0)Q(q_0), \xi_Q(q_0) ) = ( \mu_0, \xi )$ for each $\xi \in g$ and $( pr_2 \circ \tilde{h}_d^* )((\mu_0)Q(q_0)) = 0$.

It is easy to check that the action of $G$ on $Q \times (\Sigma^* \times g^*)$ induced by $\hat{\lambda}_d$ is given by

$$g \cdot (q_0, w_0, \mu_0) = (g + q_0, w_0, \mu_0), \quad g \in G, \quad (q_0, w_0, \mu_0) \in Q \times (\Sigma^* \times g^*).$$

Since $Q \times Q$, $Q \times (\Sigma \times G)$, $Q \times Q^*$ and $Q \times (\Sigma^* \times g^*)$ are vector spaces, we may identify their tangent and cotangent bundles as in [2], e.g. $T^*(Q \times Q) = Q \times Q \times Q^* \times Q^*$. Furthermore, we may consider the diagonal actions of $G$ on $T(Q \times Q)$ and $T(Q \times Q^*)$, as in [1]. Likewise, on $T^*(Q \times Q)$ and $T^*(Q \times Q^*)$ we consider the cotangent lift of the action, as in [3]. In turn, these actions may be transferred to the corresponding (co)tangent bundles of $Q \times (\Sigma \times G)$ and $Q \times (\Sigma^* \times g^*)$ using the maps $\lambda_d$ and $\hat{\lambda}_d$, and their adjoint maps, accordingly. Note that since these maps are linear, their derivatives are the maps themselves. This can be done because $\lambda_d$ and $\hat{\lambda}_d$ are linear isomorphisms. For instance, the action of $G$ on $T^*(Q \times (\Sigma^* \times g^*))$ is induced from the action on $T^*(Q \times Q^*)$ using the map $\hat{\lambda}_d$ and its adjoint, as well as their inverses.

3.2. Local expressions of the trivializations and quotients. In order to study the explicit expression of the maps introduced above, we choose a trivialization $Q = \Sigma \times G$ of $\pi_{Q,\Sigma}$ as in Section 2.2.1. Using the local expression of the discrete connection, we have

$$\lambda_d(q_0, q_1) = (q_0, x_1, g_1 - h_d(q_0, x_1)), \quad (q_0, q_1) \in Q \times Q.$$ (11)

Hence,

$$\lambda_d^{-1}(q_0, x_1, g_1) = (q_0, (x_1, g_1 + h_d(q_0, x_1))), \quad (q_0, x_1, g_1) \in Q \times (\Sigma \times G).$$ (12)

Its adjoint is given by

$$\lambda_d^*(p_0, w_1, r_1) = (p_0 - \{ r_1, h_d(\cdot, 0) \}, (w_1 - \{ r_1, h_d(0, \cdot) \}, r_1)), \quad (p_0, w_1, r_1) \in Q^* \times (\Sigma^* \times G^*).$$ (13)

Hence,

$$(\lambda_d^*)^{-1}(p_0, p_1) = (p_0 + \{ r_1, h_d(\cdot, 0) \}, w_1 + \{ r_1, h_d(0, \cdot) \}, r_1), \quad (p_0, p_1) \in Q^* \times Q^*.$$ (14)

Now we perform analogous computations for $\hat{\lambda}_d$, where we used [4] and Lemma 2.1,

$$\hat{\lambda}_d(q_0, p_0) = (q_0, w_0 + h_d^*(\Sigma)(r_0), r_0), \quad (q_0, p_0) \in Q \times Q^*.$$ (15)

The inverse is given by

$$\hat{\lambda}_d^{-1}(q_0, w_0, \mu_0) = (q_0, (w_0 - h_d^*(\Sigma)(\mu_0), \mu_0)), \quad (q_0, w_0, \mu_0) \in Q \times (\Sigma^* \times g^*).$$ (16)

Likewise, its adjoint is given by

$$\hat{\lambda}_d^*(p_1, x_1, \xi_1) = (p_1, (x_1, h_d^*(\Sigma)(x_1) + \xi_1)), \quad (p_1, x_1, \xi_1) \in Q^* \times (\Sigma \times g).$$ (17)
At last, we have

\[(\lambda_g^*)^{-1}(p_1,q_1) = (p_1,x_1,g_1 - h_{d,\Sigma}(x_1)), \quad (p_1,q_1) \in Q^* \times Q.\]

On the other hand, observe that the group action is locally given by

\[g + q_0 = (x_0,g + q_0), \quad q_0 = (x_0,g_0) \in Q, \quad g \in G,\]

where we identify \( G \simeq \{0\} \times G \subset Q \). Using the local expression for the trivializations and their adjoints, we obtain the local expression for the actions of \( G \) on the (co)tangent bundles of \( Q \times (\Sigma \times G) \) and \( Q \times (\Sigma^* \times g^*) \).

**Proposition 3.1.** The action of \( G \) on \( T^*(Q \times (\Sigma^* \times g^*)) \) is locally given by

\[g \cdot ((q_0,w_0,\mu_0),(p_1,x_1,\xi_1)) = ((g + q_0,w_0,\mu_0),(p_1,x_1,g + \xi_1))\]

for each \( g \in G \), \((q_0,w_0,\mu_0)\in Q \times (\Sigma^* \times g^*) \) and \((p_1,x_1,\xi_1) \in Q^* \times (\Sigma \times g) \). Likewise, the action on \( T(Q \times (\Sigma^* \times g^*)) \) is locally given by

\[g \cdot ((q_0,w_0,\mu_0),(q_1,w_1,\mu_1)) = ((g + q_0,w_0,\mu_0),(g + q_1,w_1,\mu_1))\]

for each \( g \in G \) and \((q_0,w_0,\mu_0),(q_1,w_1,\mu_1) \in Q \times (\Sigma^* \times g^*) \).

**Proof.** As explained at the end of the previous section, the action of \( G \) on \( T^*(Q \times (\Sigma^* \times g^*)) \) is induced from the action on \( T^*(Q \times Q^*) \). Namely,

\[
\begin{align*}
((q_0,w_0,\mu_0),(p_1,x_1,\xi_1)) \\
\text{\hspace{1cm}} \overset{\text{Invariance}}{\mapsto} \\
((q_0,(w_0 - h_{d,\Sigma}^*(\mu_0),\mu_0),(p_1,(x_1, h_{d,\Sigma}^*(x_1) + \xi_1))) \\
\text{\hspace{1cm}} \overset{G \text{ action}}{\mapsto} \\
((g + q_0,(w_0 - h_{d,\Sigma}^*(\mu_0),\mu_0),(p_1,(x_1,g + h_{d,\Sigma}^*(x_1) + \xi_1))) \\
\text{\hspace{1cm}} \overset{\text{Invariance}}{\mapsto} \\
((g + q_0,w_0,\mu_0),(p_1,x_1,g + \xi_1)).
\end{align*}
\]

The computation for \( T(Q \times (\Sigma \times g^*)) \) is analogous. \(\square\)

To conclude, we define the following local isomorphisms,

\[(q_0,w_0,\mu_0),(q_1,w_1,\mu_1) \mapsto (x_0,w_0,\mu_0,-g_0 + q_1,w_1,\mu_1),\]

and

\[(q_0,w_0,\mu_0),(p_1,x_1,\xi_1) \mapsto (x_0,w_0,\mu_0,p_1,x_1,-g_0 + \xi_1).\]

### 3.3. Invariance of the discrete induced Dirac structure.

The definition of invariance for (continuum) Dirac structures can be extended to the discrete setting. More specifically, \( D^{d^+} \) is said to be \( G \)-invariant if for each \( g \in G \) and \((z_0,z_1,\alpha_{z^+}) \in D^{d^+} \) we have

\[
\left( \Phi^T_g Q(z_0), \Phi^T_g Q(z_1), \Phi^{T^*(Q \times Q^*)}_g (\alpha_{z^+}) \right) \in D^{d^+}.
\]

**Proposition 3.2.** The \((+)-\)discrete induced Dirac structure \( D^{d^+} \) is \( G \)-invariant.
Proof. Let \( g \in G \) and \((z_0, z_1, \alpha_{z^+}) \in D^{d^+}\). By definition, we have
\[
\alpha_{z^+} = \Omega^b_{d^+}((z_0, z_1)) = (q_0, p_0, p_1, q_1).
\]
Recall that the actions are given by \( \Phi^T_g Q (q_0, p_0) = (g + q_0, p_0) \) and \( \Phi^T_g (Q \times Q^*) (q_0, p_0, p_1, q_1) = (g + q_0, p_0, p_1, g + q_1) \), respectively. Hence,
\[
\hat{z}_i = \Phi^T_g Q (z_i) = (g + q_i, p_i), \quad i = 0, 1,
\]
and
\[
\hat{\alpha}_{z^+} = \Phi^T_g (Q \times Q^*) (\alpha_{z^+}) = (g + q_0, p_0, p_1, g + q_1),
\]
where \( \hat{z} = (g + q_0, p_1) \). It immediately follows that \( \Omega^b_{d^+}((z_0, \hat{z}_1)) = \hat{\alpha}_{z^+} \), which establishes the desired result.

On the other hand, the right trivialization \( \hat{\lambda}_d \) allows \( D^{d^+} \) to induce a (+)-discrete Dirac structure on \( Q \times (\Sigma^* \times g^*) \),
\[
\hat{D}^{d^+} = \left\{ (\hat{z}_0, \hat{z}_1, \hat{\alpha}_{z^+}) \mid z_0, z_1 \in Q \times Q^*, \hat{\alpha}_{z^+} \in T^\times_{\hat{z}_2} (Q \times (\Sigma^* \times g^*)), z^+ = (q_0, p_1), (z_0, z_1, (\hat{\lambda}_{d}^{-1}, \hat{\lambda}_d^*)) \in D^{d^+} \right\},
\]
where, for the sake of simplicity, we denote \( \hat{z} = \lambda_d(z) \) for each \( z = (q, p) \in Q \times Q^* \). Equivalently, \( \Omega^b_{d^+} \) induces a map \( \hat{\Omega}^b_{d^+} \) between the trivialized spaces by imposing the commutativity of the following diagram,
\[
\begin{array}{ccc}
(Q \times Q^*) \times (Q \times Q^*) & \xrightarrow{\Omega^b_{d^+}} & T^* (Q \times Q^*) \\
\downarrow{\lambda_d, \hat{\lambda}} & & \downarrow{(\hat{\lambda}_d, (\hat{\lambda}_d^*)^{-1})} \\
(Q \times (\Sigma^* \times g^*)) \times (Q \times (\Sigma^* \times g^*)) & \xrightarrow{\hat{\Omega}^b_{d^+}} & T^* (Q \times (\Sigma^* \times g^*))
\end{array}
\]
This way, \( \hat{D}^{d^+} \) can be regarded as the (+)-discrete Dirac structure induced by \( \hat{\Omega}^b_{d^+} \), that is,
\[
\hat{D}^{d^+} = \left\{ (\hat{z}_0, \hat{z}_1, \hat{\alpha}_{z^+}) \mid z_0, \hat{z}_1 \in Q \times (\Sigma^* \times g^*), \hat{\alpha}_{z^+} \in T^\times_{\hat{z}_2} (Q \times (\Sigma^* \times g^*)), \hat{z}^+ = \hat{\lambda}_d(q_0, p_1), \hat{\Omega}^b_{d^+}((\hat{z}_0, \hat{z}_1)) = \hat{\alpha}_{z^+} \right\}.
\]

Of course, the \( G \)-invariance of \( D^{d^+} \) leads to the \( G \)-invariance of \( \hat{D}^{d^+} \), since we have constructed the actions on the trivialized space so that \( \hat{\lambda}_d \) is equivariant.

3.4. Reduced discrete Dirac structure. The \( G \)-invariance of \( \hat{D}^{d^+} \) ensures that it descends to a discrete Dirac structure on the quotient space. Since \( \Omega^b_{d^+} \) and \( \hat{\lambda}_d \) are equivariant, so is \( \hat{\Omega}^b_{d^+} \), which induces a well-defined map between the quotient spaces,
\[
[\hat{\Omega}^b_{d^+}] : [(Q \times (\Sigma^* \times g^*)) \times (Q \times (\Sigma^* \times g^*))] / G \longrightarrow [T^* (Q \times (\Sigma^* \times g^*))] / G.
\]
Then, the reduced (+)-discrete Dirac structure is the structure induced by this map, i.e.,
\[
[\hat{D}^{d^+}] = \left\{ [(\hat{z}_0, \hat{z}_1), [\hat{\alpha}_{z^+}]] \mid [\hat{z}_0, \hat{z}_1] \in [(Q \times (\Sigma^* \times g^*)) \times (Q \times (\Sigma^* \times g^*))] / G, [\hat{\alpha}_{z^+}] \in [T^* (Q \times (\Sigma^* \times g^*))] / G, [\hat{\Omega}^b_{d^+}] ([\hat{z}_0, \hat{z}_1]) = [\hat{\alpha}_{z^+}] \right\}.
\]
Locally, identifications \( (19) \) and \( (20) \) enable us to regard \( [\hat{\Omega}^b_{d^+}] \) as a map between the trivializations,
\[
[\hat{\Omega}^b_{d^+}] : \Sigma \times (\Sigma^* \times g^*) \times Q \times (\Sigma^* \times g^*) \longrightarrow \Sigma \times (\Sigma^* \times g^*) \times Q^* \times (\Sigma \times g).
\]
\textbf{Lemma 3.1.} Working in a trivialization and using the above identification, we have

\[ [\hat{\Omega}_d^+](x_0, w_0, \mu_0, q_1, w_1, \mu_1) = \left( x_0, w_1, \mu_1, (w_0 - h^*_d, \Sigma)(\mu_0), x_1, g_1 - h_d, \Sigma(x_1) \right), \]

for each \((x_0, w_0, \mu_0, q_1, w_1, \mu_1) \in \Sigma \times (\Sigma^* \times g^*) \times Q \times (\Sigma^* \times g^*)\).

\textit{Proof.} We employ the explicit local expression computed in the previous sections,

\[
\begin{align*}
(x_0, w_0, \mu_0, q_1, w_1, \mu_1) & \xrightarrow{(19)} \left( ((x_0, 0), w_0, \mu_0), (q_1, w_1, \mu_1) \right) \\
& \xrightarrow{(19)} \left( ((x_0, 0), (w_0 - h^*_d, \Sigma)(\mu_0), (q_1, (w_1 - h^*_d, \Sigma)(\mu_1), \mu_1)) \right) \\
& \xrightarrow{\Omega_d^+} \left( ((x_0, 0), (w_1 - h^*_d, \Sigma)(\mu_1)), ((w_0 - h^*_d, \Sigma)(\mu_0), q_1) \right) \\
& \quad \xrightarrow{(19), (18)} \left( ((x_0, 0), w_1, \mu_1), ((w_0 - h^*_d, \Sigma)(\mu_0), x_1, g_1 - h_d, \Sigma(x_1)) \right) \\
& \quad \xrightarrow{(20)} (x_0, w_1, \mu_1, (w_0 - h^*_d, \Sigma)(\mu_0), x_1, g_1 - h_d, \Sigma(x_1)).
\end{align*}
\]

\[ \square \]

Observe that \([\hat{\Omega}_d^+]\) is a bundle morphism covering the identity if we regard the previous maps as bundles over \(\Sigma\) with the projection onto the first component.

\textbf{Proposition 3.3.} Locally, the reduced (+)-discrete Dirac structure is given by

\[
[\hat{D}^d^+] = \left\{ ((x_0, w_0, \mu_0, q_1, w_1, \mu_1), (x_0, w_1, \mu_1, p, x_1, \xi)) \mid p = (w_0 - h^*_d, \Sigma)(\mu_0), \xi = g_1 - h_d, \Sigma(x_1) \right\}
\subset (\Sigma \times (\Sigma^* \times g^*) \times Q \times (\Sigma^* \times g^*)) \times (\Sigma \times (\Sigma^* \times g^*) \times Q^* \times (\Sigma \times g^*)).\]

\section{Discrete Lagrange–Poincaré–Dirac reduction}

Making use of the reduced discrete Dirac structure, we will compute the reduced equations corresponding to a discrete Lagrange–Dirac system. Let \(Q\) be a vector space and \(G \subset Q\) be a vector subspace acting by addition on \(Q\), and let \((L_d, X_d)\) be a (+)-discrete Lagrange–Dirac system. Suppose that \(L_d\) is \(G\)-invariant, i.e.,

\[ L_d(g + q_0, g + q_1) = L_d(q_0, q_1), \quad g \in G, \quad (q_0, q_1) \in Q \times Q. \]

The invariance of the discrete Lagrangian leads to the invariance of its partial derivatives.

\textbf{Lemma 4.1.} If \(L_d\) is \(G\)-invariant, then so are its partial derivatives \(D_i L_d : Q \times Q \to Q^*\), \(i = 1, 2\), i.e.,

\[ D_i L_d(g + q_0, g + q_1) = D_i L_d(q_0, q_1), \quad (q_0, q_1) \in Q \times Q, \quad g \in G. \]
Proof. For each \( q \in Q \) we have

\[
\langle D_1 L_d(g + q_0, g + q_1), q \rangle = \lim_{h \to 0} \frac{L_d(g + q_0 + hq, g + q_1) - L_d(g + q_0, g + q_1)}{h}
\]

\[
= \lim_{h \to 0} \frac{L_d(q_0 + hq, q_1) - L_d(q_0, q_1)}{h}
\]

\[
= \langle D_1 L_d(q_0, q_1), q \rangle
\]

An analogous computation establishes the result for \( D_2 L_d \). \( \square \)

**Remark 4.1.** Working on a trivialization \( Q = \Sigma \times G \) of \( \pi_Q \Sigma \), we may regard \( L_d \) as a function defined on \( (\Sigma \times G) \times (\Sigma \times G) \). This way, its partial derivatives can be written as

\[ D_1 L_d(q_0, q_1) = \frac{\partial L_d}{\partial q_0}(q_0, q_1) = \left( \frac{\partial L_d}{\partial x_0}(x_0, q_0, x_1, g_1), \frac{\partial L_d}{\partial g_0}(x_0, q_0, x_1, g_1) \right) \]

for each \( (q_0, q_1) = (x_0, q_0, x_1, g_1) \in (\Sigma \times G) \times (\Sigma \times G) \), and analogously for \( D_2 L_d(q_0, q_1) \).

Since the discrete vector field \( X_d \) is a solution of the (+)-discrete Lagrange–Dirac equations, we may regard it as a map

\[
X_d : \ Q \times Q \quad \longrightarrow \quad T(Q \times Q^*) = (Q \times Q^*) \times (Q \times Q^*)
\]

\[
(q_0, q_1) \quad \longmapsto \quad \left( (q_0, -D_1 L_d(q_0, q_1)), (q_1, D_2 L_d(q_0, q_1)) \right)
\]

It follows from Lemma 4.1 that this map is \( G \)-equivariant. Similarly to \( \Omega_{d+}^b \), this induces a map between the trivialized spaces by imposing the commutativity of the following diagram,

\[
\begin{array}{ccc}
Q \times Q & \xrightarrow{X_d} & (Q \times Q^*) \times (Q \times Q^*) \\
\downarrow \lambda_d & & \downarrow (\hat{\lambda}_d, \tilde{\lambda}_d) \\
Q \times (\Sigma \times G) & \xrightarrow{\hat{X}_d} & (Q \times (\Sigma^* \times g^*)) \times (Q \times (\Sigma^* \times g^*))
\end{array}
\]

Since \( \hat{X}_d \) is \( G \)-equivariant, it descends to a reduced discrete vector field,

\[
[\hat{X}_d] : (Q \times (\Sigma \times G))/G \to T(Q \times (\Sigma^* \times g^*))/G.
\]

As above, locally we may regard it as \( [\hat{X}_d] : \Sigma \times (\Sigma \times G) \to \Sigma \times (\Sigma^* \times g^*) \times Q \times (\Sigma^* \times g^*) \).

**4.1 Reduced discrete Dirac differential.** Analogous to \( \Omega_{d+}^b \), \( \gamma_{Q}^{d+} \) induces a map between the trivialized spaces by imposing the commutativity of the following diagram,

\[
\begin{array}{ccc}
T^*(Q \times Q) & \xrightarrow{\gamma_{Q}^{d+}} & T^*(Q \times Q^*) \\
(\lambda_d, (\lambda_d^*)^{-1}) & & (\hat{\lambda}_d, (\tilde{\lambda}_d^*)^{-1}) \\
T^*(Q \times (\Sigma \times G)) & \xrightarrow{\hat{\gamma}_{Q}^{d+}} & T^*(Q \times (\Sigma^* \times g^*))
\end{array}
\]

Furthermore, since \( \gamma_{Q}^{d+} \), \( \lambda_d \) and \( \hat{\lambda}_d \) are \( G \)-equivariant, so is \( \hat{\gamma}_{Q}^{d+} \), by construction. In the same vein, we induce the exterior derivative of the discrete Lagrangian by imposing the commutativity of the
following diagram,

\[
\begin{array}{c}
Q \times Q \xrightarrow{dL_d} T^*(Q \times Q) \\
\lambda_d \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
4.2. Discrete Lagrange–Poincaré–Dirac equations. Define \( \hat{L}_d : Q \times (\Sigma \times G) \to \mathbb{R} \) by the condition \( \hat{L}_d \circ \lambda_d = L_d \). Clearly, it is \( G \)-invariant, thus inducing the reduced discrete Lagrangian,

\[
l_d : (Q \times (\Sigma \times G))/G \to \mathbb{R}.
\]

Locally, we may regard it as \( l_d : \Sigma \times \Sigma \times G \to \mathbb{R} \). It is easy to check that

\[
l_d(x_0, x_1, g_1) = L_d(x_0, 0, x_1, g_1 + h_0^d(x_0, x_1), \quad (x_0, x_1, g_1) \in \Sigma \times \Sigma \times G.
\]

**Lemma 4.2.** Locally, for each \((x_0, x_1, g_1) \in \Sigma \times \Sigma \times G\) we have

\[
\frac{\partial L_d}{\partial q_0} = \left( \frac{\partial l_d}{\partial x_0} - \left[ \frac{\partial l_d}{\partial g_1}, h_0^d(\cdot, 0) \right] \right),
\]

\[
\frac{\partial L_d}{\partial x_1} = \frac{\partial l_d}{\partial x_1} - \left[ \frac{\partial l_d}{\partial g_1}, h_0^d(0, \cdot) \right],
\]

\[
\frac{\partial L_d}{\partial g_1} = \frac{\partial l_d}{\partial g_1},
\]

where the partial derivatives of \( L_d \) and \( l_d \) are evaluated at \((x_0, 0, x_1, g_1 + h_0^d(x_0, x_1))\) and \((x_0, x_1, g_1)\), respectively.

**Proof.** Let \((x_0, g_0, x_1, g_1') \in (\Sigma \times G) \times (\Sigma \times G)\). Using (11), we obtain

\[
L_d((x_0, g_0), (x_1, g_1')) = (\hat{L}_d \circ \lambda_d)((x_0, g_0), (x_1, g_1'))
\]

\[
= \hat{L}_d((x_0, g_0), x_1, g_1' - h_d((x_0, g_0), x_1))
\]

\[
= l_d(x_0, x_1, g_1' - h_d((x_0, g_0), x_1)).
\]

Recall that the partial derivatives of \( h_d \) are given by (6). Using this fact, as well as the chain rule and the relation above, we obtain

\[
\frac{\partial L_d}{\partial x_0} = \frac{\partial l_d}{\partial x_0} - \left[ \frac{\partial l_d}{\partial g_1}, h_0^d(\cdot, 0) \right],
\]

\[
\frac{\partial L_d}{\partial x_1} = \frac{\partial l_d}{\partial x_1} - \left[ \frac{\partial l_d}{\partial g_1}, h_0^d(0, \cdot) \right],
\]

\[
\frac{\partial L_d}{\partial g_1} = \frac{\partial l_d}{\partial g_1}.
\]

In the above expressions, the partial derivatives of \( L_d \) and \( l_d \) are evaluated at \((x_0, g_0, x_1, g_1')\) and \((x_0, x_1, g_1' - h_d((x_0, g_0), x_1))\), respectively. To conclude, we choose \( g_0 = 0 \) and \( g_1' = g_1 + h_0^d(x_0, x_1) \). \( \square \)

Gathering the results of the previous sections, we arrive at the main result of this paper, which are the reduced discrete equations.

**Theorem 4.1** (Discrete Lagrange–Poincaré–Dirac equations). Let \((L_d, X_d)\) be a \((+)-\) discrete Lagrange–Dirac system and suppose that \( L_d \) is \( G \)-invariant. Let \([\hat{D}^d] \) and \([\hat{D}^d L_d] \) be the reduced \((+)-\) discrete Dirac structure, the reduced discrete vector field and the reduced \((+)-\) discrete Dirac differential defined in (21), (22) and (23), respectively. Then, they satisfy the \((+)-\) discrete Lagrange–Poincaré–Dirac equations, i.e., for each \( 0 \leq k \leq N - 1 \) we have

\[
([\hat{X}_d]|(q_k, x_{k+1}, g_{k+1})), [\hat{D}^d L_d]|(q_k, x^+_k, g^+_k)) \in [\hat{D}^d].
\]

In order to obtain its local expression, we write the discrete vector field as

\[
[\hat{X}_d] = \{[\hat{X}_d^k] = (x_k, w_k, \mu_k, q_{k+1}, w_{k+1}, \mu_{k+1}) \mid 0 \leq k \leq N - 1 \}.
\]

Subsequently, the reduced equations read as

\[
([\hat{X}_d^k], [\hat{D}^d L_d]|(x_k, x^+_k, g^+_k)) \in [\hat{D}^d], \quad 0 \leq k \leq N - 1.
\]
Making use of Proposition 3.3, Proposition 4.1 and Lemma 4.2, we arrive at the local expression of the equations,

\[
\begin{align*}
\frac{\partial l_d}{\partial x_1} - \left( \frac{\partial l_d}{\partial g_1}, h^0_d(0, \cdot) \right) + h^*_d,\Sigma \left( \frac{\partial l_d}{\partial g_1} \right) &= w_{k+1} \\
\frac{\partial l_d}{\partial g_1} &= \mu_{k+1} \\
- \frac{\partial l_d}{\partial x_0} + \left( \frac{\partial l_d}{\partial g_1}, h^0_d(\cdot, 0) \right) &= w_k - h^*_d,\Sigma(\mu_k) \\
\left( \frac{\partial l_d}{\partial g_1}, h_d((0, \cdot), 0) \right) &= \mu_k \\
x_k^+ &= x_{k+1} \\
g_k^+ + h_d,\Sigma(x, 0) &= g_{k+1} - h_d,\Sigma(x_{k+1})
\end{align*}
\]

(25)

In the above equations, partial derivatives of \( l_d \) are evaluated at \((x_k, x^+_k, g^+_k)\).

5. Reduction of the discrete variational principle

In this section, we perform reduction of discrete Lagrange–Dirac systems from the variational point of view. As expected, we will recover the discrete Lagrange–Poincaré–Dirac equations obtained from the geometric reduction of the discrete Dirac structure. Let \( Q \) be a vector space and \( G \subset Q \) be a vector subspace acting by addition on \( Q \).

5.1. Trivialization of the Pontryagin bundle. Let \( \omega_d : Q \times Q \to G \) be a discrete connection form. Using the trivializations defined in Section 3.1, we define the following map

\[
\Lambda_d = (\lambda_d, \hat{\lambda}_d) : Q \times Q \times Q^* \to Q \times (\Sigma \times G) \times Q \times (\Sigma^* \times g^*).
\]

Again, the \( G \)-action on \( Q \times Q \times Q^* \) given by

\[
g \cdot (q_0, q_0^+, q_1, p_1) = (g + q_0, g + q_0^+, g + q_1, p_1), \quad g \in G, \quad (q_0, q_0^+, q_1, p_1) \in Q \times Q \times Q^*
\]

induces an action on the trivialized space by \( \Lambda_d \).

Let \( Q = \Sigma \times G \) be a trivialization of \( \Sigma \times G \), as in Section 2.2.1. For each \( q_0 = (x_0, q_0) \), \( q_0^+ = (x_0^+, q_0) \), \( q_1 = (x_1, g_1) \in Q = \Sigma \times G \) and \( p_1 = (w_1, r_1) \in Q^* = (\Sigma^* \times G, r_1) \), the local expression of \( \Lambda_d \) is

\[
\Lambda_d(q_0, q_0^+, q_1, p_1) = \left( q_0, x_0^+, g_0^+ + h_d(q_0, x_0^+, q_1, w_1 + h^*_d,\Sigma(r_1), r_1) \right).
\]

(26)

Similarly, for each \( q_0 = (x_0, q_0) \), \( q_1 = (x_1, g_1) \in Q \), \( (x_0^+, g_0^+) \in \Sigma \times G \) and \( (w_1, \mu_1) \in \Sigma^* \times g^* \) we have

\[
\Lambda_d^{-1}(q_0, x_0^+, g_0^+, q_1, w_1, \mu_1) = \left( q_0, \left( x_0^+, g_0^+ + h_d(q_0, x_0^+) \right), q_1, (w_1 - h^*_d,\Sigma(\mu_1), \mu_1) \right).
\]

(27)

At last, locally we may identify the quotient spaces as

\[
\frac{(Q \times Q \times Q \times Q^*) / G}{[q_0, q_0^+, q_1, p_1]} = \frac{\Sigma \times Q \times Q^*}{[q_0, q_0^+, q_1, p_1]}
\]

and

\[
\frac{(Q \times (\Sigma \times G) \times Q \times (\Sigma^* \times g^*)) / G}{[q_0, x_0^+, g_0^+, q_1, w_1, \mu_1]} = \frac{\Sigma \times (\Sigma \times G) \times Q \times (\Sigma^* \times g^*)}{[q_0, x_0^+, g_0^+, q_1, w_1, \mu_1]}.
\]

(28)
5.2. Discrete reduced variational principle. Let \( L_d : Q \times Q \rightarrow \mathbb{R} \) be a \( G \)-invariant discrete Lagrangian and consider the corresponding reduced Lagrangian \( l_d : (Q \times (\Sigma \times G))/G \rightarrow \mathbb{R} \). The (+)-discrete generalized energy is the map

\[
E_d : Q \times Q \times Q^* \rightarrow \mathbb{R}, \quad (q_0, q_0^+, q_1, p_1) \mapsto L_d(q_0, q_0^+) + \langle p_1, q_1 - q_0^+ \rangle
\]

Since \( L_d \) is \( G \)-invariant, so is \( E_d \). Analogous to the discrete Lagrangian, we consider the trivilalized energy, that is, \( \tilde{E}_d : Q \times (\Sigma \times G) \times Q \times (\Sigma^* \times \mathfrak{g}^*) \rightarrow \mathbb{R} \), which is defined by the condition \( \tilde{E}_d \circ \Lambda_d = E_d \).

Again, \( \tilde{E}_d \) is \( G \)-invariant, what enables us define the reduced (+)-discrete generalized energy,

\[
ed : (Q \times (\Sigma \times G) \times Q \times (\Sigma^* \times \mathfrak{g}^*)) / G \rightarrow \mathbb{R}.
\]

Using a trivialization \( Q = \Sigma \times G \) of \( \pi_{Q, \Sigma} \) we may regard it as \( e_d : \Sigma \times (\Sigma \times G) \times Q \times (\Sigma^* \times \mathfrak{g}^*) \rightarrow \mathbb{R} \).

Lemma 5.1. Locally, for each \( (x_0, x_0^+, g_0^+, q_1, w_1, \mu_1) \in \Sigma \times (\Sigma \times G) \times Q \times (\Sigma^* \times \mathfrak{g}^*) \) we have

\[
e_d(x_0, x_0^+, g_0^+, q_1, w_1, \mu_1) = l_d(x_0, x_0^+, g_0^+) + \langle x_0^+ q_1 - h_d^* (\mu_1), x_1 - x_0^+ \rangle
\]

Proof. It is a straightforward computation, making use of the local expressions of the maps and actions that we have computed previously,

\[
(x_0, x_0^+, g_0^+, q_1, w_1, \mu_1) \underset{[29]}{\downarrow} \gamma \}
\]

\[
\left( (x_0, 0), x_0^+, g_0^+, q_1, (w_1 - h_d^* (\mu_1), x_1) \right) \underset{[27]}{\downarrow} \gamma \}
\]

\[
L_d((x_0, x_0^+, g_0^+, q_1, (w_1 - h_d^* (\mu_1), x_1)) + \left( w_1 - h_d^* (\mu_1), q_1 - (\mu_1, g_0^+ - h_d((x_0, 0), x_0^+)) \right) \underset{[29]}{\downarrow} \gamma \}
\]

\[
l_d((x_0, x_0^+, g_0^+, q_1, (w_1 - h_d^* (\mu_1), x_1)) + \left( w_1 - h_d^* (\mu_1), q_1 - (\mu_1, g_0^+ - h_d((x_0, 0), x_0^+)) \right) \underset{[29]}{\downarrow} \gamma \}
\]

The following result relates the variational principles in both the original and the reduced space.

Theorem 5.1 (Reduced variational principle). Let \( L_d : Q \times Q \rightarrow \mathbb{R} \) be a \( G \)-invariant discrete Lagrangian and \( \{(q_k, q_k^+, p_{k+1}) \in Q \times Q \times Q^* \mid 0 \leq k \leq N \} \) be a trajectory on the (+)-discrete Pontryagin bundle. Consider the reduced trajectory, that is,

\[
\{(\hat{q}_k, \hat{x}_k^+, \hat{g}_k^+, \hat{q}_{k+1}, \hat{w}_{k+1}, \hat{\mu}_{k+1}) \in (Q \times (\Sigma \times G) \times Q \times (\Sigma^* \times \mathfrak{g}^*))/G \mid 0 \leq k \leq N - 1 \},
\]

where \( (\hat{q}_k, \hat{x}_k^+, \hat{g}_k^+, \hat{q}_{k+1}, \hat{w}_{k+1}, \hat{\mu}_{k+1}) = \Lambda_d(q_k, q_k^+, q_{k+1}, p_{k+1}) \) for \( 0 \leq k \leq N - 1 \). Then the (+)-discrete Lagrange–Pontryagin principle \( [9] \) is satisfied if and only if the reduced (+)-discrete Lagrange–Pontryagin principle is satisfied, i.e.,

\[
\delta \sum_{k=0}^{N-1} e_d((\hat{q}_k, \hat{x}_k^+, \hat{g}_k^+, \hat{q}_{k+1}, \hat{w}_{k+1}, \hat{\mu}_{k+1})) = 0,
\]

for free variations \( \{(\delta \hat{q}_k, \delta \hat{x}_k^+, \delta \hat{g}_k^+, \delta \hat{q}_{k+1}, \delta \hat{w}_{k+1}, \delta \hat{\mu}_{k+1}) \in Q \times (\Sigma \times G) \times (\Sigma^* \times \mathfrak{g}^*) \mid 0 \leq k \leq N \} \) with fixed endpoints, i.e., \( \delta \hat{q}_0 = \delta \hat{q}_N = 0 \).
Proof. By construction, we have
\[
S_{L_d}\left[(q_k, q_k^+, p_{k+1})_{k=0}^N\right] = \sum_{k=0}^{N-1} (\hat{E}_d \circ \Lambda_d)(q_k, q_k^+, q_{k+1}, p_{k+1})
\]
\[
= \sum_{k=0}^{N-1} \hat{E}_d(q_k, \dot{x}_k^+, \dot{g}_k^+, \dot{w}_{k+1}, \dot{\mu}_{k+1})
\]
\[
= \sum_{k=0}^{N-1} e_d\left([\dot{q}_k, \dot{x}_k^+, \dot{g}_k^+, \dot{w}_{k+1}, \dot{\mu}_{k+1}]\right).
\]

To arrive at our conclusion, note that the free variations of the Lagrange–Pontryagin principle yield free variations on the trivialized space (with fixed endpoints), since \(\Lambda_d\) is a linear isomorphism. □

Lastly, we show that the reduced variational equations agree with the reduced geometric equations obtained in the previous section by making use of the reduced discrete Dirac structure.

**Proposition 5.1.** The variational equations obtained from the reduced (+)-discrete Lagrange–Pontryagin principle are the (+)-discrete Lagrange–Poincaré–Dirac equations.

Proof. Since the equations are local, we may work in a trivialization \(Q \simeq \Sigma \times G\) of \(\pi_{Q, \Sigma}\). Observe that locally, (30) reads as
\[
\delta X = \nabla_{\mu} h_d \cdot \nabla_{\mu} \mu - \langle \mu, h_d(0, 0) \rangle = 0,
\]
for free variations \(\{(\delta x_k, \delta g_k, \delta x_k^+, \delta g_k^+, \delta w_{k+1}, \delta \mu_{k+1}) \in Q \times (\Sigma \times G) \times (\Sigma^* \times G^*) \mid 0 \leq k \leq N\}\) with fixed endpoints, i.e., \(\delta x_0 = \delta x_N = 0\) and \(\delta g_0 = \delta g_N = 0\). Making use of Lemma 5.1 and taking variations \(\delta x_k, 1 \leq k \leq N - 1\), with fixed endpoints, we get
\[
\frac{\partial l_d}{\partial x_0} + \mu_k - \langle \mu, h_d(0, 0) \rangle = 0.
\]
Analogously, taking variations \(\delta g_k, 1 \leq k \leq N - 1\), with fixed endpoints, yield
\[
\mu_k - \langle \mu, h_d(0, 0) \rangle = 0.
\]
Now, we consider variations \(\delta x_k^+, 0 \leq k \leq N - 1\),
\[
\frac{\partial l_d}{\partial x_1} - \mu_{k+1} = 0.
\]
Similarly, for \(\delta g_k^+, 0 \leq k \leq N - 1\),
\[
\frac{\partial l_d}{\partial g_1} - \mu_{k+1} = 0.
\]
Next, for \(\delta w_{k+1}, 0 \leq k \leq N - 1\),
\[
x_{k+1} - x_k^+ = 0.
\]
In the end, for \(\delta \mu_{k+1}, 0 \leq k \leq N - 1\),
\[
h_d(\Sigma(x_{k+1} - x_k^+) + g_{k+1} - g_k^+) = 0.
\]
In the above equations, partial derivatives of \(l_d\) are evaluated at \((x_k, x_k^+, g_k^+)\). By gathering all the equations and rearranging terms, it is easy to check that these equations are exactly (25). □
6. NONLINEAR THEORY

The previous reduction theory has been developed for the linear setting, that is, when $Q$ is a vector space and $G \subset Q$ is a vector subspace acting by addition on $Q$. Nevertheless, it can be applied when $Q$ is an arbitrary smooth manifold and $G$ is an abelian Lie group acting freely and properly on $Q$. In order to see this, we use retractions and compatible charts (see, for example, [2] [14]).

**Definition 6.1.** A retraction of a smooth manifold $M$ is a smooth map $\mathcal{R} : TM \to M$ such that for each $m \in M$ we have $\mathcal{R}_m(0_m) = m$ and $(d\mathcal{R}_m)_0 = 1_{T_m M}$, where $\mathcal{R}_m = \mathcal{R}|_{T_m M}$ and we make the identification $T_0_m(T_m M) \simeq T_m M$.

Observe that the second condition ensures that $\mathcal{R}_m : T_m M \to M$ is invertible around $0_m$.

**Definition 6.2.** Let $M$ be an $n$-dimensional smooth manifold and $\mathcal{R} : TM \to M$ be a retraction of $M$. A coordinate chart $(U, \phi)$ on $M$ is said to be compatible at $m \in M$ with $\mathcal{R}$ if $\phi(m) = 0$ and $\mathcal{R}(v_m) = \phi^{-1}((d\phi)_m(v_m))$ for each $v_m \in T_m M$, where we identify $T_m \mathbb{R}^n = \mathbb{R}^n$ using the standard coordinates in $\mathbb{R}^n$.

For the particular case of a Lie group $G$ and $g \in G$, it was shown in [14], §9 that the canonical coordinates of the first kind at $g \in G$ (cf. [23, 34]) are a coordinate chart compatible with the retraction defined as $\mathcal{R}_g^\Sigma = L_g \circ \exp(dL_g^{-1}) g$, where $L_g : G \to G$ denotes the left multiplication by $g$.

**Proposition 6.1.** Let $\mathcal{R} : TM \to M$ be a retraction of an $n$-dimensional smooth manifold $M$ and $(U, \phi)$ be a compatible coordinate chart on $M$ at $m \in M$. Then for each $r \in U$ and $p_m \in T_m^* M$ we have

$$\langle p_m, \mathcal{R}_m^{-1}(r) \rangle = \sum_{i=1}^n p_i r^i,$$

where $\mathcal{R}_m^{-1}(r) \simeq r^i \partial_i$ and $p_m \simeq p_i dq^i$ in this chart.

In other words, the previous result says that the dual pairing of $T_m M$ and $T_m^* M$ reduces to the usual Euclidean inner product on $\mathbb{R}^n$ when using retraction compatible charts.

6.1. **Retractions and abelian Lie group actions.** In what follows, let $Q$ be a smooth manifold and $G$ be a connected, abelian Lie group acting freely and properly (on the left) on $Q$, thus yielding a principal bundle $\pi_{Q, \Sigma} : Q \to \Sigma = Q/G$. Consider retractions $\mathcal{R}^\Sigma : T\Sigma \to \Sigma$ and $\mathcal{R}^G : TG \to G$ of $\Sigma$ and $G$, respectively, and a trivializing set $U \subset \Sigma$ of $\pi_{Q, \Sigma}$. For simplicity, we write $U = \Sigma$, so we have an identification $Q \simeq \Sigma \times G$. Under this identification, a straightforward check shows that the map

$$\mathcal{R} = (\mathcal{R}^\Sigma, \mathcal{R}^G) : TQ \to Q$$

is a retraction of $Q$. We may use it to define (at least locally) a discrete Lagrange–Pontryagin action. Namely, given a discrete Lagrangian $L_d : Q \times Q \to \mathbb{R}$, the (+)-discrete Lagrange–Pontryagin action is defined as

$$\mathcal{S}_{L_d} \left( (q_k, q^+_k, p_{k+1}, k=0 \right) = \sum_{k=0}^{N-1} \left( L_d(q_k, q^+_k) + \langle p_{k+1}, \mathcal{R}^{-1}_{q_k} - \mathcal{R}^{-1}_{q_{k+1}}(q^+_k) \rangle \right),$$

where $q_k, q^+_k \in Q$ and $p_{k+1} \in T^*_{q^+_k+1} Q$. The (+)-discrete Lagrange–Pontryagin principle is obtained by enforcing free variations vanishing at the endpoints. In order for the previous expression to be well-defined, $q^+_k$ must be in the neighbourhood of $q_{k+1}$ where $\mathcal{R}^{-1}_{q_{k+1}}$ is invertible for each $0 \leq k \leq N-1$. This can always be achieved by reducing the size of the time step. Furthermore, Proposition 6.1 ensures that using retraction compatible coordinate charts, this discrete action reduces to the one considered in the linear case [8].

---

2We will assume that the inverse of $\mathcal{R}_m$ is defined on the whole of $U$ by choosing a smaller coordinate domain if necessary.
On the other hand, recall that the $G$ action on the trivialized principal bundle is given by the left multiplication, i.e. $g \cdot q_0 = (x, g g_0)$ for each $q \in G$ and $q_0 = (x_0, g_0) \in Q \simeq \Sigma \times G$. Furthermore, suppose that the retraction on $G$ is given by $R^G_{g_0} = L_{g_0} \circ \text{exp}(dL_{g_0})$, as mentioned when describing the canonical coordinates of the first kind. Since $G$ is abelian and connected, the exponential map is surjective and, hence, the inverse of $R^G_{g_0}$ exists locally around every element of the group. As a result, when $g_0 = (x_0, g_0)$ is fixed, an action of $T_{g_0} G$ on $T_{g_0} \Sigma$ may be built as follows

$$u_{g_0} \cdot (v_{x_0}, v_{g_0}) = \left(v_{x_0}, (R^G_{g_0})^{-1} (R^G_{g_0}(u_{g_0}) R^G_{g_0}(v_{g_0})) \right), \quad v_{x_0} \in T_{x_0} \Sigma, \quad u_{g_0}, v_{g_0} \in T_{g_0} G.$$ 

When $g_0 = 0$, this action is of the form

$$\xi \cdot (v_{x_0}, \xi_0) = (v_{x_0}, \log(\exp(\xi) \cdot (\xi_0))) = (v_{x_0}, \xi + \xi_0), \quad v_{x_0} \in T_{x_0} \Sigma, \quad \xi, \xi_0 \in \mathfrak{g},$$

where we have used that $G$ is abelian. This is the case considered in the previous sections, a vector subspace acting by addition. Note that if $g_0 \neq 0$, we obtain the same result by using the left translation.

In summary, the linear theory developed above is the coordinate representation of a general system with abelian group of symmetries when retraction compatible charts are used. Furthermore, if $\Sigma$ and $G$ admit both retraction compatible atlases, then a retraction compatible atlas for $Q$ can be built such that each coordinate domain is a trivializing set for $\pi_Q \Sigma$. Hence, we can make computations on the whole $Q$ by starting from a specific compatible chart and changing to another one whenever it is necessary. Since the variational principle, as well as the group action, have the same expressions in each compatible chart, the local preservation of geometric properties extends to the global setting. This is an important feature of retraction compatible atlases, since computation on local charts might otherwise lead to dynamics that is not globally well-defined.

7. Examples

7.1. Charged particle in a magnetic field. We analyze the dynamics of a charged particle moving in a magnetic field, as presented in [22] for the continuous case. To account for gauge symmetry, we consider the Kaluza-Klein configuration space,

$$Q_K = \mathbb{R}^3 \times S^1,$$

with coordinates $(q, \theta)$, where $q = (q^1, q^2, q^3) \in \mathbb{R}^3$ and $\theta = e^{i\theta} \in S^1$. The Kaluza-Klein Lagrangian is defined as

$$L_K(q, \dot{q}, \theta, \dot{\theta}) = \frac{1}{2} m \langle \dot{q}, \dot{q} \rangle + \frac{1}{2} \left( \langle A(q), \dot{q} \rangle + \dot{\theta} \right)^2,$$

where $m \in \mathbb{R}^+$ is the mass of the particle, $A \in \Omega^1(\mathbb{R}^3)$ is the magnetic potential, and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^3$. The conjugate momenta are given by

$$p = \frac{\partial L_K}{\partial \dot{q}} = m \langle \dot{q}, \cdot \rangle + \langle A(q), \dot{q} \rangle + \dot{\theta} \rangle A(q), \quad p_\theta = \frac{\partial L_K}{\partial \dot{\theta}} = \langle A(q), \dot{q} \rangle + \dot{\theta}.$$

It is clear that this Lagrangian is invariant under the tangent lifted action of $S^1$ on $Q_K$ given by $\theta' \cdot (q, \theta) = (q, \theta + \theta)$ for each $(q, \theta) \in Q_K$ and $\theta' \in S^1$. Note that using these coordinates, the action is linear. The corresponding quotient is $\Sigma = Q_K/S^1 \simeq \mathbb{R}^3$. Choosing the principal connection $\omega = A + d\theta \in \Omega^1(\mathbb{R}^3)$ on $Q_K \rightarrow \mathbb{R}^3$, the corresponding reduced equations are the Lorentz force law together with the conservation of the momentum corresponding to $\theta$, that is,

$$m \frac{d\dot{q}}{dt} = -\dot{q} \times \mathbf{B}, \quad \dot{p}_\theta = 0,$$

where $\mathbf{B} \in \mathfrak{X}(\mathbb{R}^3)$ is the magnetic field corresponding to $dA = \omega \in \Omega^2(\mathbb{R}^3)$, and $e = c p_\theta$ is the electric charge of the particle.

Recall that we can associate to each 2-form $\alpha = \alpha^i d\xi_i \in \Omega^2(\mathbb{R}^3)$ a vector field $\mathbf{a} = \mathbf{a}^i \partial_i \in \mathfrak{X}(\mathbb{R}^3)$, which is known as proxy field, where $\iota_U$ denotes the left interior product by $U \in \mathfrak{X}(\mathbb{R}^3)$. \footnote{Recall that we can associate to each 2-form $\alpha = \alpha^i d\xi_i \in \Omega^2(\mathbb{R}^3)$ a vector field $\mathbf{a} = \mathbf{a}^i \partial_i \in \mathfrak{X}(\mathbb{R}^3)$, which is known as proxy field, where $\iota_U$ denotes the left interior product by $U \in \mathfrak{X}(\mathbb{R}^3)$.}
7.1.1. Discrete equations. Given $h > 0$, consider the following discrete Lagrangian

$$L_d(q_0, \theta_0, q_0^+, \theta_0^+; h) = hL_K \left(q_0, \frac{q_0^+ - q_0}{h}, \theta_0, \frac{\theta_0^+ - \theta_0}{h} \right).$$

It is easy to check from Definition 2.1 that the following is a discrete principal connection,

$$\omega_d((q_0, \theta_0), (q_0^+, \theta_0^+)) = \theta_0^+ - \theta_0, \quad (q_0, \theta_0), (q_0^+, \theta_0^+) \in Q_K.$$

Subsequently, the local map $h_d : Q_K \times \mathbb{R}^3 \rightarrow S^1$ is given by $h_d((q_0, \theta_0), (q_0^+, \theta_0^+)) = \theta_0$ and $h_{d, \Sigma} \equiv 0$. The reduced discrete Lagrangian (24) is

$$l_d(q_0, q_0^+, \theta_0^+; h) = L_d(q_0, 0, q_0^+, \theta_0^+ + h_0^0(q_0, q_0^+))$$

$$= \frac{m}{2h} \langle q_0^+ - q_0, q_0^+ - q_0 \rangle + \frac{1}{2h} (\langle A(q_0), q_0^+ - q_0 \rangle + \theta_0^+)^2.$$

Consider an interval $[0, T] \subset \mathbb{R}$ and divide it into $N = T/h$ subintervals $[t_k, t_{k+1}]$, with $t_k = kh$, $0 \leq k \leq N - 1$. As usual, we denote $q_k = q(t_k)$, $0 \leq k \leq N$, and analogously for the other quantities. Similarly, we express the discrete vector field as $[X_d^k] = (q_k, w_k, \mu_k, q_{k+1}, \theta_{k+1}, w_{k+1}, \mu_{k+1})$. By computing the partial derivatives of the reduced discrete Lagrangian we obtain the reduced discrete equations

$$\begin{cases}
\frac{m}{h} \langle q_{k+1} - q_k, \cdot \rangle + \mu_{k+1}A(q_k) = w_{k+1} \\
\frac{1}{h} (\langle A(q_k), q_{k+1} - q_k \rangle + \theta_{k+1}) = \mu_{k+1} \\
\frac{m}{h} \langle q_{k+1} - q_k, \cdot \rangle - \mu_{k+1} (\langle A, q_{k+1} - q_k \rangle - A(q_k)) = w_k \\
\mu_{k+1} = \mu_k
\end{cases}, \quad 0 \leq k \leq N - 1.
$$

(32)

7.1.2. Numerical computations. In order to implement the above equations, we need to make a particular choice of the particle mass and charge, time interval, magnetic field and initial conditions. We express all of the magnitudes in natural units, that is, $c = 1$. We choose $m = 1$, $e = 1$ and $T = 20$. For each $q = (q^1, q^2, q^3) \in \mathbb{R}^3$, we write $q = q^i \partial_i \in T_q \mathbb{R}^3 \simeq \mathbb{R}^3$ when regarded as a vector. Hence, $\langle q, \cdot \rangle = q^i dq^i$. On the other hand, we suppose that the magnetic field is constant, i.e., $B(q) = B_0 \partial_z$ for some fixed $B_0 \in \mathbb{R}$. The corresponding magnetic potential is

$$A(q) = \frac{B_0}{2} \left(-q^2 dq^1 + q^1 dq^2 \right), \quad q = (q^1, q^2, q^3) \in \mathbb{R}^3.$$

We suppose that the initial position is the origin, $q_0 = 0$, and the initial velocity is $q_0 = \partial_1 + \partial_3$. Likewise, we choose $\theta_0 = 0$. Since $A(q_0) = A(0) = 0$, the corresponding initial momenta are

$$p_0 = \langle q_0, \cdot \rangle + (\langle A(q_0), q_0 \rangle + \theta) A(q_0) = \langle q_0, \cdot \rangle = dq^1 + dq^3$$

and $(p_{\theta})_0 = e = 1$. Likewise, we have $\theta_0 = (p_{\theta})_0 - \langle A(q_0), \dot{q}_0 \rangle = 1$. The initial position and momenta may be trivialized using (15),

$$\left( (q_0, \theta_0), w_0, \mu_0 \right) = \lambda_d \left( (q_0, \theta_0), (p_0, (p_{\theta})_0) \right) = \left( (0, 0), (1, 0, 1), 1 \right).$$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|q(T) - q_N|$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.6781</td>
</tr>
<tr>
<td>50</td>
<td>0.25971</td>
</tr>
<tr>
<td>100</td>
<td>0.06626</td>
</tr>
<tr>
<td>200</td>
<td>0.01664</td>
</tr>
</tbody>
</table>

Table 1. Number of steps vs. error at the last step.
On the other hand, it is easy to check that the exact solution of (31) for $B_0 = 1$ is
$$q(t) = (\sin(t), \cos(t) - 1, t), \quad 0 \leq t \leq T.$$  

After working out the approximate solution for different values of $N$ and comparing to the exact solution, we can see that the error decreases with the number of steps at a second-order rate of convergence, as shown in Table 1. In addition, Figure 1 compares both the exact and numerical trajectories for $N = 100$, i.e., $h = 0.2$.

7.2. **Double spherical pendulum.** In this example, we investigate the double spherical pendulum. We assume that there is no friction and that the system is under a uniform gravitational field. The dynamics of the double spherical pendulum has been investigated in [19; 21] and variational integrators from different perspectives have been proposed in [10; 12].

For $i = 1, 2$, we denote by $m_i \in \mathbb{R}^+, l_i \in \mathbb{R}^+$ and $r_i \in \mathbb{R}^3$ the particle mass, the link length and the position of the $i$-th pendulum, respectively. Furthermore, we use standard coordinates $r_i = (r_i^1, r_i^2, r_i^3) \in \mathbb{R}^3$ with coordinate origin at the fixed pivot, and we suppose that the gravitational acceleration is given by $g = (0, 0, -g)$ for some fixed $g \in \mathbb{R}^+$. This way, the Lagrangian is
$$L(r_1, r_2, \dot{r}_1, \dot{r}_2) = \frac{1}{2}m_1 \langle \dot{r}_1, \dot{r}_1 \rangle + \frac{1}{2}m_2 \langle \dot{r}_2, \dot{r}_2 \rangle - m_1 g r_1^3 - m_2 g r_2^3,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^3$. In addition, the links connecting the particles yield the following constraints,
$$\langle r_1, r_1 \rangle = l_1^2, \quad \langle r_2 - r_1, r_2 - r_1 \rangle = l_2^2.$$

To avoid the constraints we use spherical coordinates for each particle, $r_i = (\rho_i, \theta_i, \varphi_i)$, $i = 1, 2$, where the origin of the first sphere is at the pivot and the origin of the second one is at the first
Figure 2. Spherical coordinates for the double spherical pendulum.

particle, as shown in Figure 2. In such case, the constraints become $\rho_i = l_i$, $i = 1, 2$, and may be introduced straightforwardly in the Lagrangian. This way, the configuration space is given by

$$Q = S^2 \times S^2$$

with angular coordinates $(\theta_1, \phi_1, \theta_2, \phi_2)$. After the change of coordinates, the Lagrangian reads

$$L((\theta_1, \varphi_1, \theta_2, \varphi_2), \dot{\theta}_1, \dot{\varphi}_1, \dot{\theta}_2, \dot{\varphi}_2) =$$

$$\frac{1}{2} l_1^2 m_1 (\dot{\varphi}_1^2 + \dot{\theta}_1^2 \sin^2 \varphi_1) + T_2 - g m_1 l_1 \cos \varphi_1 - g m_2 (l_1 \cos \varphi_1 + l_2 \cos \varphi_2),$$

where $T_2$ is the kinetic energy of the second pendulum,

$$T_2 = \frac{1}{2} m_2 \left( l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 + l_1^2 \dot{\theta}_1^2 \sin^2 \varphi_1 + l_2^2 \dot{\theta}_2^2 \sin^2 \varphi_2 + 2 l_1 l_2 (\dot{\varphi}_1 \dot{\varphi}_2 \cos \varphi_1 \cos \varphi_2 \cos(\theta_1 - \theta_2) + \dot{\varphi}_1 \dot{\theta}_2 \sin \varphi_2 \sin(\theta_1 - \theta_2) \cos \varphi_1$$

$$- \dot{\varphi}_2 \dot{\theta}_1 \sin \varphi_1 \sin(\theta_1 - \theta_2) \cos \varphi_2 + \dot{\theta}_1 \dot{\theta}_2 \sin \varphi_1 \sin \varphi_2 \cos(\theta_1 - \theta_2)) \right).$$

Observe that the system is invariant by simultaneous rotation of both pendula around the $Z$-axis, that is, the group of symmetries is $G = S^1$ with the action on $S^2 \times S^2$ given locally by

$$\theta \cdot (\theta_1, \varphi_1, \theta_2, \varphi_2) = (\theta + \theta_1, \varphi_1, \theta + \theta_2, \varphi_2),$$

for each $(\theta_1, \varphi_1, \theta_2, \varphi_2) \in S^2 \times S^2$ and $\theta \in S^1$. As usual, we denote the quotient by $\Sigma = (S^2 \times S^2)/S^1$. 
**Remark 7.1.** Observe that this action is not free. Indeed, it leaves invariant configurations with \( \varphi_1 = k_1 \pi \) and \( \varphi_2 = k_2 \pi \) for some \( k_1, k_2 \in \mathbb{Z} \). Therefore, the following is only valid for trajectories not passing through those configurations.

At last, we perform another change of coordinates,

\[
\vartheta_1 = \frac{\theta_1 + \theta_2}{2}, \quad \vartheta_2 = \frac{\theta_2 - \theta_1}{2},
\]

with \( \varphi_1 \) and \( \varphi_2 \) remaining the same. Observe that the inverse is given by \( \theta_1 = \vartheta_1 - \vartheta_2 \) and \( \theta_2 = \vartheta_1 + \vartheta_2 \). In these coordinates, the action reads

\[
\theta \cdot (\vartheta_1, \varphi_1, \vartheta_2, \varphi_2) = (\theta + \vartheta_1, \varphi_1, \vartheta_2, \varphi_2),
\]

for each \( (\vartheta_1, \varphi_1, \vartheta_2, \varphi_2) \in \mathbb{S}^2 \times \mathbb{S}^2 \) and \( \theta \in \mathbb{S}^1 \).

**7.2.1. Discrete equations.** For the sake of simplicity, we will denote \( q_0 = (\vartheta_1, \varphi_1, \vartheta_2, \varphi_2) \in \mathbb{S}^2 \times \mathbb{S}^2 \) and \( x_0 = (\varphi_1, \vartheta_2, \varphi_2) \in (\mathbb{S}^2 \times \mathbb{S}^2)/\mathbb{S}^1 \), and analogous for \( q_0^+ \) and \( x_0^+ \). Given \( h > 0 \), we define the discrete Lagrangian as

\[
L_d(q_0, q_0^+; h) = hL \left( q_0, \frac{q_0 - q_0^+}{h} \right), \quad q_0, q_0^+ \in \mathbb{S}^2 \times \mathbb{S}^2.
\]

Likewise, we choose the following discrete principal connection

\[
\omega_d(q_0, q_0^+) = \vartheta_1^+ - \vartheta_1, \quad q_0, q_0^+ \in \mathbb{S}^2 \times \mathbb{S}^2.
\]

In particular, the map \( h_d : Q \times \Sigma \rightarrow G \) is given by \( h_d(q_0, x_0) = \vartheta_1 \). Thus, \( h_d, \Sigma \equiv 0 \), \( h_d, \Sigma \equiv 0 \) and \( h_d^0 \equiv 0 \). The reduced discrete Lagrangian is

\[
l_d(x_0, x_0^+; h) = L_d((0, \varphi_1, \vartheta_2, \varphi_2), (\vartheta_1^+, \varphi_1^+, \vartheta_2^+, \varphi_2^+); h)
\]

for each \( x_0, x_0^+ \in \Sigma \) and \( \vartheta_1^+ \in \mathbb{S}^1 \). Given the interval \([0, T] \subset \mathbb{R} \), we divide it into \( N = T/h \) subintervals \([t_k, t_{k+1}]\), with \( t_k = kh \), \( 0 \leq k \leq N - 1 \). We denote the discrete vector field by \( [\dot{X}^h] = (x_k, w_k, \mu_k, q_{k+1}, w_{k+1}, \mu_{k+1}) \), \( 0 \leq k \leq N - 1 \). Observe that \( w_k = (w_{1k}, w_{2k}, w_{3k}) \in \mathbb{S}^* \approx \mathbb{R}^3 \) and \( \mu_k \in G^* \approx \mathbb{R} \). Since \( h_d((\vartheta_1, 0), 0) = \vartheta_1 \) and \( h_d, Q(x_0, 0) = 0 \), the reduced discrete equations read

\[
\begin{cases}
\frac{\partial l_d}{\partial \varphi_1^+} - \frac{\partial l_d}{\partial \varphi_2^+} = (w_1^+, w_1^+, w_3^+), \\
\frac{\partial l_d}{\partial \varphi_2^+} + \frac{\partial l_d}{\partial \varphi_1^+} = (w_2^+, w_2^+, w_3^+), \\
\frac{\partial l_d}{\partial \vartheta_1^+} = \mu_0,
\end{cases}
\]

where \( (\varphi_1, \vartheta_2, \varphi_2), \mu_0 \) and \( w_0 \) are the initial conditions, whereas \( (\vartheta_1^+, \varphi_1^+, \vartheta_2^+, \varphi_2^+) \), \( \mu_1 \) and \( w_1 \) are the unknowns. As in the previous example, by using \( [\dot{X}^h] \), the momentum is given by \( p_0 = (\mu_0, w_0) \).

**7.2.2. Numerical computations.** For specific computations we need to fix the parameters of the system, as well as the initial conditions. Using SI units, we pick \( T = 100 \), \( N = 10^4 \), \( m_1 = 20 \), \( m_2 = 35 \), \( l_1 = 500 \), \( l_2 = 800 \), \( g = 9.8 \), \( q_0 = (0, 3\pi/4, 1, 3\pi/4) \) and \( (\mu_0, w_0) = (0, 0.1, 0.1, -0.1) \). Recall that the \(-\)-discrete generalized energy of the system is given by \( \frac{29}{29} \), that is, \( (E_d)_k = L_d(q_k, q_k^+) \), where we have used that \( q_{k+1} = q_k^+ \), \( 0 \leq k \leq N - 1 \). The evolution of the energy is plotted in Figure 3. Observe that it exhibits good near energy conservation, since it oscillates around a fixed value instead of exhibiting a spurious drift. This is typical of symplectic, and in particular, variational integrators, but is not generally true of standard integrators.
Figure 3. Evolution of $\varphi_1$ (top left), projection of the trajectory of the second pendulum on the $XY$-plane (top right) and evolution of the discrete energy (bottom) for the double spherical pendulum.

8. Conclusions

In this paper, we develop the theory of discrete Dirac reduction of discrete Lagrange–Dirac systems with an abelian symmetry group acting on a linear configuration space. This involves the use of the notion of discrete principal connections to coordinatize the quotient spaces, which allows us to study the reduction of the Dirac structure and the discrete variational principle expressed in terms of the discrete generalized energy. Both the reduced discrete Dirac structure and the reduced discrete variational principle leads to the same reduced discrete equations of motion. We also discussed the role of retractions and the atlas of retraction compatible maps in allowing us to generalize the local theory that was discussed to a discrete reduction theory that is globally well-defined on a manifold.

Framing the discrete reduction theory in the context of discrete Dirac mechanics is significant, as it is the natural setting for studying discrete Hamiltonian mechanics on manifolds, particularly when expressed in terms of the discrete generalized energy. This is because the notion of discrete Hamiltonians $[11, 15]$, which are Type 2 and 3 generating functions, do not make intrinsic sense on a nonlinear manifold, since it is not possible to specify a covector without also specifying a base point. In contrast, the discrete generalized energy does make intrinsic sense, and is a more promising foundation on which to construct a discrete analogue of Hamiltonian mechanics on manifolds. Discrete Dirac mechanics is also the basis of a discrete theory of interconnections $[30]$, which allows one to construct discretizations of complex multiphysics systems by interconnecting simpler subsystems.

For future work, we will extend this to the setting of nonabelian symmetry groups, and to discrete analogues of Routh reduction, where the discrete dynamics is also restricted to the level sets of the discrete momentum.
ACKNOWLEDGEMENTS

ARA was supported by a FPU grant from the Spanish Ministry of Science, Innovation and Universities (MICIU). ML was supported in part by the NSF under grants DMS-1411792, DMS-1345013, DMS-1813635, CCF-2112665, by AFOSR under grant FA9550-18-1-0288, and by the DoD under grant HQ00342010023 (Newton Award for Transformative Ideas during the COVID-19 Pandemic).

REFERENCES


Email address: alvrod06@ucm.es

Department of Mathematics, UC San Diego, 9500 Gilman Drive, La Jolla, CA 92093-0112, USA.

Email address: mleok@ucsd.edu