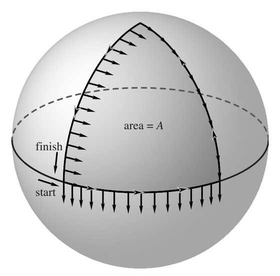
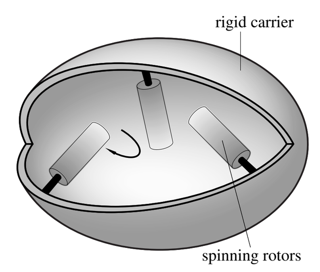
Computational Geometric Mechanics and Geometric Control

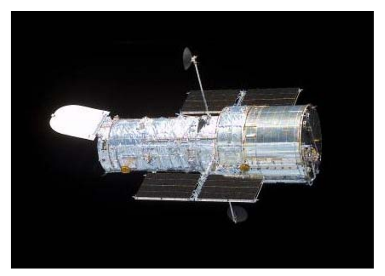
Melvin Leok
Department of Mathematics, University of California, San Diego

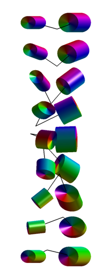
The Geometry of Falling Cats and Satellite Control

- Cats are able to control their orientation while falling by changing their shape, so as to land on their feet.
- There is a nontrivial coupling between the shape and orientation due to the curvature of the space of zero angular momentum.
- This is described mathematically by a connection, which provide a means of comparing elements of a fiber based at different points on the manifold.
- This approach can be used to control the orientation of satellites by using internal momentum wheels and gyroscopes, and is more precise than methods based on chemical propulsion.







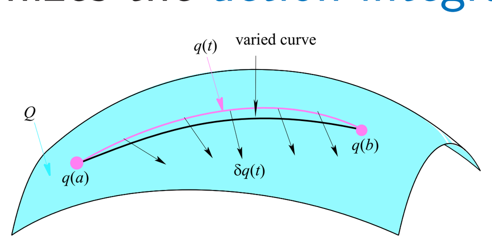


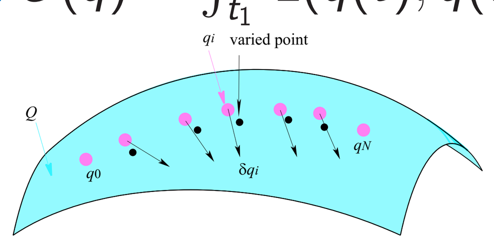
Geometry and Numerical Methods

- Many continuous dynamical systems have conserved geometric invariants:
  - Energy
  - Symmetries, Reversibility, Monotonicity
  - Momentum - Angular, Linear, Kelvin Circulation Theorem.
  - Symplectic Form
  - Integrability
- At other times, the equations themselves are defined on a manifold, such as a Lie group, or more generally, a configuration manifold of a mechanical system, and we require numerical methods that automatically remain on the manifold.
- Geometric invariants affect the qualitative properties of dynamical systems, and geometric numerical integrators conserve discrete geometric invariants.

Discrete Variational Mechanics

- Mechanics can be described covariantly by considering a Lagrangian, $L : TQ \to \mathbb{R}$. that is given by the difference of kinetic and potential energies.
- Hamilton's principle states that the trajectory $q(t)$ that joins two points $q(t_1)$ and $q(t_2)$ extremizes the action integral $\mathcal{S}(q) = \int_{t_1}^{t_2} L(q(t), \dot{q}(t))dt$.





- We introduce a discrete Lagrangian, $L_d(q_0, q_1) \approx \int_0^h L(q(t), \dot{q}(t))dt$.
- The discrete Hamilton's principle states that $\mathcal{S}_d = \sum_{k=0}^{N-1} L_d(q_k, q_{k+1})$ is stationary. This leads to the discrete Euler–Lagrange equations,

$$D_2L_d(q_{k-1}, q_k) + D_1L_d(q_k, q_{k+1}) = 0,$$

which induces a map $F_{L_d} : (q_{k-1}, q_k) \mapsto (q_k, q_{k+1})$, that is automatically symplectic, momentum-preserving, and exhibits good energy behavior.

Comparing representations of the rotation group $SO(3)$

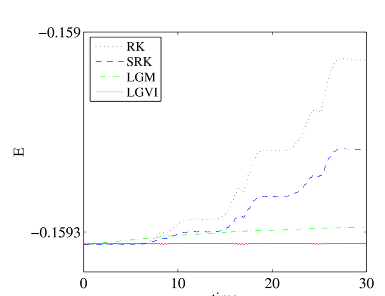
- Euler Angles
  - Local coordinate chart, exhibits singularities.
  - Requires change of charts to simulate large attitude maneuvers.
- Unit Quaternions
  - Reprojection used to stay on unit 3-sphere.
  - The 3-sphere is a double-cover of $SO(3)$ which causes topological problems for optimization.
- Rotation Matrices
  - 9 dimensional space (3×3 matrices) with a 6 dimensional constraint (orthogonality), but the exponential map saves the day.

Variational Lie Group Techniques

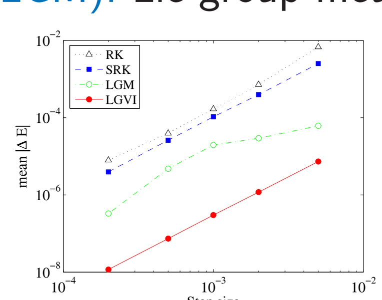
- To stay on the Lie group, we parametrize the curve by the initial point g_0, and elements of the Lie algebra ξ_i, such that, $g_d(t) = \exp\left(\sum \xi^s \tilde{l}_{k,s}(t)\right) g_0$.
- The Lie algebra is a linear space, and we use standard approximation methods on the Lie algebra and lift to the group by using the exponential map.
- Automatically stays on $SO(n)$ without the need for reprojection, constraints, or local coordinates.
- Cayley transform based methods perform 5-6 times faster, without loss of geometric conservation properties.

Numerical Simulations

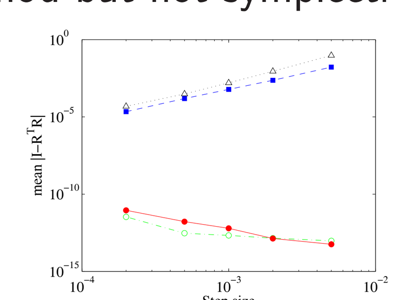
- Our Lie group variational integrator (LGVI) is a Lie Störmer–Verlet method, so it is a second-order symplectic Lie group method.
- We compare it to other second-order accurate methods:
  - Explicit Midpoint Rule (RK): Preserves neither symplectic nor Lie group properties.
  - Implicit Midpoint Rule (SRK): Symplectic but does not preserve Lie group properties.
  - Crouch-Grossman (LGM): Lie group method but not symplectic.



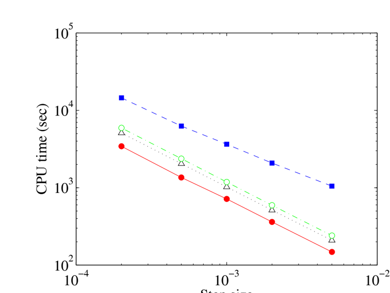
Computed total energy for 30 seconds



Mean total energy error $|E - E_0|$ vs. step size



Mean orthogonality error $\|I - R^T R\|$ vs. step size



CPU time vs. step size

Geometric Optimal Control Algorithms

- Traditional approach
  - Local analysis of the connection near the desired shape position.
  - Gives a closed form expression for the geometric phase associated with infinitesimally small loops in shape space.
  - Resulting shape trajectories are often suboptimal and slow.
- Proposed approach
  - Homotopy-based optimal control algorithm using geometrically exact numerical schemes.
  - Allows for large-amplitude trajectories that are global in nature, and more efficient than infinitesimal loops.

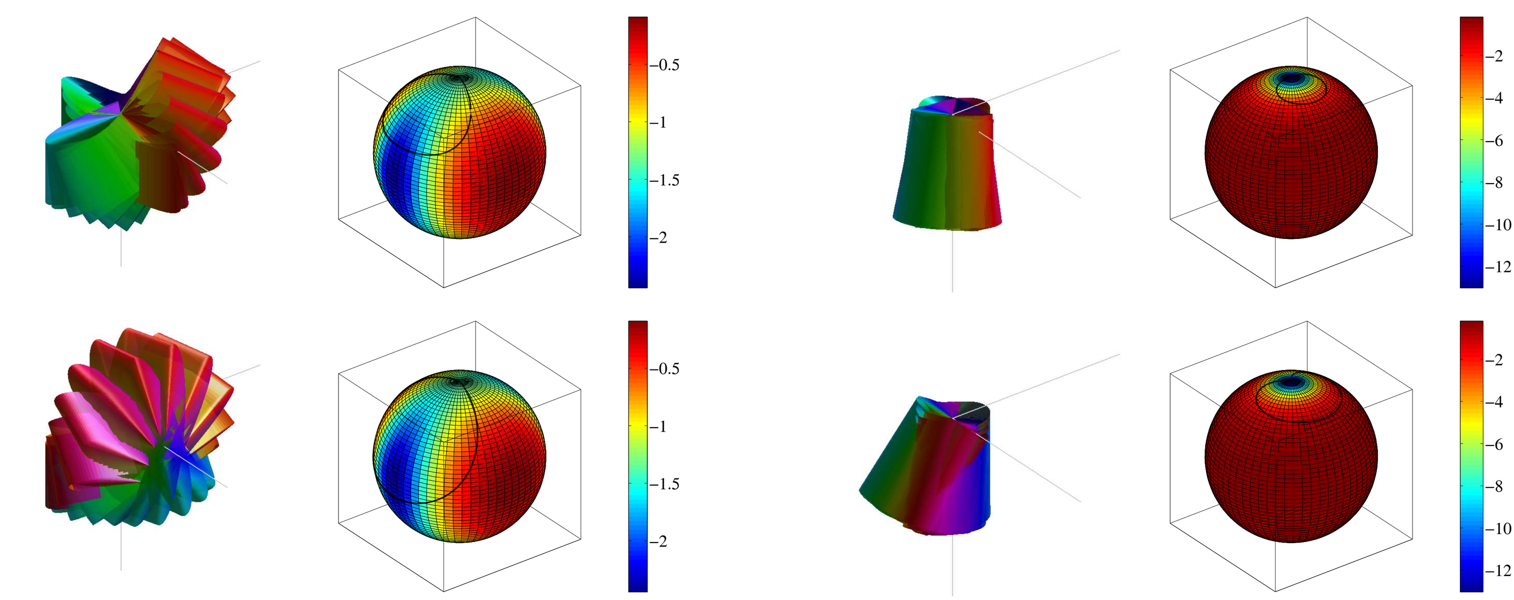
Discrete Geometric Optimal Control

- Use the discrete Lagrange–d'Alembert principle,

$$\delta \sum L_d(q_k, q_{k+1}) + \sum F_d(q_k, q_{k+1}) \cdot (\delta q_k, \delta q_{k+1}) = 0,$$

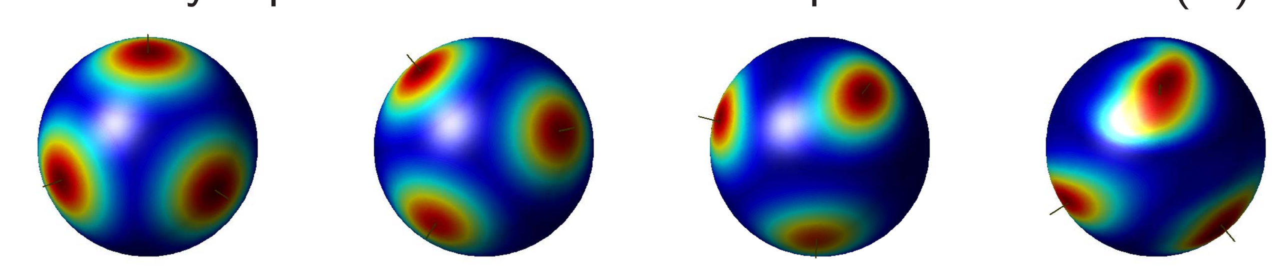
to derive the discrete forced Euler–Lagrange equations, and impose these as constraints at every time-step.
- This yields greater fidelity to the equations of motion than imposing the dynamical constraints using the method of collocation.
- The resulting numerical solutions are group-equivariant, which implies that the numerical solutions are independent of the choice of coordinate frame.

Underactuated Control of a 3D Pendulum



Uncertainty Propagation on Lie Groups

- Gromov's nonsqueezing theorem from symplectic geometry implies that there is a lower bound to the projected volume onto position-momentum planes that depends on the initial projected volume of the ensemble.
- The proposed method generalizes the generalized polynomial chaos approach, and involves solving the Liouville equation by using sample trajectories generated by Lie group variational integrators to reconstruct the distribution.
- We construct an approximation of the distribution using noncommutative harmonic analysis, in particular, the Peter–Weyl theorem, which relates irreducible unitary representations with a complete basis for $L^2(G)$.



Summary

- Geometry has an important role in nonlinear control and numerical methods.
- Geometric control theory takes into account the interaction between shape and group variables.
- Discrete geometry and mechanics is important for developing accurate and efficient computational schemes.

Supported by NSF Grants DMS-0726263, DMS-1001521, DMS-1010687 (CAREER), CMMI-1029445, DMS-1065972 (FRG)

18th Annual German-American Frontiers of Science Symposium, Potsdam, Germany, May 10-13, 2012