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Geometric Formulations of Furuta Pendulum Control Problems

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Abstract. A Furuta pendulum is a serial connection of two thin, rigid links, where the first link is actuated by a vertical control torque while it is constrained to rotate in a horizontal plane; the second link is not actuated. The second link of the conventional Furuta pendulum is constrained to rotate in a vertical plane orthogonal to the first link, under the influence of gravity. Methods of geometric mechanics are used to formulate a new global description of the Lagrangian dynamics on the configuration manifold $(\mathbb{S}^1)^2$. In addition, two modifications of the Furuta pendulum, viewed as double pendulums, are introduced. In one case, the second link is constrained to rotate in a vertical plane that contains the first link; global Lagrangian dynamics are developed on the configuration manifold $(\mathbb{S}^1)^2$. In the other case, the second link can rotate without constraint; global Lagrangian dynamics are developed on the configuration manifold $\mathbb{S}^1 \times \mathbb{S}^2$. The dynamics of the Furuta pendulum models can be viewed as under-actuated nonlinear control systems. Stabilization of an inverted equilibrium is the most commonly studied nonlinear control problem for the conventional Furuta pendulum. Nonlinear, under-actuated control problems are introduced for the two modifications of the Furuta pendulum introduced in this paper, and these problems are shown to be extremely challenging.

1 Introduction

A serial connection of two thin, rigid links, where the first link is constrained to rotate in a horizontal plane about a fixed vertical axis and the second link is constrained to rotate in a vertical plane that is orthogonal to the first link, was introduced in 1992 by K. Furuta and his co-authors [1]. The Furuta pendulum is an example of a double pendulum. It has two degrees of freedom; uniform, constant gravity affects the second link but not the first link.

This example has served for both theoretical and experimental investigation of nonlinear control. Typically, a control torque is applied about the first vertical axis, while the second axis is not actuated;

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the objective is most commonly to stabilize the second link to an inverted equilibrium. Thus the Furuta pendulum is an under-actuated dynamic system that is unstable at any inverted equilibrium.

In the following, several different versions of the Furuta pendulum, based on idealized rigid, massless links with concentrated masses, are studied; we assume control actuation of the first link only. It is assumed throughout that there are no collisions of the links.

Essentially all of the existing publications make use of local angle coordinates to describe the dynamics of the Furuta pendulum; this necessarily introduces singularities into the models, thereby limiting the validity of the model [1, 2, 3, 4, 5, 6, 7]. In addition, the equations of motion for the Furuta pendulum, and in fact for most serial connections of pendulums, are complicated and this complication is exaggerated when using local angle coordinates. In particular, [8] identifies several errors in the published literature.

In contrast, the following developments are based on a geometric formulation that avoids the use of local angle coordinates. This geometric formulation leads to equations of motion without singularities or ambiguities [9]. Although the equations of motion are still complicated, they do have a geometric structure and they avoid complicated trigonometric functions. Once the Lagrangian function is identified, the subsequent computations required to obtain the equations of motion are amenable to symbolic computations.

As usual, we denote the unit sphere, centered at the origin in $\mathbb{R}^2$, by $\mathbb{S}^1$ and we denote the unit sphere, centered at the origin in $\mathbb{R}^3$, by $\mathbb{S}^2$. These are each a differentiable manifold with relatively simple geometry. We denote the standard basis for $\mathbb{R}^2$ by e_1, e_2 and the standard basis for $\mathbb{R}^3$ by e_1, e_2, e_3; in the subsequent development the meaning of e_1 or e_2 should be clear from the context. We use g to denote the constant acceleration of gravity.

The following matrix notation is used subsequently:

$$\mathcal{S} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

In addition, we use the skew-symmetric matrix function $S : \mathbb{R}^3 \to \mathfrak{so}(3)$ given by

$$\mathbb{S}(\gamma) = \begin{bmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{bmatrix}.$$

2 The 2 DOF conventional Furuta pendulum

We consider the standard Furuta pendulum. Assume that the first link is constrained by an inertially fixed pivot to rotate in a fixed horizontal plane, while the second link is constrained to rotate in a vertical plane that is orthogonal to the first link.

Assume the first link is a thin, rigid body idealized as a massless link; one end is pinned to an inertial frame so that the first link is constrained to rotate in a horizontal plane; the length of the massless link is L_1 and a mass m_1 is concentrated at the end of the first link. The second link is also a thin, rigid body idealized as a massless link; the length of the massless link is L_2 and a mass m_2 is concentrated at the end of the second link.

Introduce a Euclidean inertial frame for $\mathbb{R}^3$ where the origin is located at the inertially fixed pivot of the first link; the first two axes are assumed to be horizontal while the third axis is assumed to be vertical.

Since the mass at the end of the first link moves in a horizontal plane, the position vector $x_1 \in \mathbb{R}^3$ of the first mass element in the inertial frame is

$$x_1 = L_1 \zeta_1,$$

where $\zeta_1 \in \mathbb{S}^2$ is expressed in terms of $q_1 = (q_{11}, q_{12}) \in \mathbb{S}^1$ by using the relation

$$\zeta_1 = C q_1.$$

Here, q_1 can be thought of as the unit direction vector of the first link in the horizontal plane within which the first mass element moves, and this relation enforces the constraint that the first link rotates in the horizontal plane.

The position vector $x_2 \in \mathbb{R}^3$ of the second mass element in the inertial frame is most easily described in terms of the basis for $\mathbb{R}^3$:

$$Cq_1, \quad C\mathcal{S}q_1, \quad e_3.$$

This can be shown to be an orthonormal basis set for $\mathbb{R}^3$. This basis has the property that the direction of the first link is always along the first basis vector, while the direction of the second link lies in the span of the second and third basis vectors.

The position vector $x_2 \in \mathbb{R}^3$ can be expressed as

$$x_2 = x_1 + L_2 \zeta_2,$$

where $\zeta_2 \in \mathbb{S}^2$ is expressed in terms of $q_2 = (q_{21}, q_{22}) \in \mathbb{S}^1$ by using the relation

$$\zeta_2 = q_{21} C\mathcal{S} q_1 + q_{22} e_3,$$

Here, q_2 can be thought of as the unit direction vector for the second link in its instantaneous plane of rotation, which is spanned by $C\mathcal{S}q_1$ and e_3. Then, using the fact that $q_{21} = e_1^T q_2$ and $q_{22} = e_2^T q_2$, we can check that

$$x_2 = L_1 C q_1 + L_2 (e_1^T q_2 C\mathcal{S} q_1 + D q_2) = L_1 C q_1 + L_2 (C\mathcal{S} q_1 e_1^T + D) q_2.$$

Since the position vectors for the two mass elements that define the Furuta pendulum can be expressed in terms of $q = (q_1, q_2) \in (\mathbb{S}^1)^2$, the configuration manifold for the Furuta pendulum is $(\mathbb{S}^1)^2$; the dimension of the configuration manifold is two, so there are two degrees of freedom. The following development follows the approach developed in [9] for Lagrangian dynamics that evolve on the manifold $(\mathbb{S}^1)^2$.

The velocity vectors of the two mass elements in the inertial frame are

$$\begin{aligned} \dot{x}_1 &= L_1 C \dot{q}_1, \\ \dot{x}_2 &= L_1 C \dot{q}_1 + L_2 (e_1^T \dot{q}_2 C\mathcal{S} q_1 + e_1^T q_2 C\mathcal{S} \dot{q}_1 + D \dot{q}_2) \\ &= (L_1 C + L_2 e_1^T q_2 C\mathcal{S}) \dot{q}_1 + L_2 (C\mathcal{S} q_1 e_1^T + D) \dot{q}_2. \end{aligned}$$

The expression for the kinetic energy of the mass element of the first link is

$$T_1(q,\dot{q}) = \frac{1}{2}m_1\|\dot{x}_1\|^2 = \frac{1}{2}m_1L_1^2\|\dot{q}_1\|^2.$$

The kinetic energy of the mass element of the second link is

$$T_2(q,\dot{q}) = \frac{1}{2}m_2\|\dot{x}_2\|^2 = \frac{1}{2}m_2\left\|(L_1C+L_2e_1^Tq_2C\mathcal{S})\dot{q}_1+L_2(C\mathcal{S}q_1e_1^T+D)\dot{q}_2\right\|^2.$$

Thus, the expression for the kinetic energy of the Furuta pendulum is

$$\begin{aligned} T(q,\dot{q}) &= \frac{1}{2}m_1L_1^2\|\dot{q}_1\|^2 + \frac{1}{2}m_2\left\|(L_1C+L_2e_1^Tq_2C\mathcal{S})\dot{q}_1+L_2(C\mathcal{S}q_1e_1^T+D)\dot{q}_2\right\|^2 \\ &= \frac{1}{2}\dot{q}_1^Tm_{11}\dot{q}_1+\dot{q}_1^Tm_{12}\dot{q}_2+\frac{1}{2}\dot{q}_2^Tm_{22}\dot{q}_2, \end{aligned}$$

where the inertia terms are

$$\begin{aligned} m_{11} &= \left\{(m_1+m_2)L_1^2+m_2L_2^2(e_1^Tq_2)^2\right\}I_{2\times2}, \\ m_{12} &= m_2L_2\left\{L_1\mathcal{S}+L_2e_1^Tq_2I_{2\times2}\right\}q_1e_1^T, \\ m_{22} &= m_2L_2^2I_{2\times2}. \end{aligned}$$

The gravitational potential energy of the Furuta pendulum is

$$U(q) = m_2ge_3^Tx_2 = m_2gL_2e_3^TDq_2,$$

where we have used the facts that $e_3^Tx_1=0$ and $e_3^Tx_2=L_2e_3^TDq_2$.

The Lagrangian function $L:\mathrm{T}(\mathbb{S}^1)^2\to\mathbb{R}^1$ for the Furuta pendulum is given by

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}_1^Tm_{11}\dot{q}_1+\dot{q}_1^Tm_{12}\dot{q}_2+\frac{1}{2}\dot{q}_2^Tm_{22}\dot{q}_2-mgL_2e_3^TDq_2.$$

Let $u\in\mathbb{R}^1$ denote the control torque on the first link in the vertical direction. The corresponding virtual work done by this control torque is easily determined, so that the Lagrange-d'Alembert principle can be written as

$$\delta\int_{t_0}^{t_f}L(q,\dot{q})dt = \int_{t_0}^{t_f}uq_1^T\mathcal{S}\delta q_1\,dt,$$

for all possible infinitesimal variations $\delta q=(\delta q_1,\delta q_2):[t_0,t_f]\to\mathrm{T}_q(\mathbb{S}^1)^2$ that vanish at the endpoints, that is $\delta q(t_0)=\delta q(t_f)=0$. Following the results in [9], the infinitesimal variations can be expressed as

$$\delta q_i = \gamma_i\mathcal{S}q_i,\ i=1,2$$

where $\gamma_i:[t_0,t_f]\to\mathbb{R}^1$, $i=1,2$ vanish at t_0 and t_f. Substitute these expressions into the Lagrange-d'Alembert principle, integrate by parts, use the boundary conditions, and apply the fundamental lemma of the calculus of variations. After using several identities, the Euler–Lagrange equations are given by

$$m_{11}\ddot{q}_1 + (I_{2\times 2} - q_1 q_1^T) m_{12}\ddot{q}_2 + m_{11}\|\dot{q}_1\|^2 q_1 + (I_{2\times 2} - q_1 q_1^T) F_1(q,\dot{q}) = uSq_1, \tag{2.1}$$

$$(I_{2\times 2} - q_2 q_2^T) m_{12}^T \ddot{q}_1 + m_{22}\ddot{q}_2 + m_{22}\|\dot{q}_2\|^2 q_2 + (I_{2\times 2} - q_2 q_2^T) F_2(q,\dot{q}) + m_2 g L_2 (I_{2\times 2} - q_2 q_2^T) e_2 = 0. \tag{2.2}$$

Here, the terms F_1, F_2 are quadratic in the time derivatives of the configurations. These vector functions are

$$F_1(q,\dot{q}) = \dot{m}_{11}\dot{q}_1 + \dot{m}_{12}\dot{q}_2 - \frac{\partial T(q,\dot{q})}{\partial q_1},$$

$$F_2(q,\dot{q}) = \dot{m}_{12}^T \dot{q}_1 - \frac{\partial T(q,\dot{q})}{\partial q_2}.$$

These Euler–Lagrange equations (2.1) and (2.2) describe the global evolution of the controlled dynamics of the Furuta pendulum on the tangent bundle $\mathrm{T}(\mathbb{S}^1)^2$.

The equilibrium solutions of the standard Furuta pendulum are easily determined assuming zero control input. They occur when the time derivatives of the configurations are zero and when the configuration of the second link satisfies $(I_{2\times 2} - q_2 q_2^T)e_2 = 0$; the configuration of the first link is arbitrary. Consequently, there are two equilibrium manifolds in $(\mathbb{S}^1)^2$ given by

$$\left\{(q_1,q_2) \in (\mathbb{S}^1)^2 : q_2 = e_2\right\},$$

referred to as the inverted equilibrium manifold, and

$$\left\{(q_1,q_2) \in (\mathbb{S}^1)^2 : q_2 = -e_2\right\},$$

referred to as the hanging equilibrium manifold.

3 A 2 DOF modified Furuta pendulum

We now consider the dynamics of a double pendulum, viewed as a modification of the Furuta pendulum, acting under gravity. The pendulum is an interconnection of two thin, rigid links with each rigid link idealized as being massless with fixed length and a mass element at the end of the link. The first link rotates in a fixed horizontal plane. The second link is connected by a frictionless pivot to the end of the first link, so that the plane of rotation of the second link is always spanned by the direction of the first link and the vertical direction. The motion of the mass element of the second link is necessarily in three-dimensions, and in particular, it is constrained to move on the surface of a torus in $\mathbb{R}^3$.

Uniform gravity acts on the mass elements of the two pendulum links. The distance from the fixed pivot to the mass element of the first link is L_1 and m_1 denotes its mass. The distance from the pivot connecting the two links to the mass element of the second link is L_2 and m_2 denotes its mass.

The first link is actuated by a control torque $u \in \mathbb{R}^1$ in the vertical direction; the second link is not actuated.

Introduce a Euclidean inertial frame for $\mathbb{R}^3$ where the origin is located at the inertially fixed pivot of the first link; the first two axes are assumed to be horizontal while the third axis is assumed to be vertical. Since the mass at the end of the first link moves in a horizontal plane, the position vector $x_1 \in \mathbb{R}^3$ of the first mass element in the inertial frame is

$$x_1 = L_1\zeta_1,$$

where $\zeta_1 \in \mathbb{S}^2$ is expressed in terms of $q_1 = (q_{11}, q_{12}) \in \mathbb{S}^1$ by using the relation

$$\zeta_1 = Cq_1.$$

Here, q_1 can be thought of as the unit direction vector of the first link in the horizontal plane within which the first mass element moves, and this expression enforces the constraint that the first link rotates in the horizontal plane.

The position vector $x_2 \in \mathbb{R}^3$ of the second mass element in the inertial frame is most easily described in terms of the basis for $\mathbb{R}^3$:

$$Cq_1, \quad C\mathcal{S}q_1, \quad e_3.$$

This can be shown to be an orthonormal basis set for $\mathbb{R}^3$. This basis has the property that the direction of the first link is always along the first basis vector, while the direction of the second link lies in the span of the first and third basis vectors.

The position vector $x_2 \in \mathbb{R}^3$ can be expressed as

$$x_2 = x_1 + L_2\zeta_2,$$

where $\zeta_2 \in \mathbb{S}^2$ is expressed in terms of $q_2 = (q_{21}, q_{22}) \in \mathbb{S}^1$ by using the relation

$$\zeta_2 = q_{21}Cq_1 + q_{22}e_3.$$

Here, q_2 can be thought of as the unit direction vector for the second link in its instantaneous plane of rotation. Then, using the fact that $q_{21} = e_1^T q_2$ and $q_{22} = e_2^T q_2$, we can check that

$$x_2 = L_1Cq_1 + L_2(e_1^T q_2 Cq_1 + e_2^T q_2 e_3) = L_1Cq_1 + L_2(Cq_1e_1^T + D)q_2.$$

Since the position vectors for the two mass elements that define the modified Furuta pendulum can be expressed in terms of $q = (q_1, q_2) \in (\mathbb{S}^1)^2$, the configuration manifold for the Furuta pendulum is $(\mathbb{S}^1)^2$; the dimension of the configuration manifold is two, so there are two degrees of freedom. The following development follows the approach developed in [9] for Lagrangian dynamics that evolve on the manifold $(\mathbb{S}^1)^2$.

The velocity vectors of the two mass elements in the inertial frame are

$$\begin{aligned}
\dot{x}_1 &= L_1C\dot{q}_1, \\
\dot{x}_2 &= L_1C\dot{q}_1 + L_2(e_1^T \dot{q}_2 Cq_1 + e_1^T q_2 C\dot{q}_1 + D\dot{q}_2) \\
&= (L_1C + L_2e_1^T q_2 C)\dot{q}_1 + L_2(Cq_1e_1^T + D)\dot{q}_2.
\end{aligned}$$

The expression for the kinetic energy of the mass element of the first link is

$$T_1(q,\dot{q}) = \frac{1}{2}m_1\|\dot{x}_1\|^2 = \frac{1}{2}m_1L_1^2\|\dot{q}_1\|^2.$$

The kinetic energy of the mass element of the second link is

$$\begin{aligned}
T_2(q,\dot{q}) &= \frac{1}{2}m_2\|\dot{x}_2\|^2 \\
&= \frac{1}{2}m_2(L_1 + L_2e_1^T q_2)^2\|\dot{q}_1\|^2 + \frac{1}{2}m_2L_2^2\|\dot{q}_2\|^2 + m_2L_2(L_1 + L_2e_1^T q_2)(e_1^T q_1)\dot{q}_1^T e_1^T \dot{q}_2.
\end{aligned}$$

Thus, the expression for the kinetic energy of the modified Furuta pendulum is

$$T(q,\dot{q}) = \frac{1}{2}m_1L_1^2\|\dot{q}_1\|^2 + \frac{1}{2}m_2(L_1+L_2e_1^Tq_2)^2\|\dot{q}_1\|^2 + \frac{1}{2}m_2L_2^2\|\dot{q}_2\|^2$$
$$+ m_2L_2(L_1+L_2e_1^Tq_2)(e_1^Tq_1)\dot{q}_1^Te_1^T\dot{q}_2.$$

This can be written as

$$T(q,\dot{q}) = \frac{1}{2}\dot{q}_1^Tm_{11}\dot{q}_1 + \dot{q}_1^Tm_{12}\dot{q}_2 + \frac{1}{2}\dot{q}_2^Tm_{22}\dot{q}_2,$$

where the inertia terms are

$$m_{11} = \left\{m_1L_1^2 + m_2(L_1+L_2e_1^Tq_2)^2\right\}I_{2\times2},$$
$$m_{12} = m_2L_2\left\{L_1+L_2e_1^Tq_2\right\}q_1e_1^T,$$
$$m_{22} = m_2L_2^2I_{2\times2}.$$

Note that the kinetic energy function for this modified Furuta pendulum has the identical structure as the kinetic energy function for the conventional Furuta pendulum, but with different specifications for the inertia terms.

The gravitational potential energy of the modified Furuta pendulum is

$$U(q) = m_2ge_3^Tx_2 = m_2gL_2e_2^Tq_2,$$

so that the Lagrangian function $L : \mathrm{T}(\mathbb{S}^1)^2 \to \mathbb{R}^1$ is

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}_1^Tm_{11}\dot{q}_1 + \dot{q}_1^Tm_{12}\dot{q}_2 + \frac{1}{2}\dot{q}_2^Tm_{22}\dot{q}_2 - m_2gL_2e_2^Tq_2.$$

Let $u \in \mathbb{R}^1$ denote the control torque on the first link in the vertical direction. The corresponding virtual work done by this control torque is easily determined and the Lagrange-d'Alembert principle can be written as

$$\delta\int_{t_0}^{t_f} L(q,\dot{q})dt = \int_{t_0}^{t_f} uq_1^T\mathcal{S}\delta q_1\,dt,$$

for all possible infinitesimal variations $\delta q = (\delta q_1, \delta q_2) : [t_0,t_f] \to \mathrm{T}_q(\mathbb{S}^1)^2$ that vanish at the endpoints, that is $\delta q(t_0) = \delta q(t_f) = 0$. Following the results in [9], the infinitesimal variations can be expressed as

$$\delta q_i = \gamma_i\mathcal{S}q_i,\ i = 1,2$$

where $\gamma_i : [t_0,t_f] \to \mathbb{R}^1$, $i = 1,2$ vanish at t_0 and t_f. Substitute these expressions into the Lagrange-d'Alembert principle, integrate by parts, use the boundary conditions, and apply the fundamental lemma of the calculus of variations. After using several identities, the Euler–Lagrange equations are given by

$$m_{11}\ddot{q}_1 + (I_{2\times2} - q_1q_1^T)m_{12}\ddot{q}_2 + m_{11}\|\dot{q}_1\|^2q_1 + (I_{2\times2} - q_1q_1^T)F_1(q,\dot{q}) = u\mathcal{S}q_1, \tag{3.1}$$

$$(I_{2\times2} - q_2q_2^T)m_{12}^T\ddot{q}_1 + m_{22}\ddot{q}_2 + m_{22}\|\dot{q}_2\|^2q_2 + (I_{2\times2} - q_2q_2^T)F_2(q,\dot{q})$$
$$+ m_2gL_2(I_{2\times2} - q_2q_2^T)e_2 = 0. \tag{3.2}$$

Here, the terms F_1, F_2 are quadratic in the time derivatives of the configurations. These vector functions are

$$F_1(q,\dot{q}) = \dot{m}_{11}\dot{q}_1 + \dot{m}_{12}\dot{q}_2 - \frac{\partial T(q,\dot{q})}{\partial q_1},$$
$$F_2(q,\dot{q}) = \dot{m}_{12}^T\dot{q}_1 - \frac{\partial T(q,\dot{q})}{\partial q_2}.$$

Thus, the controlled dynamics of the modified Furuta pendulum is described by the evolution of $(q,\dot{q}) \in \mathrm{T}(\mathbb{S}^1)^2$ on the tangent bundle of $(\mathbb{S}^1)^2$.

The equilibrium solutions of the modified Furuta pendulum are easily determined assuming zero control input. They occur when the time derivatives of the configurations are zero and when the configuration of the second link satisfies $(I_{2\times 2} - q_2 q_2^T)e_2 = 0$; the configuration of the first link is arbitrary. Consequently, there are two equilibrium manifolds in $(\mathbb{S}^1)^2$ given by

$$\left\{(q_1,q_2) \in (\mathbb{S}^1)^2 : q_2 = e_2\right\},$$

referred to as the inverted equilibrium manifold, and

$$\left\{(q_1,q_2) \in (\mathbb{S}^1)^2 : q_2 = -e_2\right\},$$

referred to as the hanging equilibrium manifold.

4 A 3 DOF modified Furuta pendulum

We next consider a natural modification of the Furuta pendulum that does not seem to have been previously investigated. In particular, assume that the first link is constrained to rotate about a fixed pivot in a fixed horizontal plane, while the second link is constrained to rotate with respect to a pivot point fixed in the first link as a spherical pendulum, that is without constraint.

Assume the first link is a thin rigid body idealized as a massless link; the length of the link is L_1 and a mass m_1 is concentrated at the end of the first link. The second link is also a thin rigid body idealized as a massless link; the length of the link is L_2 and a mass m_2 is concentrated at the end of the second link.

The first link is actuated by a control torque $u \in \mathbb{R}^1$ in the vertical direction; the second link is not actuated.

Introduce a Euclidean inertial frame for $\mathbb{R}^3$ where the origin is located at the inertially fixed pivot of the first link; the first two axes are assumed to be horizontal while the third axis is assumed to be vertical.

Since the mass at the end of the first link moves in a horizontal plane, the position vector $x_1 \in \mathbb{R}^3$ of the first mass element in the inertial frame is

$$x_1 = L_1 C q_1,$$

where $q_1 = (q_{11}, q_{12}) \in \mathbb{S}^1$ can be thought of as the unit direction vector of the first link in the horizontal plane within which the first mass element moves.

The position vector $x_2 \in \mathbb{R}^3$ of the second mass element in the inertial frame is

$$x_2 = x_1 + L_2 q_2,$$

where $q_2 = (q_{21}, q_{22}, q_{23}) \in \mathbb{S}^2$ can be thought of as the unit direction vector for the second link in the inertial frame.

Since the position vectors for the two mass elements that define the modified Furuta pendulum can be expressed in terms of $q = (q_1, q_2) \in \mathbb{S}^1 \times \mathbb{S}^2$, the configuration manifold for the modified Furuta pendulum is $\mathbb{S}^1 \times \mathbb{S}^2$; the dimension of the configuration manifold is three, so there are three degrees of freedom. The following development follows the approach developed in [10, 9] for Lagrangian dynamics that evolve on the manifold $\mathbb{S}^1 \times \mathbb{S}^2$.

Thus, the velocity vectors of the two mass elements in the inertial frame are

$$\begin{aligned} \dot{x}_1 &= L_1 C \dot{q}_1, \\ \dot{x}_2 &= L_1 C \dot{q}_1 + L_2 \dot{q}_2. \end{aligned}$$

The expression for the kinetic energy of the mass element of the first link is

$$T_1(q, \dot{q}) = \frac{1}{2} m_1 \|\dot{x}_1\|^2 = \frac{1}{2} m_1 L_1^2 \|\dot{q}_1\|^2 .$$

The kinetic energy of the mass element of the second link is

$$\begin{aligned} T_2(q, \dot{q}) &= \frac{1}{2} m_2 \|\dot{x}_2\|^2 \\ &= \frac{1}{2} m_2 \|L_1 C \dot{q}_1 + L_2 \dot{q}_2\|^2 \\ &= \frac{1}{2} m_2 L_1^2 \|\dot{q}_1\|^2 + \frac{1}{2} m_2 L_2^2 \|\dot{q}_2\|^2 + m_2 L_1 L_2 \dot{q}_1^T C^T \dot{q}_2. \end{aligned}$$

Thus, the expression for the kinetic energy of the modified Furuta pendulum is

$$T(q, \dot{q}) = \frac{1}{2}(m_1 + m_2) L_1^2 \|\dot{q}_1\|^2 + \frac{1}{2} m_2 L_2^2 \|\dot{q}_2\|^2 + m_2 L_1 L_2 \dot{q}_1^T C^T \dot{q}_2.$$

The gravitational potential energy of the modified Furuta pendulum is

$$U(q) = m_2 g e_3^T x_2 = m_2 g L_2 e_2^T q_2.$$

The Lagrangian function $L : \mathrm{T}(\mathbb{S}^1 \times \mathbb{S}^2) \to \mathbb{R}^1$ for the modified Furuta pendulum is given by

$$L(q, \dot{q}) = \frac{1}{2}(m_1 + m_2) L_1^2 \|\dot{q}_1\|^2 + \frac{1}{2} m_2 L_2^2 \|\dot{q}_2\|^2 + m_2 L_1 L_2 \dot{q}_1^T C^T \dot{q}_2 - m_2 g L_2 e_3^T q_2.$$

Let $u \in \mathbb{R}^1$ denote the control torque on the first link in the vertical direction. The corresponding virtual work done by this control torque is easily determined, so that the Lagrange-d'Alembert principle can be written as

$$\delta \int_{t_0}^{t_f} L(q, \dot{q}) dt = \int_{t_0}^{t_f} u q_1^T S \delta q_1 \, dt,$$

for all possible infinitesimal variations $\delta q = (\delta q_1, \delta q_2) : [t_0, t_f] \to \mathrm{T}_q(\mathbb{S}^1 \times \mathbb{S}^2)$ that vanish at the endpoints, that is $\delta q(t_0) = \delta q(t_f) = 0$. Following the results in [9], the infinitesimal variations can be expressed as

$$\delta q_1 = \gamma_1 \mathcal{S} q_1,$$
$$\delta q_2 = S(\gamma_2) q_2,$$

where $\gamma_1 : [t_0, t_f] \to \mathbb{R}^1$ and $\gamma_2 : [t_0, t_f] \to \mathbb{R}^3$ vanish at t_0 and t_f. Substitute these expressions into the Lagrange-d'Alembert principle, integrate by parts, use the boundary conditions, and apply the fundamental lemma of the calculus of variations. After using several identities, the Euler–Lagrange equations are given by

$$(m_1+m_2)L_1^2\ddot{q}_1 + m_2L_1L_2(I_{2\times 2} - q_1q_1^T)C^T\ddot{q}_2 + (m_1+m_2)L_1^2\|\dot{q}_1\|^2 q_1 = u\mathcal{S}q_1, \tag{4.1}$$
$$m_2L_1L_2(I_{3\times 3} - q_2q_2^T)C\ddot{q}_1 + m_2L_2^2\ddot{q}_2 + m_2L_2^2\|\dot{q}_2\|^2 q_2 + m_2gL_2(I_{3\times 3} - q_2q_2^T)e_3 = 0. \tag{4.2}$$

These Euler–Lagrange equations describe the global evolution of the controlled dynamics of the modified Furuta pendulum on the tangent bundle $\mathrm{T}(\mathbb{S}^1 \times \mathbb{S}^2)$.

It is interesting that the Euler–Lagrange equations (4.1) and (4.2) for this three degree of freedom version of the Furuta pendulum has a more compact form than do the Euler–Lagrange equations for the prior two degree of freedom versions of the Furuta pendulum. This feature may not have importance in terms of the complexity of the resulting dynamics for the three Furuta pendulum versions.

The equilibrium solutions of this modified Furuta pendulum are easily determined assuming zero control input. They occur when the time derivatives of the configurations are zero and when the configuration of the second link satisfies $(I_{3\times 3} - q_2q_2^T)e_3 = 0$; the configuration of the first link is arbitrary. Consequently, there are two equilibrium manifolds in $\mathbb{S}^1 \times \mathbb{S}^2$ given by

$$\left\{(q_1, q_2) \in \mathbb{S}^1 \times \mathbb{S}^2 : q_2 = e_3\right\},$$

referred to as the inverted equilibrium manifold, and

$$\left\{(q_1, q_2) \in \mathbb{S}^1 \times \mathbb{S}^2 : q_2 = -e_3\right\},$$

referred to as the hanging equilibrium manifold.

5 Nonlinear control of the Furuta pendulums

Nonlinear control problems can be readily defined for each of the controlled Furuta pendulum formulations previously introduced. Here, we do not provide comprehensive control analyses, but we do introduce several interesting new control problems. These control problems are especially challenging due to the under-actuation assumption and the nonlinear features of the global dynamics that evolve on the tangent bundle of the configuration manifold.

5.1 Nonlinear control of the two degree-of-freedom conventional Furuta pendulum

The conventional Furuta pendulum has served as a test bed example for nonlinear control to stabilize an underactuated, unstable equilibrium. There is an extensive published literature in this area [1, 2, 3, 4, 5, 6, 7]. All of the cited stabilization results are local in the sense that the domain of stability is a (perhaps) small neighborhood of the stabilized equilibrium. The global model developed previously for the standard Furuta pendulum can be used to verify those local nonlinear control results, and it can be used for purposes of simulation without concern for singularities that arise in using angle coordinates.

In order to connect the prior equations of motion (2.1) and (2.2) with the literature on stabilization of the two degree-of-freedom conventional Furuta pendulum, we linearize the equations of motion about the inverted equilibrium $(e_1, e_2) \in (\mathbb{S}^1)^2$, following the development in [9]. In terms of the local perturbations $q_1 = e_1 + (\xi_{11}, \xi_{12}) \in \mathbb{S}^1$ and $q_2 = e_2 + (\xi_{21}, \xi_{22}) \in \mathbb{S}^1$, it can be shown that (ξ_{12}, ξ_{21}) are local coordinates for the manifold $(\mathbb{S}^1)^2$ in a neighborhood of the equilibrium, and the linearized equations can be expressed as

$$\begin{bmatrix} (m_1+m_2)L_1^2 & m_2L_1L_2 \\ m_2L_1L_2 & m_2L_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\xi}_{12} \\ \ddot{\xi}_{21} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -m_2gL_2 \end{bmatrix} \begin{bmatrix} \xi_{12} \\ \xi_{21} \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix}.$$

These linearized equations of motion, valid in a neighborhood of the inverted equivalent, are equivalent to what is given in the published literature. It is clear that in a neighborhood of the inverted equilibrium, the coupling between the two local coordinate variables is due, at least to first order, to the nonzero off-diagonal terms in the inertial matrix.

The linearized equations imply that the inverted equilibrium is unstable and it is completely controllable and hence stabilizable. Both linear and nonlinear controllers have been proposed to stabilize the inverted equilibrium of the conventional Furuta pendulum and local stabilization can be verified using the linearized equations in local coordinates. It is not so common in the literature to study the resulting domain of attraction of the stabilized inverted equilibrium, but this can be done using the nonlinear equations of motion.

5.2 Nonlinear control of the two degree-of-freedom modified Furuta pendulum

The two degree-of-freedom modified Furuta pendulum seems not to have been previously studied as a nonlinear control problem. Here we make a few comments about the problem of stabilization of an inverted equilibrium.

We linearize the equations of motion (4.1) and (4.2) about the inverted equilibrium $(e_1, e_2) \in (\mathbb{S}^1)^2$, following the development in [9]. In terms of the local perturbations $q_1 = e_1 + (\xi_{11}, \xi_{12}) \in \mathbb{S}^1$ and $q_2 = e_2 + (\xi_{21}, \xi_{22}) \in \mathbb{S}^1$, it can be shown that (ξ_{12}, ξ_{21}) are local coordinates for the manifold $(\mathbb{S}^1)^2$ in a neighborhood of the equilibrium, and the linearized equations can be expressed as

$$\begin{bmatrix} (m_1+m_2)L_1^2 & 0 \\ 0 & m_2L_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\xi}_{12} \\ \ddot{\xi}_{21} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -m_2gL_2 & 0 \end{bmatrix} \begin{bmatrix} \xi_{12} \\ \xi_{21} \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix}.$$

It is clear that in a neighborhood of the inverted equilibrium, there is no coupling between the two local coordinate variables, at least to first order; this is a consequence of the fact that the off-diagonal terms in the inertial matrix are, to first order, zero. This implies that the control input, to first order, does not influence the dynamics of the second local coordinate variable. The linearized equations imply that the inverted equilibrium is unstable. The linearized equations are not locally controllable.

The nonlinear equations of motion must be analyzed to determine if the modified Furuta pendulum is controllable or stabilizable in a neighborhood of the inverted equilibrium. We do not go into the details of such nonlinear control analysis, except to mention that existing sufficient conditions for controllability of under-actuated nonlinear control systems, see for example [11], are not satisfied. Our expectation is that the inverted equilibrium can not be stabilized. Some weaker notion of stabilization of the inverted equilibrium, perhaps in some averaged sense, may be possible utilizing the available nonlinear control authority. These are open questions that we leave for future investigations.

5.3 Nonlinear control of the three degree-of-freedom modified Furuta pendulum

The three degree-of-freedom modified Furuta pendulum has not been previously studied as a nonlinear control problem. In physical terms, it should exhibit some of the features of the conventional Furuta pendulum and some of the features of the two degree-of-freedom modified Furuta pendulum.

We linearize the equations of motion (4.1) and (4.2) about the inverted equilibrium $(e_1, e_3) \in (\mathbb{S}^1 \times \mathbb{S}^2)$, following the development in [9]. In terms of the local perturbations $q_1 = e_1 + (\xi_{11}, \xi_{12}) \in \mathbb{S}^1$ and $q_2 = e_3 + (\xi_{21}, \xi_{22}, \xi_{23}) \in \mathbb{S}^2$, it can be shown that $(\xi_{12}, \xi_{21}, \xi_{22})$ are local coordinates for the manifold $(\mathbb{S}^1 \times \mathbb{S}^2)$ in a neighborhood of the inverted equilibrium, and the linearized equations can be expressed as

$$\begin{bmatrix} (m_1+m_2)L_1^2 & 0 & m_2L_1L_2 \\ 0 & m_2L_2^2 & 0 \\ m_2L_1L_2 & 0 & m_2L_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\xi}_{12} \\ \ddot{\xi}_{21} \\ \ddot{\xi}_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -m_2gL_2 \end{bmatrix} \begin{bmatrix} \xi_{12} \\ \xi_{21} \\ \xi_{22} \end{bmatrix} = \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix}.$$

It is clear that in a neighborhood of the inverted equilibrium, there is no coupling between the first two local coordinate variables, at least to first order; this is a consequence of the fact that the relevant off-diagonal terms in the inertial matrix are, to first order, zero. This implies that the control input, to first order, does not influence the dynamics of the second local coordinate variable. The linearized equations imply that the inverted equilibrium is unstable. The linearized equations are not locally controllable.

The nonlinear equations of motion must be analyzed to determine if the modified Furuta pendulum is controllable or stabilizable in a neighborhood of the inverted equilibrium. We again avoid the details of such nonlinear control analysis, except to mention that existing sufficient conditions for controllability of under-actuated nonlinear control systems [11] are not satisfied. We expect that the inverted equilibrium can not be stabilized, except perhaps in a weak, averaged sense. As for the two degree-of-freedom Furuta pendulum, these are open control questions.

6 Conclusions

The Furuta pendulum, in the several forms considered in this paper, is a physically simple double pendulum example. It provides a basis for the formulation of several under-actuated nonlinear control problems. The controlled equations of motion in each case are globally defined, and they can be used to characterize extremely complicated controlled (or uncontrolled) dynamics, especially large pendulum motions. The stabilization and control problem for the conventional Furuta pendulum is briefly reviewed; stabilization and control problems for the modified Furuta pendulums are extremely challenging, since they are not linearly controllable at the inverted equilibrium. There are many open questions for the modified Furuta pendulum control systems.

We emphasize that the development has been expressed in terms of a geometric form of the equations of motion that is globally defined everywhere on the configuration manifold. These globally defined equations of motion can provide the basis for analytical or computational (simulation based) investigations of closed loop properties.

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